COMPLEX NUMBERS

BY CHAKUAMBA FORTUNE T(CHAKS)

+263771580933

OBJECTIVES OF THE TOPIC

~Perform simple arithmetics of complex numbers

~Finding the modulus and argument of complex numbers

~Represent complex numbers on the Argand diagram

~Express complex numbers in polar form and exponential form

~Perform operations with complex numbers in polar form and exponential form

~Derive and prove DeMoivre's theorem

~Solving equations using DeMoivre's theorem

~Prove trigonometric identities using DeMoivre's theorem

~Solve polynomial equations with real coefficients and at least one non~real roots

~ Loci of equations and inequalities involving complex numbers

COMPLEX NUMBERS SYSTEM

If $x^2 + 4 = 0 \rightarrow x^2 = -4$

$x = \sqrt{-4}i = \sqrt{-1 \times 4}$ $\qquad$ $but \rightarrow i = \sqrt{-1}$

$x = \sqrt{4}i$

$x = \pm 2i$

SINCE $\quad i = \sqrt{-1}$

$\therefore i^2 = -1$

~A complex number can be denoted by $\boldsymbol{z}$ where $\boldsymbol{z} = x + yi$

x *is the real part* $Re(z)$

y *is imaginary part* $Im(z)$

ADDITION AND SUBSTRACTION OF COMPLEX NUMBERS

If $z_1 = a + bi$ *and* $z_2 = c + di$

then

$z_1 + z_2 = (a + bi) + (c + di)$

$= a + c + bi + di$ $\qquad$ *add the real parts together and the imaginary parts together*

$= (a + c) + (b + d)i$

$z_1 - z_2 = (a + bi) - (c + di)$

$= a - c + bi - di$ $\qquad$ *subtract the real parts together and imaginary parts together*

$= (a - c) + (b - d)i$

Example 1

$z_1 = 4 + 3i \qquad z_2 = 1 - 3i \qquad$ *Find (i)* $z_1 + z_2$

(ii) $z_1 - z_2$

Solution

(i) $z_1 + z_2 = (4 + 3i) + (1 - 3i)$

$= (4 + 1) + \big(3 + (-3)\big)i$

$= 5 + 0i = 5$

(ii) $z_1 - z_2 = (4+3i) - (1-3i)$

$= (4-1) + (3-(-3))i$

$= 3 + 6i$

FOLLOW UP EXERCISE

1) $(3-7i) + (-6+7i)$
2) $(3+4i) + (2+2i) + (5+6i)$
3) $(-4-6i) - (-8-8i)$
4) $5(4+3i) - 4(-1+2i)$
5) $(3\sqrt{2}+i) - (\sqrt{2}-i)$

MULTIPLICATION OF COMPLEX NUMBERS

~just operate as if you expanding algebra expressions but take note that $i^2 = -1$

$(a+bi)(c+di) = a(c+di) + a(c+di)$

Example 1

$z_1 = 4+3i$ *and* $z_2 = 3-2i$ *Find* z_1z_2

Solution

$z_1z_2 = (4+3i)(3-2i)$

$= 4(3-2i)3i(3-2i)$

$= 12 - 8i + 9i - 6i^2$

$= 12 + i - 6(-1)$

$= 18 + i$

Example 2

$z_1 = 3-2i$ *and* $z_2 = 4-i$ *Find* z_1z_2

$z_1z_2 = (3-2i)(4-i)$

$= 3(4-i) - 2i(4-i))$

$= 12 - 3i - 8i + 2i^2$

$= 12 + 11i + 2(-1)$

$= 10 - 11i$

COMPLEX CONJUGATE

If $\boldsymbol{z} = x + yi$ *the conjugate of z can be denoted by* z^* *or* $\bar{z}$

$$so\ \boldsymbol{z}^* = x - yi$$

Relationship between **z** and $\mathbf{z}^*$ is

$$z\,z^* = x^2 + y^2$$

$$z + z^* = 2Re(z)$$

$$z - z^* = i2Im(z)$$

NOTES

~*Geometrical relationships between complex number and its conjugate is a reflection in the x~axis (check on worked examples page 10)*

Zimsec November 2017 Paper 1(edited on part c)

~The property of complex conjugate is important when we are dealing with division of complex numbers

DIVISION OF COMPLEX NUMBERS

~Find the conjugate of the denominator and multiply both the numerator and denominator with conjugate of the denominator

Example 1

Given that $z_1 = -3 + 5i$ *and* $z_2 = -3 + i$ *Find* $\frac{z_1}{z_2}$

Solution

$\frac{z_1}{z_2} = \frac{-3+5i}{-3+i}$

$= \frac{(-3+5i)(-3-i)}{(-3+i)(-3-i)}$ NB: $\boldsymbol{z} \times \boldsymbol{z}^* = x^2 + y^2$

denominator

$= \frac{9+3i-15i-5i^2}{(-3)^2+1^2}$

$= \frac{9+3i-15i-(5)(-1)}{(-3)^2+1^2}$ Remember $i^2 = -1$

$= \frac{14-12i}{10}$

$= \frac{7}{5} - \frac{6}{5}i$

Example 2

The complex number is $z = \frac{-1+3i}{2+i}$. Express z in the form x + yi

Solution

$z = \frac{(-1+3i)(2-i)}{(2+i)(2-i)}$

$= \frac{-2+i+6i-3i^2}{2^2+1^2}$

$= \frac{1+7i}{5}$

$= \frac{1}{5} + \frac{7}{5}i$

FOLLOW UP EXERCISE

1) $(6+3i)(7+2i)$

2) $(2+3i)^4$ *hint use binomial expansion or pascals*

3) $Given\ that\ z = 2\sqrt{2} + \sqrt{2}i\ \ find\ i)\ z + 2z^*\ \ ii) zz^*\ \ iii)\ \sqrt{2} - z^*\ \ iv)\ 2z - 2z^*$

4) $if\ z_1 = 2 + i\ and\ z_2 = 3 + i\ \ find\ i)\ \frac{z_2}{z_1 - 1}\ \ ii) \frac{2z_2 + z_1}{z_1}\ \ iii) \frac{(z_2)^2 - 1}{z_1 + 1}$

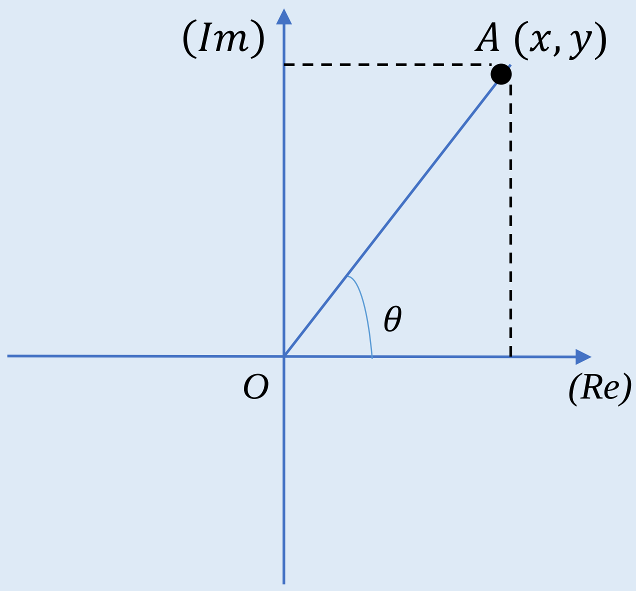
5) $The\ complex\ z\ satisfy\ the\ equation\ \frac{z}{5+2i} = \frac{1}{2-i}\ \ find\ z\ in\ form\ a + bi$

NB: Number 2 of the above exercise will be dealt with it using short method later on this chapter(under DeMoivres theorem)

MODULUS AND ARGUMENT OF COMPLEX NUMBERS

~A complex can be represented on a diagram called Argand diagram

$x{\sim}axis$ represents the real axis and $y{\sim}axis$ represents the imaginary axis



If $\mathbf{z} = x + yi$,

x and y are points of the complex number on the argand diagram relative to origin i.e OA

~the length of **OA** is the modulus of **z** which is denoted by **|z|**

$$|z| = \sqrt{x^2 + y^2}$$

~ The angle θ between the line OA and real axis is called argument of z (which is usually referred as the principal argument)

Argument of z it lies in the range $-\pi < \theta \leq \pi$ or $-180 < \theta \leq 180$

NOTE: the argument of complex is not unique it can be given by $\theta \pm 2\pi \boldsymbol{n}$

Notes on how to find the argument of a complex number

~The argument of a complex number is determined by the position of complex number on argand diagram

measured from positive real axis

~Angles below the real axis are negative are measured in a clockwise direction

~Angles above the real axis are positive and are measured in a anticlockwise direction

~ *Finding the principal argument of complex number*

First find $\alpha = tan^{-1}\left|\frac{y}{x}\right|$,

where $\left|\frac{y}{x}\right|$ is the absolute value [or you just take values of y *and* x ignoring the signs]

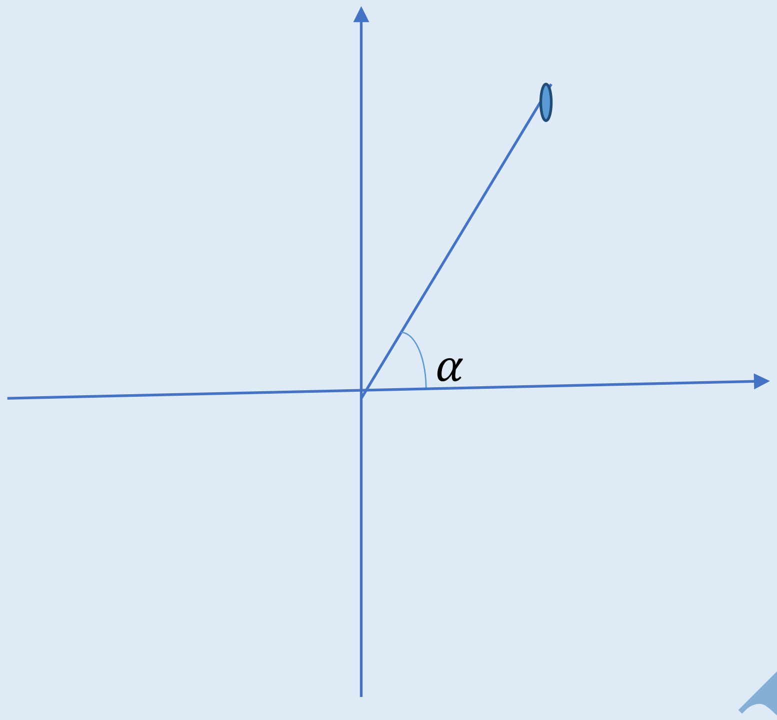
NB: α is not the real principal argument except for first quadrant angles where $\theta = \alpha$

[principal argument is determined by position of complex in argand diagram]

CHECK BELOW

Angles of complex numbers in the first quadrant

$\theta = \alpha$

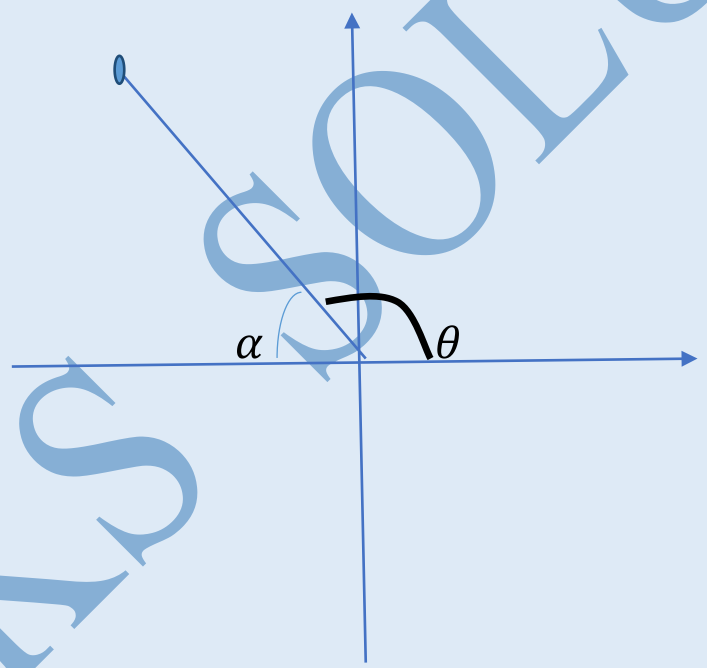


Angles of complex numbers in the second quadrant

$\boldsymbol{\theta = \pi - \alpha}$

OR

$\boldsymbol{\theta = 180° - \alpha}$

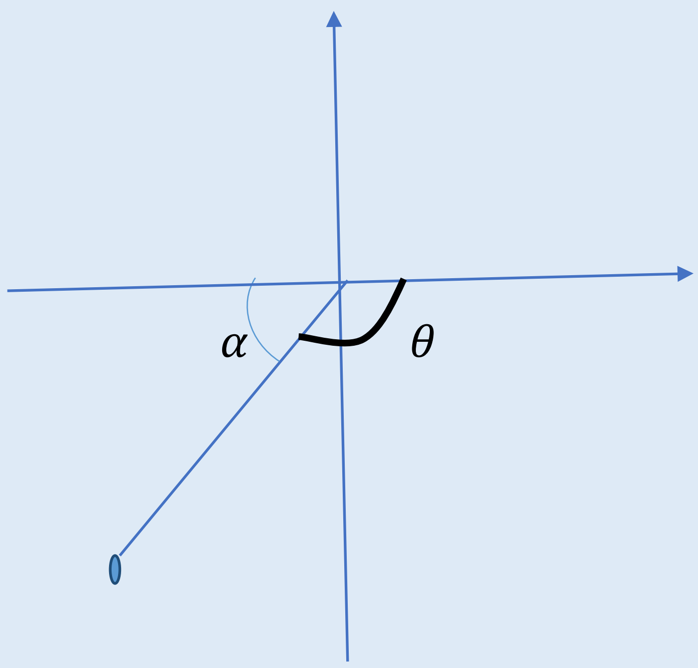


Angles of complex numbers in the third quadrant

$\boldsymbol{\theta = \alpha - \pi}$

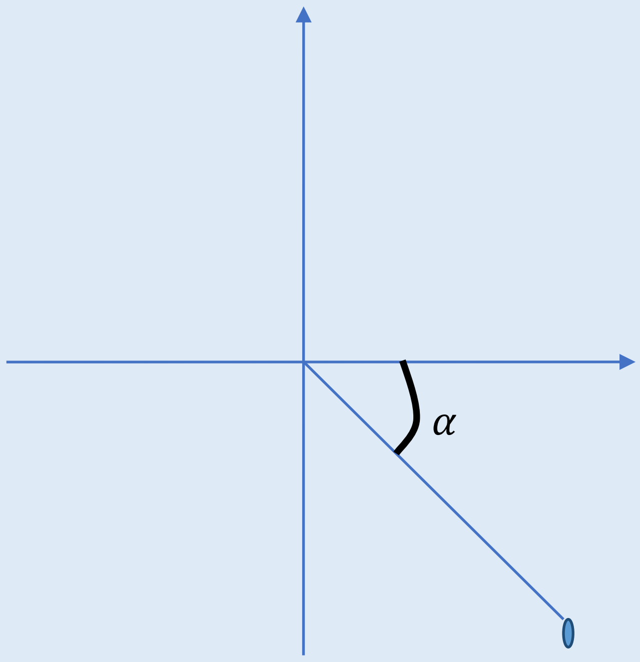
OR

$\boldsymbol{\theta = \alpha - 180°}$



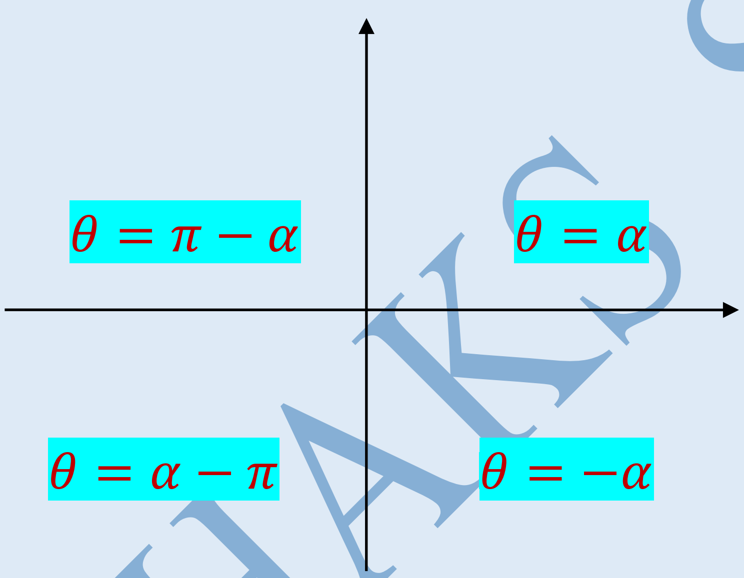
~***Angles of complex numbers in fourth quadrant***

$\boldsymbol{\theta = -\alpha}$



NB: α is acute angle between the real axis and line which joining O and position point of the complex number

For easy remembrance the diagram below can help you



Worked Examples

Zimsec June 2019 Paper 1

The complex

$$u = \frac{4 - 8i}{i}$$

i) Express u in the form $x + yi$

ii) Find the magnitude of u and argument of u

iii) Sketch u on an Argand diagram

Solution

$i)\ u = \dfrac{4-8i}{i}$

$= \dfrac{(4-8i)(-i)}{i(-i)}$

$= \dfrac{-4i+8i^2}{-i^2}$

$= \dfrac{-4i+8i^2}{-i^2}$

$= \dfrac{-8-4i}{1}$

$= -8-4i$

$ii)\ |u| = \sqrt{(-8)^2+(-4)^2}$

$= \sqrt{80}$

$= 4\sqrt{5}$

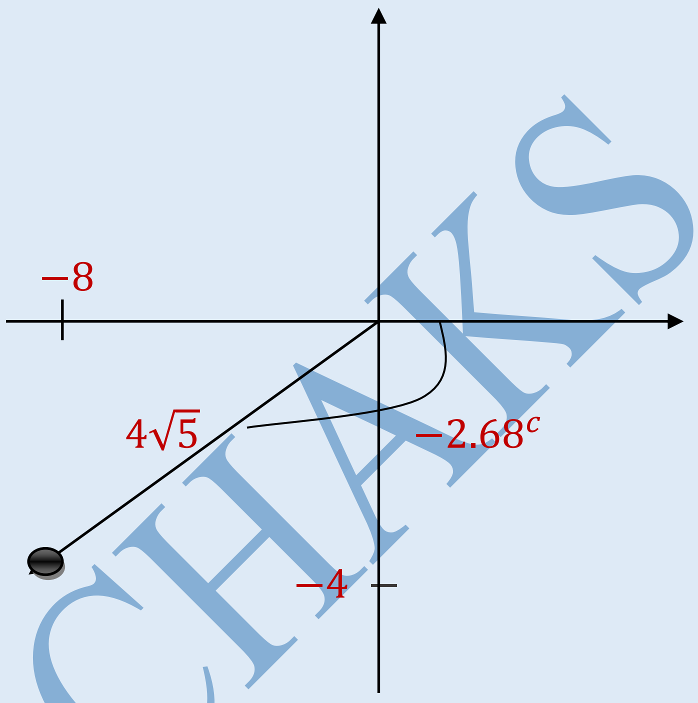
u is in third quadrant

$\theta = \alpha - \pi$

$\theta = \tan^{-1}\left(\dfrac{4}{8}\right) - \pi$

$\theta = -2.68^c \quad (3\ s.f)$

$iii)$ Sketch of u on an Argand diagram



<u>Zimsec November 2016 Paper 1</u>

The complex $\frac{1}{1+2i}$ is denoted by u

i) Find the modulus of u

ii) Find the argument of u

Solution

$$u = \frac{1}{1+2i} = \frac{1(1-2i)}{(1+2i)(1-2i)} = \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i$$

Now

$$i)\ |u| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{2}{5}\right)^2} = \frac{\sqrt{5}}{5}$$

ii) u is in the fourth quadrant

$$\theta = -\alpha$$

$$\theta = -\tan^{-1}\left(\frac{\frac{2}{5}}{\frac{1}{5}}\right)$$

$$\theta = -\tan^{-1}(2)$$

$\theta = -1{,}11^{c}$ or equivalent to degrees

Zimsec November 2017 Paper 1(edited on part c)

Given the complex numbers $w = 1 + 2i$ and $u = 3 - i$, find

a) In the form $a + bi$ where a and b are real

i) $u + w$

ii) uw

b) Find the argument of uw

c) Represent on the same Argand diagram complex u and u^*, where u^* is the conjugate of u , hence state the geometrical relationship between u and u^*

Solution

i) $u + w = (1 + 2i) + (3 - i)$

$= (1 + 3) + \big(2 + (-1)\big)i$

$= 4 + i$

ii) $uw = (1 + 2i)(3 - i)$

$= 1(3 - i) + 2i(3 - i)$

$= 3 - i + 6i - 2i^2$

$= 3 - i + 6i + 2$

$= 5 + 5i$

b) $arg\ uw = \arg(5 + 5i)$

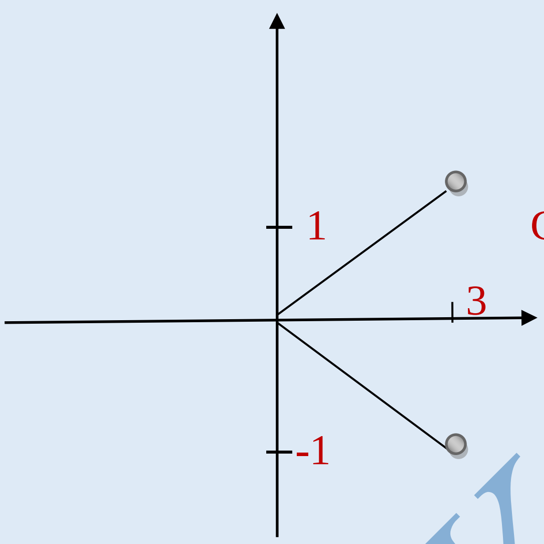
$uw = 5 + 5i$ is in the first quadrant

$\theta = \alpha$

$\theta = \tan^{-1}\left(\frac{5}{5}\right)$

$\theta = \frac{\pi}{4}\ or\ 45°$

c) $u = 3 - i$ $\qquad u^* = 3 + i$(conjugate)



Geometrical relationship of $u\ and\ u^*$ is a reflection in the $x - axis$ (real axis)

(under relationship between z and z^*)

Some special properties of modulus and argument of complex numbers

Modulus

1) $|z^n| = |z|^n$ ***2)*** $|z_1 z_2| = |z_1| \times |z_2|$ ***3)*** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Argument

1) $argz^{\boldsymbol{n}} = \boldsymbol{n}argz$ ***2)*** $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ ***3)*** $arg\left(\frac{z_1}{z_2}\right) = argz_2 - argz_2$

Example 1

If $u = i \quad and\ w = 1 + \sqrt{3}i$

a) Find modulus of

i) w^5 ii) u^2w^4 iii) $\left|\frac{u^2}{w^2}\right|$

b) find the argument of

i) w^5 ii) u^2w^4 iii) $\frac{u^2}{w^2}$

Solution

a) $|w| = \left|1 + \sqrt{3}i\right|$

$= \sqrt{1^2 + (\sqrt{3})^2}$

$= \ 2$

$|u| = |i|$

$= \sqrt{(-1)^2}$

$= 1$

Now

i) $|w^5| = |w|^5$

$= 2^5$

$= 32$

ii) $|u^2w^4| = |u^2| \times |w^4|$

$= |u|^2 \times |w|^4$

$= 1^2 \times 2^4$

$= 16$

iii) $\left|\frac{u^2}{w^2}\right| = \frac{|u^2|}{|w^2|}$

$= \frac{|u|^2}{|w|^2}$

$= \frac{1^2}{2^2}$

$= \frac{1}{4}$

b) $arg\mathbf{w}$

since $\mathbf{w}$ is in the first quadrant

$\theta = \alpha$

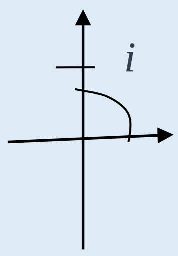
$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$arg\mathbf{w} = \frac{\pi}{3}$

$argu$ note :argu u check on the argand diagram

$\theta = \frac{\pi}{2}$

$arg\mathbf{u} = \frac{\pi}{2}$



Now

i) $arg\mathbf{w}^5 = 5arg\mathbf{w}$

$= 5\left(\frac{\pi}{3}\right)$

$= \frac{5}{3}\pi$

ii) $arg\ (u^2w^4) = arg\boldsymbol{u}^2 + arg\boldsymbol{w}^4$

$= 2arg\boldsymbol{u} + 4arg\boldsymbol{w}$

$= 2(\frac{\pi}{2}) + 4\left(\frac{\pi}{3}\right)$

$= \pi + \frac{4}{3}\pi$

$= \frac{7}{3}\pi$

iii) $\arg\left(\frac{u^2}{w^2}\right) = arg\boldsymbol{u}^2 - arg\boldsymbol{w}^2$

$= 2\text{arg}\mathbf{u} - 2\text{arg}\mathbf{w}$

$= 2\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right) - \mathbf{2}\left(\frac{\pi}{3}\right)$

$= \pi - \frac{2}{3}\pi$

$= \frac{1}{3}\pi$

PROBLEMS INVOLVING COMPLEX NUMBERS

Equal complex numbers

We can solve problems of complex numbers by equating real parts and imaginary parts from each side of an equation involving complex numbers

If $a_1 + ib_1 = a_2 + ib_2$, then

$a_1 = a_2 \Longrightarrow$ *real parts are equal*

$b_1 = b_2 \Longrightarrow$ *imaginary parts are equal*

Example 1

Given that $(a - b) + (a + b)i = 9 + 13i$ find the value of a and b

Solution

$(a - b) + (a + b)i = 9 + 13i$

$a - b = 9 \quad \ldots..i$

$a + b = 13 \ldots..ii$

Subtracting $ii\ from\ i$

$\Longrightarrow -2b = -4$

$b = 2$

Substituting b in ii

$\Longrightarrow a + 2 = 13$

$a = 13 - 2$

$a = 11$

$\therefore\ a = 11\ \ and\ b = 2$

Example 2

Given that $(-3 + 7i) = (5 - 2i)(x + yi)$.Find the value of x and y

Hence find the modulus and argument of $x + yi$

Solution

$(-3 + 7i) = (5 - 2i)(x + yi)$

$-3 + 7i = 5x + 5yi - 2xi - 2yi^2$

$-3 + 7i = 5x + 5yi - 2xi - 2y(-1)$

$-3 + 7i = 5x + 5yi - 2xi + 2y$

$-3 + 7i = 5x + 2y + (5y - 2x)i$

Remember for EQUAL COMPLEX numbers

⟹REAL PARTS ARE EQUAL

⟹IMAGINARY PARTS ARE EQUAL

Now

$-3 = 2y + 5x \ldots\ldots i \quad \times 5$

$7 = 5y - 2x \ldots\ldots ii \quad \times 2$

$-15 = 10y + 25x \ldots\ldots i$

$14 = 10y - 4x \ldots\ldots ii$

Subtracting ii from i

$\Longrightarrow -29 = 29x$

$\Longrightarrow x = -1$

Substituting the value of x in ii

$\Longrightarrow 7 = 5y - 2(-1)$

$\Longrightarrow 7 - 2 = 5y$

$\Longrightarrow 5 = 5y$

$\Longrightarrow y = 1$

$\therefore x = -1\,, y = 1$

Now

$\Longrightarrow let\ z = -1 + i$

$Arg\,(z)$

$\theta = \pi - \alpha$ *second quadrant complex number*

$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right)$

$\theta = \pi - \frac{\pi}{4}$

$\theta = \frac{3}{4}\pi$

$|z| = \sqrt{(-1)^2 + 1^2}$

$|z| = \sqrt{2}$

<u>*Example 3*</u>

<u>Zimsec November 2014 Paper</u>

The complex number z satisfies the equation

$$z + 2\overline{z} = \frac{13}{-2 + 3i}$$

Find

i) z in the form $x + yi$

ii) modulus and argument of $\frac{1}{z}$

<u>Solution</u>

$z + 2\overline{z} = \frac{13}{-2+3i}$

Let $z = x + yi$

$\bar{z} = x - yi$

$\Longrightarrow (x + yi) + 2(x - yi) = \frac{13}{-2+3i}$

$\Longrightarrow x + yi + 2x - 2yi = \frac{13(-2-3i)}{(-2+3i)(-2-3i)}$

$\Longrightarrow 3x - yi = \frac{-26-39i}{13}$

$\Longrightarrow 3x - yi = -2 - 3i$

NOW

Remember for equal complex numbers

⟹REAL PARTS ARE EQUAL

⟹IMAGINARY PARTS ARE EQUAL

$3x = -2 \Longrightarrow x = -\frac{2}{3}$

$-y = -3 \Longrightarrow y = 3$

$\therefore z = -\frac{2}{3} + 3i$

Now $\frac{1}{z} = \frac{1}{-\frac{2}{3}+3i}$

$\therefore |z| = \sqrt{\left(-\frac{2}{3}\right)^2 + (3)^2} = \frac{\sqrt{85}}{3}$

Example 4

Zimsec June 2012 Paper 2

The complex number a whose conjugate a^* satisfies the equations $4aa^* + 12i = 8a + 6$ find the two

possible values of a giving your answer in the form $p + qi$ where p and q are real

NB: On the original question paper the equation is $4aa^* + 12i = 8a + 16$ 16 was supposed to be 6

Solution

Let

$a = p + qi$

$a^* = p - qi$

Substituting the values of a and a^*

$\Longrightarrow 4aa^* + 12i = 8a + 6$

$\Longrightarrow 4(p + qi)(p - qi) + 12i = 8(p + qi) + 6$

$\Longrightarrow 4(p^2 + q^2) + 12i = 8p + 8qi + 6$

$\Longrightarrow 4p^2 + 4q^2 + 12i = 8p + 8qi + 6$

$\Longrightarrow 4p^2 + 4q^2 + 12i = 8p + 6 + 8qi$ *dividing by 2 throughout the equation*

$\Longrightarrow 2p^2 + 2q^2 + 6i = 4p + 3 + 4qi$

Remember for equal complex numbers
$\Longrightarrow$*REAL PARTS ARE EQUAL*
$\Longrightarrow$*IMAGINARY PARTS ARE EQUAL*

$2p^2 + 2q^2 = 4p + 3 \ldots \ldots i$

$6 = 4q \ldots \ldots ii$

from ii

$q = \frac{3}{2}$

Substituting the values of q in i

$\Longrightarrow 2p^2 + 2\left(\frac{3}{2}\right)^2 = 4p + 3$

$\Longrightarrow 2p^2 + \frac{9}{2} = 4p + 3$ *Multiplying by 2 throughout*

$\Longrightarrow 4p^2 - 8p + 9 - 6 = 0$

$\Longrightarrow 4p^2 - 8p + 3 = 0$

$\Longrightarrow 4p^2 - 6p - 2p + 3 = 0$

$\Longrightarrow 2p(2p - 3) - 1(2p - 3) = 0$

$\Longrightarrow (2p - 1)(2p - 3) = 0$

$\therefore p = \frac{1}{2} \quad or \quad \frac{3}{2}$

Now

$a = \frac{1}{2} + \frac{3}{2}i \quad or \quad a = \frac{3}{2} + \frac{3}{2}i$

COMPLEX NUMBERS IN POLAR FORM AND EXPONENTIAL FORM

If $z = x + yi$, z *can be written be as*

Polar form

$z = \boldsymbol{r}(cos\boldsymbol{\theta} + isin\boldsymbol{\theta})$

Exponential form

$z = \boldsymbol{r}e^{\boldsymbol{\theta}i}$

$\Longrightarrow Z = \boldsymbol{r}(cos\boldsymbol{\theta} + isin\boldsymbol{\theta}) = \boldsymbol{r}e^{\boldsymbol{\theta}i}$

Where $\boldsymbol{r}$ *is the modulus of* z *and* θ *is argument of* z

Important fact

$(cos\boldsymbol{\theta} - isin\boldsymbol{\theta}) = [cos(-\boldsymbol{\theta}) + isin(-\boldsymbol{\theta})]$

Example 1

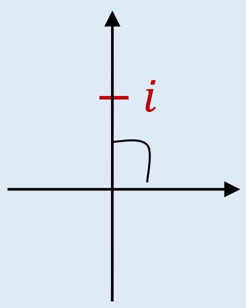
Express $z = \frac{1+i}{1-i}$ in the form $\boldsymbol{r}(cos\theta + isin\theta)$

Solution

$z = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+2i+i^2}{1^2+1^2} = \frac{2i}{2} = i$

Argument of z

$\theta = \frac{\pi}{2}$



modulus of z

$\boldsymbol{r} = \sqrt{1^2}$

$\boldsymbol{r} = 1$

Now $\frac{1+i}{1-i} = cos\frac{\pi}{2} + isin\frac{\pi}{2}$

Example 2

Express $\frac{3}{1+i\sqrt{3}}$ in

i) Exponential form

ii)Polar form

Solution

$$\frac{3(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})}=\frac{3-i3\sqrt{3})}{1^2+\left(\sqrt{3}\right)^2}=\frac{3-i3\sqrt{3})}{4}=\frac{3}{4}-\frac{i3\sqrt{3}}{4}$$

$$\Rightarrow\left|\frac{3}{4}-\frac{i3\sqrt{3}}{4}\right|=\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{3\sqrt{3}}{4}\right)^2}=\sqrt{\frac{36}{16}}=\frac{3}{2}$$

Complex in fourth quadrant

$\theta=-\alpha$

$$\text{Arg}\left(\frac{3}{4}-\frac{i3\sqrt{3}}{4}\right)=-\tan^{-1}\left(\frac{\frac{3\sqrt{3}}{4}}{\frac{3}{4}}\right)$$

$$=-\tan^{-1}\left(\frac{3\sqrt{3}}{3}\right)$$

$$=-\frac{\pi}{3}\quad or\ -60^{\circ}$$

Exponential form

$$\therefore\ \frac{3}{1+i\sqrt{3}}=\frac{3}{2}e^{-\frac{\pi}{3}i}$$

Polar form

$$\therefore\ \frac{3}{1+i\sqrt{3}}=\frac{3}{2}\left[cos\left(-\frac{\pi}{3}\right)+isin\left(-\frac{\pi}{3}\right)\right]$$

Remember $(cos\theta-isin\theta)=[\cos(-\theta)+isin(-\theta)]$

$$=\frac{3}{2}\left[cos\left(\frac{\pi}{3}\right)-isin\left(\frac{\pi}{3}\right)\right]$$

NB: You can return a complex number from polar or exponential form to the form of $x + yi$

CHECK EXAMPLES BELOW

Example 1

Express $2\left(cos\frac{\pi}{4} - isin\frac{\pi}{4}\right)$ in form $x + yi$

Solution

$$2\left(cos\frac{\pi}{4} - isin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2} - i\sqrt{2}$$

Example 2

Express $\sqrt{2}e^{\frac{19}{4}\pi}$ in the form

$i)\ r(cos\theta + isin\theta)\ \ where\ \ -\pi < \theta \leq \pi$

$ii)\ x + yi$ where x and y are real

Solution

i) θ needed by the question must lie in this range $-\pi < \theta \leq \pi$

Remember the argument for the complex number is not unique and it is given by

$argz = \theta \pm 2\pi n$

Now

$\frac{19}{4}\pi \Longrightarrow \frac{11}{4}\pi \Longrightarrow \frac{3}{4}\pi$ by substracting 2π to return argument in the range $-\pi < \theta \leq \pi$

$$\therefore \sqrt{2}e^{\frac{19}{4}\pi} = \sqrt{2}\left(cos\left(\frac{3}{4}\pi\right) + isin\left(\frac{3}{4}\pi\right)\right)$$

$$ii)\ \sqrt{2}e^{\frac{19}{4}\pi} = \sqrt{2}\left(cos\left(\frac{19}{4}\pi\right) + isin\left(\frac{19}{4}\pi\right)\right)$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= -1 + i$$

FOLLOW UP EXERCISE

1)Express the following in the form r$(cos\theta + isin\theta)$ giving the exact value of θ and r where possible

a) $-2\sqrt{3} - 2i$

b) $-8 + 2i$

c) $\sqrt{3} + 2i$

d) $3 - 4i$

e) $-2\sqrt{3} - 2\sqrt{3}i$

2)Express the following in the form $x + yi$ where $x \in R$ and $y \in R$

a) $8\left(cos\frac{9}{4}\pi - isin\frac{9}{4}\pi\right)$

b) $-4\left(cos\frac{7}{6}\pi - isin\frac{7}{6}\pi\right)$

c) $3\sqrt{2}e^{-\frac{3}{4}\pi i}$

d) $e^{\frac{5}{6}\pi i}$

3)Express the following in the form r$(cos\theta + isin\theta)$ $where -\pi < \theta \leq \pi$,

a) $6e^{-\frac{13}{6}\pi i}$

b) $3\sqrt{2}e^{\frac{17}{5}\pi i}$

PRODUCT AND QUOTIENTS OF COMPLEX NUMBERS IN POLAR AND EXPONENTIAL FORM

If $z_1 = \boldsymbol{r_1}(cos\boldsymbol{\theta_1} + isin\boldsymbol{\theta_1})$ and $z_2 = \boldsymbol{r_2}(cos\boldsymbol{\theta_2} + isin\boldsymbol{\theta_2})$

POLAR FORM

NOW

Multiplying z_1 *and* z_2

$$\Rightarrow z_1z_2 = \boldsymbol{r_1r_2}[\cos(\boldsymbol{\theta_1} + \boldsymbol{\theta_2}) + isin(\boldsymbol{\theta_1} + \boldsymbol{\theta_2})]$$

multiply the modulus and add the arguments

Dividing z_1 and z_2

$$\Longrightarrow \frac{z_1}{z_2} = \frac{\boldsymbol{r_1}}{\boldsymbol{r_2}}[\cos(\boldsymbol{\theta_1} - \boldsymbol{\theta_2}) + i\sin(\boldsymbol{\theta_1} - \boldsymbol{\theta_2})]$$

divide the modulus and substrate the arguments

NB: BEFORE ANY OPERATION COMPLEX IN THE FORM $\boldsymbol{\cos\theta - i\sin\theta}$

MUST BE FIRST EXPRESSED TO $\boldsymbol{\cos}(-\boldsymbol{\theta}) + \boldsymbol{i\sin}(-\boldsymbol{\theta})$

$(\cos\theta - i\sin\theta) = [\cos(-\theta) + i\sin(-\theta)]$

EXPONENTIAL FORM

If

$z_1 = \boldsymbol{r_1}e^{\boldsymbol{\theta_1 i}}$ and $z_2 = \boldsymbol{r_2}e^{\boldsymbol{\theta_2 i}}$

NOW

Multiplying z_1 and z_2

$$\Longrightarrow z_1 z_2 = \boldsymbol{r_1 r_2}e^{(\boldsymbol{\theta_1} + \boldsymbol{\theta_2})\boldsymbol{i}}$$

multiply the modulus and add the arguments

Dividing z_1 and z_2

$$\Longrightarrow \frac{z_1}{z_2} = \frac{\boldsymbol{r_1}}{\boldsymbol{r_2}}e^{(\boldsymbol{\theta_1} - \boldsymbol{\theta_2})\boldsymbol{i}}$$

divide the modulus and substrate the arguments

Example 1

Given that

$$z_1 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right) \quad and \quad z_2 = \sqrt{2}\left(\cos\left(\frac{12}{5}\pi\right) - i\sin\left(\frac{12}{5}\pi\right)\right)$$

Find $z_1 z_2$ in the form $x + yi$

Solution

$$z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right) \times \sqrt{2}\left(\cos\left(\frac{12}{5}\pi\right) - i\sin\left(\frac{12}{5}\pi\right)\right)$$

Remember $(\cos\theta - i\sin\theta) = [\cos(-\theta) + i\sin(-\theta)]$

$$\Longrightarrow z_1z_2 = 2\sqrt{2}\left(cos\left(\frac{\pi}{15}\right) + isin\left(\frac{\pi}{15}\right)\right) \times \sqrt{2}\left(cos\left(-\frac{12}{5}\pi\right) + isin\left(-\frac{12}{5}\pi\right)\right)$$

$$\Longrightarrow z_1z_2 = 2\sqrt{2}(\sqrt{2})\left[cos\left(\frac{\pi}{15} + \left(-\frac{12}{5}\pi\right)\right) + isin\left(\frac{\pi}{15} + \left(-\frac{12}{5}\pi\right)\right)\right]$$

multiply the modulus — add the arguments

$$= 4\left[cos\left(\frac{\pi}{15} - \frac{12}{5}\pi\right) + isin\left(\frac{\pi}{15} - \frac{12}{5}\pi\right)\right]$$

$$= 4\left[cos\left(\frac{\pi}{15} - \frac{12}{5}\pi\right) + isin\left(\frac{\pi}{15} - \frac{12}{5}\pi\right)\right]$$

$$= 4\left[cos\left(-\frac{7}{3}\pi\right) + isin\left(-\frac{7}{3}\pi\right)\right]$$

$$= 4\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$= 2 - i\sqrt{3}$$

DEMOIVRE'S THEOREM

$If\ z = \boldsymbol{r}(cos\theta + isin\theta)$

Then

$$z^{\boldsymbol{n}} = \boldsymbol{r}^{\boldsymbol{n}}(cos\theta + isin\theta)^{\boldsymbol{n}}$$
$$= \boldsymbol{r}^{\boldsymbol{n}}(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta)$$

where $\boldsymbol{n}$ is an interger or a fraction i.e (positive or negative interger , positive or negative fraction)

Note: ***Demoivre's theorem only applies for complex number in the form*** $(\boldsymbol{cos\theta + isin\theta})$ ***so any complex number which is not in this form must be changed to that form***

i) complex in the form $\boldsymbol{x + yi}$ ***must be changed to*** $\boldsymbol{r(cos\theta + isin\theta)}$

OTHER HELPFUL CHANGING

$\boldsymbol{ii)\ (cos\theta - isin\theta) = [cos(-\theta) + isin(-\theta)]}$

$\boldsymbol{iii)\ sin\theta + icos\theta = (i)(cos\theta - isin\theta) = (i)(cos(-\theta) + isin(-\theta))}$

$\boldsymbol{iv)\ sin\theta - icos\theta = (-i)(cos\theta + isin\theta)}$

Also

$If\ z = \boldsymbol{r}e^{\theta i}$

Then $z^{\boldsymbol{n}} = \left(\boldsymbol{r}e^{\theta i}\right)^{\boldsymbol{n}}$

$= \boldsymbol{r}^{\boldsymbol{n}}e^{\boldsymbol{n}\theta i}$

where $\boldsymbol{n}$ is an interger or a fractions i.e (positive or negative interger, positive or negative fraction)

WE CAN PROVE BY INDUCTION THE DEMOIVRE'S THEOREM CHECK CHAKS PROOF BY INDUCTION

<u>*Example 1*</u>

Express the following in the form $x + yi$

$$\left(-2\sqrt{3} - 2i\right)^5$$

<u>Solution</u>

$Let\ z = -2\sqrt{3} - 2i$

$|z| = \sqrt{\left(-2\sqrt{3}\right)^2 + (-2)^2} \quad = 4$

Complex z is in the third quadrant so arg z (do a sketch of complex z)

$\theta = \alpha - \pi$

$\arg z = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) - \pi$

$= -\frac{5\pi}{6}$

Now $z = 4\left[\cos(5)\left(\frac{-5\pi}{6}\right) + isin(5)\left(\frac{-5\pi}{6}\right)\right]$

$$z^5 = \left(-2\sqrt{3} - 2i\right)^5 = \left[4\left(\cos\left(\frac{-5\pi}{6}\right) + isin\left(\frac{-5\pi}{6}\right)\right)\right]^5$$

apply $\boldsymbol{r}^{\boldsymbol{n}}(cos\theta + isin\theta)^{\boldsymbol{n}} = \boldsymbol{r}^{\boldsymbol{n}}(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta)$

$$= 4^5\left[\cos(5)\left(\frac{-5\pi}{6}\right) + i\sin(5)\left(\frac{-5\pi}{6}\right)\right]$$

$$= 4^5\left[\cos\left(\frac{-25\pi}{6}\right) + i\sin\left(\frac{-25\pi}{6}\right)\right]$$

$$= 512\sqrt{3} - 512i$$

$$= 512(\sqrt{3} - i)$$

Example 2

Simplify

$$\frac{\left(cos\frac{7\pi}{13} + isin\frac{7\pi}{13}\right)^4}{\left(cos\frac{4\pi}{13} - sin\frac{4\pi}{13}\right)^6}$$

Solution

$$\frac{\left(cos\frac{7\pi}{13} + isin\frac{7\pi}{13}\right)^4}{\left(cos\frac{4\pi}{13} - sin\frac{4\pi}{13}\right)^6}$$

change the denominator $(\boldsymbol{cos\theta - isin\theta}) = [\boldsymbol{cos(-\theta) + isin(-\theta)}]$

$$= \frac{\left(cos\frac{7\pi}{13} + isin\frac{7\pi}{13}\right)^4}{\left(cos\left(-\frac{4\pi}{13}\right) + sin\left(-\frac{4\pi}{13}\right)\right)^6}$$

apply $\boldsymbol{r^n}(cos\theta + isin\theta)^{\boldsymbol{n}} = \boldsymbol{r^n}(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta)$

$$= \frac{\left(\cos(4)\left(\frac{7\pi}{13}\right) + isin(4)\left(\frac{7\pi}{13}\right)\right)}{\left(\cos(6)\left(-\frac{4\pi}{13}\right) + isin(6)\left(-\frac{4\pi}{13}\right)\right)}$$

$$= \frac{\left(cos\frac{28\pi}{13} + isin\frac{28\pi}{13}\right)}{\left(\cos\left(-\frac{24}{13}\pi\right) + isin\left(-\frac{24\pi}{13}\right)\right)}$$

$$= \cos\left(\frac{28\pi}{13} - \left(-\frac{24}{13}\pi\right)\right) + isin\left(\frac{28\pi}{13} - \left(-\frac{24}{13}\pi\right)\right)$$

$$= \cos\left(\frac{52\pi}{13}\right) + isin\left(\frac{52\pi}{13}\right)$$

$= 1$

Example 3

Simplify

$$\frac{(1+i)^4}{(2-2i)^3}$$ giving answer in the form $a + bi$

Solution

Let $u = 1 + i$

$z = 2 - 2i$

$|u| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$arg u = \frac{\pi}{4}$

$arg z = -\frac{\pi}{4}$

Now

$u = \sqrt{2}\left[cos\frac{\pi}{4} + isin\frac{\pi}{4}\right]$

$z = 2\sqrt{2}\left[cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\right]$

$$\Rightarrow \frac{(1+i)^4}{(2-2i)^3} = \frac{\left[\sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)\right]^4}{\left[2\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\right)\right]^5}$$

apply $\boldsymbol{r^n}(cos\theta + isin\theta)^{\boldsymbol{n}} = \boldsymbol{r^n}(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta)$

$$= \frac{\left(\sqrt{2}\right)^4\left(cos4\left(\frac{\pi}{4}\right) + isin4\left(\frac{\pi}{4}\right)\right)}{\left(2\sqrt{2}\right)^3\left(cos3\left(-\frac{\pi}{4}\right) + isin3\left(-\frac{\pi}{4}\right)\right)}$$

$$= \frac{4(cos\pi + isin\pi)}{16\sqrt{2}\left(cos\left(-\frac{3\pi}{4}\right) + isin\left(-\frac{3\pi}{4}\right)\right)}$$

Divide the modulus

subsract the arguments

$$= \frac{4}{16\sqrt{2}}\left[cos\left(\pi - \left(-\frac{3\pi}{4}\right)\right) + isin\left(\pi - \left(-\frac{3\pi}{4}\right)\right)\right]$$

$$= \frac{1}{4\sqrt{2}}\left[cos\left(\frac{7\pi}{4}\right) + isin\left(\frac{7\pi}{4}\right)\right]$$

$$= \frac{1}{8} - \frac{1}{8}i$$

NOTE: THE ABOVE EXAMPLE CAN BE WORKED AS FOLLOWS

$$\frac{(1+i)^4}{(2-2i)^3} = \frac{(1+i)^4}{2^3(1-i)^3}$$

By observing the denominator $\Rightarrow -i(1+i) = 1 - i$

$$\Rightarrow \left(-i(1+i)\right)^3 = (1-i)^3$$

$$\Rightarrow (-i)^3(1+i)^3 = (1-i)^3$$

$$\frac{(1+i)^4}{2^3(1-i)^3} = \frac{(1+i)^4}{2^3(-i)^3(1+i)^3}$$

$$= \frac{(1+i)^4}{8i(1+i)^3}$$

$$= \frac{1+i}{8i} = \frac{(1+i)(-8i)}{8i(-8i)}$$

$$= \frac{8-8i}{64}$$

$$= \frac{1}{8} - \frac{1}{8}i$$

Example 4

Find the real part of

$$\frac{1}{\left(sin\frac{\pi}{3} - icos\frac{\pi}{3}\right)^5}$$

Solution

$$\frac{1}{\left(sin\frac{\pi}{3}-icos\frac{\pi}{3}\right)^5}=\frac{1}{(-i)^5(cos\frac{\pi}{3}+isin\frac{\pi}{3})^5}$$

remember $sin\theta - icos\theta = (-i)(cos\theta + isin\theta)$

$$=\frac{\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{-5}}{-i}$$

apply $\boldsymbol{r^n}(cos\theta + isin\theta)^{\boldsymbol{n}} = \boldsymbol{r^n}(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta)$ *on the numerator*

$$=\frac{(cos\frac{-5\pi}{3}+isin\frac{-5\pi}{3})}{-i}$$

$$=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\times\frac{1}{-i}$$

$$=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\times\frac{1i}{-i(i)}$$

$$=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

∴ The real part is $-\frac{\sqrt{3}}{2}$

FOLLOW UP EXERCISE

1)$Simplify$

a) $(cos3\theta - isin3\theta)^5$

$b)\left(-2\sqrt{3} - 2\sqrt{3}i\right)^6$

c) $i)\dfrac{(2-2i)^7}{(1+i)^5}$ $\quad ii)\dfrac{(2+3i)^7}{(3-2i)^6}$ $\quad iii)\dfrac{(2+5i)^6}{(5-2i)^5}$

2) Find the imaginary part for $\left(-3 + i3\sqrt{3}\right)^5$

3) Simplify

$i)\dfrac{(cos2\theta + isin2\theta)^3}{(cos3\theta + isin3\theta)^4}$ $\quad ii)\dfrac{(cos3\theta - isin3\theta)^5}{(cos2\theta + isin2\theta)^2}$ $\quad iii)\dfrac{100e^{-\frac{1}{4}\pi}}{25e^{-\frac{3}{4}\pi}}$

4) Simplify

$i)\dfrac{\left(cos\frac{1}{4}\pi + isin\frac{1}{4}\pi\right)^5}{\left(cos\frac{1}{3}\pi + isin\frac{1}{3}\pi\right)^2}$ $\quad ii)\dfrac{\left(cos\frac{9}{4}\pi - isin\frac{9}{4}\pi\right)^5}{\left(cos\frac{7}{6}\pi + isin\frac{7}{6}\pi\right)^2}$

5)$if\ z = 3\sqrt{2}e^{-\frac{3}{4}\pi i}$. Find z^5 in the form $x + yi$

APPLICATION OF DEMOIVRE'S THEOREM IN PROVING TRIGONOMETRIC IDENTITIES

Expressing trigonometric identities of $Cosn\theta$ and $Sinn\theta$ in powers of $Cos\theta$ and $Sin\theta$

We know that by Demovre's theorem

$$(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta) = (cos\theta + isin\theta)^{\boldsymbol{n}}$$

*We can expand the **RHS** then*

~equate the real parts of the expansion to $cos\boldsymbol{n}\theta$

~equate the imaginary parts of the expansion to $sin\boldsymbol{n}\theta$

You can expand using pascals or binomial expansion

Example 1

Show that $Sin3\theta = 3sin\theta - 4sin^3\theta$

Solution

Using $(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta) = (cos\theta + isin\theta)^{\boldsymbol{n}}$

$\Longrightarrow (cos\mathbf{3}\theta + isin\mathbf{3}\theta) = (cos\theta + isin\theta)^{\mathbf{3}}$

RHS

$(cos\theta + isin\theta)^{\mathbf{3}}$ let $cos\theta = c$ and $sin\theta = s$

$\Longrightarrow (c + is)^{\mathbf{3}}$

$= c^3 + 3c^2(is)^1 + 3c^1(is)^2 + (is)^3$ *by pascals*

$= c^3 + i3c^2s - 3cs^2 - is^3$

NOW

$(cos\mathbf{3}\theta + isin\mathbf{3}\theta = c^3 + i3c^2s - 3cs^2 - is^3)$

Comparing the imaginary parts

$\Longrightarrow sin\mathbf{3}\theta = 3c^2s - s^3$

But $cos\theta = c$ and $sin\theta = s$

$\Longrightarrow sin\mathbf{3}\theta = 3cos^2\theta sin\theta - sin^3\theta$

Appying $cos^2\theta + sin^2\theta = 1$

$\Longrightarrow sin\mathbf{3}\theta = 3(1 - sin^2\theta)\theta sin\theta - sin^3\theta$

$sin\mathbf{3}\theta = 3sin\theta - 3sin^3\theta - sin^3\theta$

$Sin3\theta = 3sin\theta - 4sin^3\theta$ shown

Example 2

Express

$\frac{Sin4\theta}{sin\theta}$

in powers of $cos\theta$

Solution

Using $(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta) = (cos\theta + isin\theta)^{\boldsymbol{n}}$

Considering the numerator

$\Longrightarrow (cos\mathbf{4}\theta + isin\mathbf{4}\theta) = (cos\theta + isin\theta)^{\mathbf{4}}$

RHS

$(cos\theta + isin\theta)^{\mathbf{4}}$ let $cos\theta = c$ and $sin\theta = s$

$\Longrightarrow (c + is)^{\mathbf{4}}$

$= c^4 + 4c^3(is)^1 + 6c^2(is)^2 + 4c^1(is)^3 + (is)^4$ *by Pascals triangle*

$= c^4 + i4c^3s - 6c^2s^2 - i4cs^3 + s^4$

Comparing the imaginary parts

$\Longrightarrow sin\mathbf{4}\theta = 4c^3s - 4cs^3$ but $cos\theta = c$ and $sin\theta = s$

$sin\mathbf{4}\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

NOW

$$\frac{Sin4\theta}{sin\theta} = \frac{4cos^3\theta sin\theta - 4cos\theta sin^3\theta}{sin\theta}$$

$$\frac{Sin4\theta}{sin\theta} = 4cos^2\theta - 4cos\theta sin^2\theta$$

Applying $cos^2\theta + sin^2\theta = 1$

$$\frac{Sin4\theta}{sin\theta} = 4cos^2\theta - 4cos\theta\left(1 - cos^2\theta\right)$$

$$\frac{Sin4\theta}{sin\theta} = 4cos^2\theta - 4cos\theta\left(1 - cos^2\theta\right)$$

$= 4cos^2\theta - 4cos\theta + 4cos^3\theta$

$= 4cos^3\theta + 4cos^2\theta - 4cos\theta$ as required

Example 3

Show that $\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$

Solution

Using $(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$

$\Rightarrow (\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$

LHS

$\Rightarrow (\cos\theta + i\sin\theta)^5$ let $\cos\theta = c$ and $\sin\theta = s$

$\Rightarrow (c + is)^5$

$= c^5 + 5(c)^4(is)^1 + 10(c)^3(is)^2 + 10(c)^2(is)^3 + 5(c)^1(is)^4 + (is)^5$ by Pascals triangles

$= c^5 + i5c^4s - 10c^3s^2 - i10c^2s^3 + 5cs^4 + is^5$

Comparing real parts

$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$

but $\cos\theta = c$ and $\sin\theta = s$

$\Rightarrow \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Using $\cos^2 x + \sin^2 x = 1$

$= \cos^5\theta - 10\cos^3(1 - \cos^2) + 5\cos\theta(1 - \cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta)$

$= (16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)$

$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$ *shown*

SOME USEFUL TIPS

*All proving of **Cosnθ and Simθ to powers of Cosθ and Sinθ** by Demoivre's identities are derived from*

$$(cos\boldsymbol{n}\theta + isin\boldsymbol{n}\theta) = (cos\theta + isin\theta)^{\boldsymbol{n}}$$

then

$$tan\boldsymbol{n}\theta = \frac{sin\boldsymbol{n}\theta}{cos\boldsymbol{n}\theta}$$ *divide the IMAGINARY part over REAL part of the expansion*

$$cot\boldsymbol{n}\theta = \frac{cos\boldsymbol{n}\theta}{sin\boldsymbol{n}\theta}$$ *divide the REAL part over IMAGINARY part of the expansion*

Expressing trigonometrics of $Cos^n\theta$ and $Sin^n\theta$ in multiples angles of Cosθ and Sinθ

If $z = cos\boldsymbol{\theta} + isin\boldsymbol{\theta}$

Then

$$\frac{1}{z} = z^{-1} = (cos\boldsymbol{\theta} + isin\boldsymbol{\theta})^{-1}$$

applying DeMoivre's theorem

$$\frac{1}{z} = (cos(-\boldsymbol{\theta}) + isin(-\boldsymbol{\theta}))$$

$$= cos\boldsymbol{\theta} - isin\boldsymbol{\theta}$$

Now

$$z + \frac{1}{z} = cos\boldsymbol{\theta} + isin\boldsymbol{\theta} + cos\boldsymbol{\theta} - isin\boldsymbol{\theta}$$

$$= 2cos\boldsymbol{\theta}$$

Also

$$z - \frac{1}{z} = cos\boldsymbol{\theta} + isin\boldsymbol{\theta} - (cos\boldsymbol{\theta} - isin\boldsymbol{\theta})$$
$$= i2sin\boldsymbol{\theta}$$

IMPORTANT RESULTS ESTABLISHED ABOVE

$z + \frac{1}{z} = 2cos\boldsymbol{\theta}$ $\qquad$ $z - \frac{1}{z} = i2sin\boldsymbol{\theta}$

ALSO

If $z = cos\boldsymbol{\theta} + isin\boldsymbol{\theta}$

Then

$z^n = cos\boldsymbol{n\theta} + isin\boldsymbol{n\theta}$

$\Rightarrow \frac{1}{z^n} = z^{-1} = (cos\boldsymbol{n\theta} + isin\boldsymbol{n\theta})^{-1}$

applying DeMoivre's theorem

$\frac{1}{z^n} = (cos\boldsymbol{n}(-\boldsymbol{\theta}) + isin\boldsymbol{n}(-\boldsymbol{\theta})) = cos\boldsymbol{n\theta} - isin\boldsymbol{n\theta}$

Now

$z^{\boldsymbol{n}} + \frac{1}{z^{\boldsymbol{n}}} = cos\boldsymbol{n\theta} + isin\boldsymbol{n\theta} + cos\boldsymbol{n\theta} - isin\boldsymbol{n\theta}$

$= \boldsymbol{2}cos\boldsymbol{n\theta}$

Also

$z^{\boldsymbol{n}} - \frac{1}{z^{\boldsymbol{n}}} = cos\boldsymbol{n\theta} + \boldsymbol{i}sin\boldsymbol{n\theta} - (cos\boldsymbol{n\theta} - isin\boldsymbol{n\theta})$

$= \boldsymbol{i2}sin\boldsymbol{n\theta}$

IMPORTANT RESULTS ESTABLISHED

$z^{\boldsymbol{n}} + \frac{1}{z^{\boldsymbol{n}}} = 2cos\boldsymbol{n\theta}$ $\qquad$ $z^{\boldsymbol{n}} - \frac{1}{z^{\boldsymbol{n}}} = i2sin\boldsymbol{n\theta}$

NOW WE CAN APPLY AS FOLLOWS WHERENEVER WE WANT TO PROVE POWERS OF COSINE AND SINE TO MULTIPLE ANGLES OF COSINE AND SINE

$$\left(z+\frac{1}{z}\right)^n = (2cos\boldsymbol{\theta})^n \qquad \left(z-\frac{1}{z}\right)^n = (i2sin\boldsymbol{\theta})^n$$

CHECK EXAMPLES BELOW

Example 1

Express $sin^4\theta$ in multiples angles of $cos\theta$,

Solution

$$Using\ \left(z-\frac{1}{z}\right)^n = (i2sin\boldsymbol{\theta})^n$$

$$\Longrightarrow \left(z-\frac{1}{z}\right)^4 = (i2sin\theta)^4$$

$$\Longrightarrow \left(z-\frac{1}{z}\right)^4 = (16sin^4\theta)$$

Expanding LHS by Pascals

$$\Longrightarrow z^4 + 4(z)^3\left(-\frac{1}{z}\right)^1 + 6(z)^2\left(-\frac{1}{z}\right)^2 + 4(z)^1\left(-\frac{1}{z}\right)^3 + \left(-\frac{1}{z}\right)^4 = 16sin^4\theta$$

$$\Longrightarrow z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} = 16sin^4\theta$$

$$\Longrightarrow \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6 = 16sin^4\theta$$

Remember

$$z^n + \frac{1}{z^n} = 2cos\boldsymbol{n\theta} \qquad z^n - \frac{1}{z^n} = i2sin\boldsymbol{n\theta}$$

Now

$$2cos4\theta - 4(2cos2\theta) + 6 = 16sin^4\theta$$

$$sin^4\theta = \frac{1}{8}cos4\theta - \frac{1}{2}cos2\theta + \frac{3}{8}$$

Example 2

Express $8cos^5\theta$ multiple angle of $cos\theta$

Solution

Using

$$\left(z + \frac{1}{z}\right)^n = (2cos\boldsymbol{\theta})^n$$

$$\Rightarrow \left(z + \frac{1}{z}\right)^5 = (2cos\boldsymbol{\theta})^5$$

$$\Rightarrow \left(z + \frac{1}{z}\right)^5 = (32cos^5\boldsymbol{\theta})$$

Expanding LHS by Pascals

$$\Rightarrow z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5 = (32cos^5\boldsymbol{\theta})$$

$$\Rightarrow z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^5}\right) = (32cos^5\boldsymbol{\theta})$$

$$\Rightarrow z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) + \left(\frac{1}{z^5}\right) = (32cos^5\boldsymbol{\theta})$$

$$\Rightarrow z^5 + \left(\frac{1}{z^5}\right) + 5z^3 + 5\left(\frac{1}{z^3}\right) + 10z + 10\left(\frac{1}{z}\right) = (32cos^5\boldsymbol{\theta})$$

$$\Rightarrow \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right) = (32cos^5\boldsymbol{\theta})$$

Remember

$$z^n + \frac{1}{z^{\boldsymbol{n}}} = 2cos\boldsymbol{n\theta} \qquad z^{\boldsymbol{n}} - \frac{1}{z^{\boldsymbol{n}}} = i2sin\boldsymbol{n\theta}$$

Now $2cos5\theta + 5(2cos3\theta) + 10(2cos\theta) = (32cos^5\theta)$ dividing by 4

$$\Longrightarrow \frac{1}{4}(2cos5\theta + 10cos3\theta + 20cos\theta) = 8cos^5\theta$$

$$\Longrightarrow 8cos^5\theta = \frac{1}{2}cos5\theta + \frac{5}{2}cos3\theta + 5cos\theta$$

WE CAN APPLY THIS TECHNIQUE TO FIND INTERGRAL VALUE OF $\boldsymbol{Cos^n\theta}$ **and** $\boldsymbol{Sin^n\theta}$

CHECK CHAKS SOLUTIONS QUESTION AND ANSWERS COMPLEX NUMBERS

FOLLOW UP EXERCISE

1) Show that $sin^3\theta = -\frac{1}{4}(sin3\theta - 3sin\theta)$

Hence the find exact value

$$\int_0^{\frac{\pi}{4}} 8sin^3\theta$$

2) Show that $sin5\theta - 5sin\theta = 16sin^5\theta - 20sin^3\theta$

Hence or otherwise find

$$\int_0^{\frac{\pi}{4}} 16sin^5\theta - 20sin^3\theta$$

3) $Show\ that\ cos7\theta = 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7cos\theta$

4) $Show\ that\ Sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

i) Hence find any expression of $\frac{Sin4\theta}{sin\theta}$ powers cosθ

ii) Solve the equation $4cos^3\theta sin\theta - 4cos\theta sin^3\theta - \sqrt{3} = 0$, for $0 < \theta < 2\pi$

5) Express $Cot5\theta$ in powers of $cot\theta$

6) Show that

$$tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$$

By using the above result find correct to 3 s.f, the 4 solutions of equation for $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

7) Show that $tan5\theta(1 - 10tan^2\theta + 5tan^4\theta) = 5tan\theta - 10tan^3\theta + tan^5\theta$

Hence write the expression of $tan5\theta\ when\ \theta\ is\ small\ in\ powers\ of\ \theta$

Use to your expression to evaluate $tan15°\ in\ terms\ of\ \pi$

8) Express $sin^5\theta$ multiples angles of $sine$

N^{th} ROOTS OF COMPLEX NUMBERS

$If\ \ z^n = a + bi$

$z^n = \boldsymbol{r}(cos\boldsymbol{\theta} + isin\boldsymbol{\theta})$

Since the argument of complex is not unique so we can use this fact

$\Rightarrow z^n = \boldsymbol{r}\big(cos(\boldsymbol{\theta} + 2\pi k) + isin(\boldsymbol{\theta} + 2\pi k)\big)$

$$\Rightarrow z_k = \boldsymbol{r}^{\frac{1}{n}}\left(cos\left(\frac{\boldsymbol{\theta} + 2\pi k}{\boldsymbol{n}}\right) + isin\left(\frac{\boldsymbol{\theta} + 2\pi k}{\boldsymbol{n}}\right)\right)$$

$For\ k = 0\,;1\,,2,3\,,\dots\dots n - 1$

ALSO

$If\ \ z^n = a + bi$

$z^n = \boldsymbol{r}e^{\boldsymbol{\theta} i}$

Since the argument of complex is not unique so we can use this fact

$\Rightarrow z^n = \boldsymbol{r}e^{(\boldsymbol{\theta}+2\pi k)i}$

$$\Rightarrow z_k = \boldsymbol{r}^{\frac{1}{n}}e^{\left(\frac{\boldsymbol{\theta}+2\pi k}{\boldsymbol{n}}\right)\boldsymbol{i}}$$

$For\ k = 0\,;1\,,2,3\,,\dots\dots n - 1$

Remember that a complex number can be change from polar form or exponential form to $x + yi$

So care need to be taken when attempting the question on form needed

The number of roots are determined by the degree power of $z^{\boldsymbol{n}}$ *, so we have* $\boldsymbol{n}$ *roots*

Some situation

We can use $k = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$ *in this case you check for* z^n *when you reach* $\boldsymbol{n}$ *roots you stop, follow this order*

$k = 0\,;\ 1; -1\,;\ 2\,; -2\,;\ 3\,;\ -3 \ldots\ldots$ *and so on until you reach* $\boldsymbol{n}$ *roots*

SUB TOPIC : N^{th}root of a unity

Solutions to the equation of form $z^n = 1$ are called n^{th} root of a unity

Important notes

~For a unity the modulus is 1

~The argument is zero

~The roots of a unity sum up to zero

THEN

$$\boldsymbol{z^n = 1}$$

$$\Longrightarrow z^n = \left(cos(2\pi k) + isin(2\pi k)\right) \quad check\ \theta = 0\ \ and\ modulus = 1$$

$$\Longrightarrow z_k = \left(cos\left(\frac{2\pi k}{\boldsymbol{n}}\right) + isin\left(\frac{2\pi k}{\boldsymbol{n}}\right)\right)$$

For $k = 0\,; 1\,, 2, 3\,, \ldots\ldots n - 1$

ALSO

$$\boldsymbol{z^n = 1}$$

$$\Longrightarrow z^n = e^{(2\pi k)i}$$

$$\Longrightarrow z_k = \boldsymbol{r^{\frac{1}{n}}}e^{\left(\frac{2\pi k}{\boldsymbol{n}}\right)\boldsymbol{i}}$$

$For\ k = 0\ ;1\ ,2,3\ ,\dots\dots n-1$

Remember that a complex number can be change from polar form or exponential form to $x + yi$
So care need to be taken when attempting the question on form needed (I REPEAT) take note

Example 1

Find the third root of a unity

Solution

$z^3 = 1$

$\arg z^3 = 0 \quad ; \quad |z^3| = 1$

$Using\ z^n = \big(cos(2\pi k) + isin(2\pi k)\big) \quad for\ n^{th}\ of\ a\ unity$

$z^3 = \cos(2\pi k) + isin(2\pi k)$

$$apply\ z_k = \left(cos\left(\frac{2\pi k}{\boldsymbol{n}}\right) + isin\left(\frac{2\pi k}{\boldsymbol{n}}\right)\right)$$

$$z_k = \cos\left(\frac{2\pi k}{3}\right) + isin\left(\frac{2\pi k}{3}\right)$$

$for\ k = 0,1,2$

$$z_o = \cos\left(\frac{2\pi(0)}{3}\right) + isin\left(\frac{2\pi(0)}{3}\right) = 1$$

$$z_1 = \cos\left(\frac{2\pi(1)}{3}\right) + isin\left(\frac{2\pi(1)}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

$$z_2 = \cos\left(\frac{2\pi(2)}{3}\right) + isin\left(\frac{2\pi(2)}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

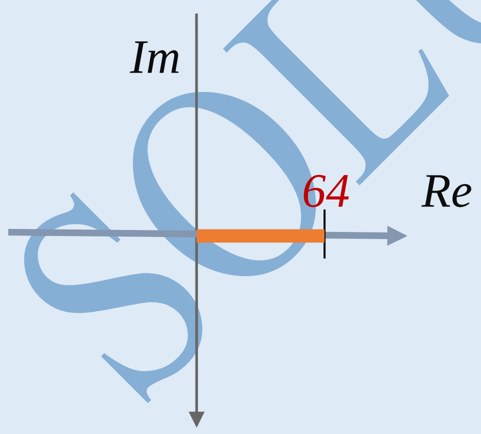
$\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) + 1 = 0$ sum of roots a unity

SOME EXAMPLE THAT CAN BE TREATED AS N^{th} $of\ a\ unity$

EXAMPLE 1

$Solve\ z^3 - 64 = 0$ and represent your solutions on an argand diargram

$z^3 = 64$



$\arg z^3 = 0$; $|z^3| = \sqrt{64^2} = 64$

$Using\ z^n = r\big(cos(2\pi k) + isin(2\pi k)\big)$

$z^3 = 64\cos(2\pi k) + isin(2\pi k)$

$$apply\ z_k = \boldsymbol{r}^{\frac{1}{n}}\left(cos\left(\frac{2\pi k}{\boldsymbol{n}}\right) + isin\left(\frac{2\pi k}{\boldsymbol{n}}\right)\right)$$

$$z_k = 64^{\frac{1}{3}}\left(\cos\left(\frac{2\pi k}{3}\right) + isin\left(\frac{2\pi k}{3}\right)\right)$$

$for\ k = 0,1,2$

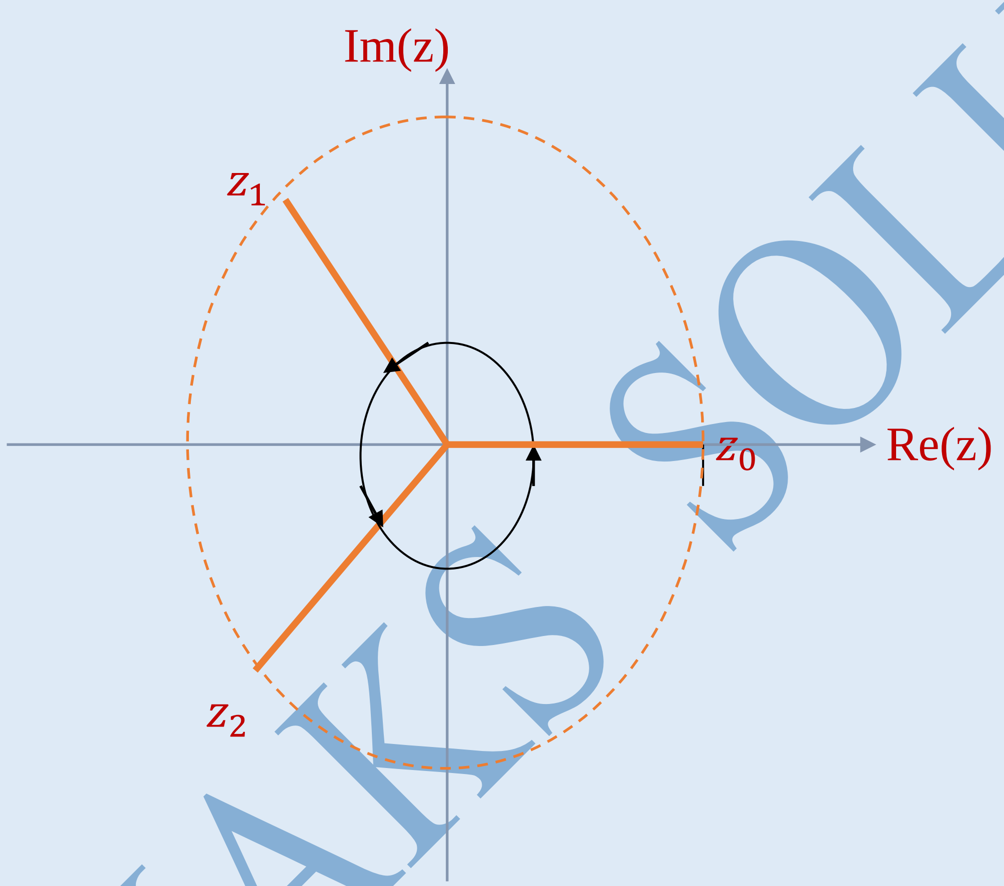
$$z_o = 64^{\frac{1}{3}}\left(\cos\left(\frac{2\pi(0)}{3}\right) + i\sin\left(\frac{2\pi(0)}{3}\right)\right) = 4$$

$$z_1 = 64^{\frac{1}{3}}\left(\cos\left(\frac{2\pi(1)}{3}\right) + i\sin\left(\frac{2\pi(1)}{3}\right)\right) = -2 + 2\sqrt{3}i$$

$$z_2 = 64^{\frac{1}{3}}\left(\cos\left(\frac{2\pi(2)}{3}\right) + i\sin\left(\frac{2\pi(2)}{3}\right)\right) = -2 - 2\sqrt{3}i$$

Representing the roots on the argand diagram

Note: the modulus of each root is 4



Label the angles between the complex roots as $\frac{2}{3}\pi$

SOLVING BINOMIAL COMPLEX NUMBERS EQUATIONS

WE HAVE DEDUCE THAT

$$If\ \ z^n = a + bi$$

$$z^n = \boldsymbol{r}(cos\boldsymbol{\theta} + isin\boldsymbol{\theta})$$

$$\Longrightarrow z^n = \boldsymbol{r}\big(cos(\boldsymbol{\theta} + 2\pi k) + isin(\boldsymbol{\theta} + 2\pi k)\big)$$

$$\Longrightarrow z_k = \boldsymbol{r}^{\frac{1}{n}}\left(cos\left(\frac{\boldsymbol{\theta} + 2\pi k}{\boldsymbol{n}}\right) + isin\left(\frac{\boldsymbol{\theta} + 2\pi k}{\boldsymbol{n}}\right)\right)$$

$For\ k = 0\,;1\,,2,3\,,\ldots\ldots n-1$

ALSO

$$\Longrightarrow z^n = \boldsymbol{r}e^{(\boldsymbol{\theta}+2\pi k)i}$$

$$\Longrightarrow z_k = \boldsymbol{r}^{\frac{1}{n}}e^{\left(\frac{\boldsymbol{\theta}+2\pi k}{\boldsymbol{n}}\right)\boldsymbol{i}}$$

$For\ k = 0\,;1\,,2,3\,,\ldots\ldots n-1$

Example 1

Zimsec November 2020 P2

i)Express $3 - 3\sqrt{3}i$ in exponential form , $\boldsymbol{r}e^{\theta i}$, where r is the modulus of the complex number and θ is the argument

ii)Hence or otherwise find all the roots of the equation $z^4 - 3 + 3\sqrt{3}i = 0$ in exponential form giving the answers correct to three significant figures

Solution

Let $u = 3 - 3\sqrt{3}i$

$$|u| = \sqrt{3^2 + \left(-3\sqrt{3}\right)^2} = 6$$

$$\arg u = -\tan^{-1}\left(\frac{3\sqrt{3}}{3}\right) = -\frac{\pi}{3}$$ *since u is the third quardrant so* $\theta = -\alpha$

NOW

$3 - 3\sqrt{3}i = \mathbf{6}e^{\left(-\frac{\pi}{3}\right)i}$

ii), $z^4 - 3 + 3\sqrt{3}i = 0$

$\Rightarrow z^4 = 3 - 3\sqrt{3}i$ from part i) $3 - 3\sqrt{3}i = \mathbf{6}e^{\left(-\frac{\pi}{3}\right)i}$

$\Rightarrow z^4 = \mathbf{6}e^{\left(-\frac{\pi}{3}\right)i}$

Since the argument of the complex is not unique

$\Rightarrow z^4 = \mathbf{6}e^{\left(-\frac{\pi}{3}+2\pi k\right)i}$

Apply $\Rightarrow z_k = \boldsymbol{r}^{\frac{1}{n}}e^{\left(\frac{\theta+2\pi k}{n}\right)i}$ *since the answers needed must be in exponential*

$\Rightarrow z_k = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{-\frac{\pi}{3}+2\pi k}{4}\right)i}$ *for* $k = 0, 1, 2, 3$

$$z_0 = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{-\frac{\pi}{3}+2\pi(0)}{4}\right)i} = \mathbf{6}^{\frac{1}{4}}e^{\left(-\frac{\pi}{12}\right)i} = 1.57e^{-0.262i}$$

$$z_1 = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{-\frac{\pi}{3}+2\pi(1)}{4}\right)i} = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{5\pi}{12}\right)i} = 1.57e^{1.31i}$$

$$z_2 = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{-\frac{\pi}{3}+2\pi(2)}{4}\right)i} = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{11\pi}{12}\right)i} = 1.57e^{2.88i}$$

$$z_3 = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{-\frac{\pi}{3}+2\pi(3)}{4}\right)i} = \mathbf{6}^{\frac{1}{4}}e^{\left(\frac{17\pi}{12}\right)i} = 1.57e^{4.45i}$$

Example 2

Solve the equation $z^3 - 32 - 32\sqrt{3}i = 0$

Giving your answers in the form $re^{i\theta}$, where $r > 0$. *and* θ *is the argument*

Solution

$z^3 - 32 - 32\sqrt{3}i = 0$

$\Longrightarrow z^3 = 32 + 32\sqrt{3}i$

$|z^3| = \sqrt{32^2 + (32\sqrt{3}\,)^2} = 64$

$argz^3 = \ tan^{-1}\left(\frac{32\sqrt{3}}{32}\right) = \frac{\pi}{3}$

$z^3 = 64e^{\left(\frac{\pi}{3}+2\pi k\right)i}$

$z_k = 64^{\frac{1}{3}}e^{\left(\frac{\frac{\pi}{3}+2\pi k}{3}\right)i} \ for\ k = 0,1,2$

$z_o = 4e^{\left(\frac{\pi}{9}\right)i}$

$z_1 = 4e^{i\left(\frac{7\pi}{9}\right)i}$

$z_2 = 4e^{i\left(\frac{13\pi}{9}\right)i}$

Example 2

Solve $z^4 + 2\sqrt{3}i = -2$ *leaving your answers in the form* $a + bi$

Solution

$z^4 + 2\sqrt{3}i = -2$

$\Longrightarrow z^4 = -2 - 2\sqrt{3}i$

$$|z^4| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4$$

$arg z^4 = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) - \pi$ *since z^4 is in the third quadrant*

$$= \frac{\pi}{3} - \pi$$

$$= -\frac{2}{3}\pi$$

$$\Rightarrow z^4 = 4\left(cos\left(-\frac{2}{3}\pi\right) + isin\left(-\frac{2}{3}\pi\right)\right)$$

Since the argument of the complex is not unique

$$\Rightarrow z^4 = \mathbf{4}\left(cos\left(-\frac{2}{3}\pi + 2\pi k\right) + isin\left(-\frac{2}{3}\pi + 2\pi k\right)\right)$$

$$\Rightarrow z_k = \mathbf{4}^{\frac{1}{4}}\left(cos\left(\frac{-\frac{2}{3}\pi + 2\pi k}{4}\right) + isin\left(\frac{-\frac{2}{3}\pi + 2\pi k}{4}\right)\right)$$

For $k = 0\,;1\,;\ 2;\ 3$

$$\Rightarrow z_0 = \sqrt{\mathbf{2}}\left(cos\left(\frac{-\frac{2}{3}\pi + 2\pi(0)}{4}\right) + isin\left(\frac{-\frac{2}{3}\pi + 2\pi(0)}{4}\right)\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

$$\Rightarrow z_1 = \sqrt{\mathbf{2}}\left(cos\left(\frac{-\frac{2}{3}\pi + 2\pi(1)}{4}\right) + isin\left(\frac{-\frac{2}{3}\pi + 2\pi(1)}{4}\right)\right) = \sqrt{\mathbf{2}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$$

$$\Rightarrow z_2 = \sqrt{\mathbf{2}}\left(cos\left(\frac{-\frac{2}{3}\pi + 2\pi(2)}{4}\right) + isin\left(\frac{-\frac{2}{3}\pi + 2\pi(1)}{4}\right)\right) = \sqrt{\mathbf{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

$$\Rightarrow z_3 = \sqrt{\mathbf{2}}\left(cos\left(\frac{-\frac{2}{3}\pi + 2\pi(3)}{4}\right) + isin\left(\frac{-\frac{2}{3}\pi + 2\pi(3)}{4}\right)\right) = \sqrt{\mathbf{2}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$$

Example 3

Solve $z^6 + z^3\sqrt{2} + 1 = 0$ *leaving your answer in the form* $cos\theta + isin\theta$

Solution

$z^6 + z^3\sqrt{2} + 1 = 0$

$\Rightarrow (z^3)^2 + z^3\sqrt{2} + 1 = 0$ this is a disguised equation

Let $u = z^3$

$\Rightarrow u^2 + u\sqrt{2} + 1 = 0$

$$u = \frac{-\sqrt{2} \pm \sqrt{\sqrt{2}^2 - 4(1)(1)}}{2(1)} \quad using\ the\ quadratic\ formula\ to\ find\ u$$

$$= \frac{-\sqrt{2} \pm \sqrt{2}i}{2}$$

$$\therefore \quad u = \frac{-\sqrt{2} + \sqrt{2}i}{2} \quad or \quad \frac{-\sqrt{2} - \sqrt{2}i}{2}$$

$$arguments \quad \theta_1 = \frac{3\pi}{4} \quad or \quad \theta_2 = -\frac{3\pi}{4} \quad and\ modulus = 1$$

but $u = z^3$

Remember argument of a complex is not unique

$$\Rightarrow z^3 = \cos\left(\frac{3\pi}{4} + 2\pi k\right) + isin\left(\frac{3\pi}{4} + 2\pi k\right) \quad or \quad \cos\left(\frac{-3\pi}{4} + 2\pi k\right) + isin\left(\frac{-3\pi}{4} + 2\pi k\right)$$

$$z_k = \cos\left(\frac{\frac{3\pi}{4} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4} + 2\pi k}{3}\right) \quad or \quad \cos\left(\frac{-\frac{3\pi}{4} + 2\pi k}{3}\right) + i\sin\left(\frac{-\frac{3\pi}{4} + 2\pi k}{3}\right)$$

$for\ k = 0,1,2$

$$z_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \quad or \quad \cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)$$

$$z_1 = \cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right) \quad or \quad \cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)$$

$$z_2 = \cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right) \quad or \quad \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)$$

FOLLOW UP EXERCISE

1) Solve leaving your answers in the form $a + bi$ where possible give exact value of a and b or giving your answers correct to three significant figures

i) $z^3 - 8i = 0$ ii) $z^5 = 1$ iii) $z^4 + 64 = 0$ iv) $z^7 - 8 - 8i = 0$

2) Solve the following giving your answer in the form $r(\cos\theta + i\sin\theta)$ where possible give exact value of θ and r or giving your answers correct to three significant figures

i) $z^7 - 2 + 2\sqrt{3}i = 0$ ii) $z^4 - \sqrt{13} + 4i = 0$ iii) $z^4 = 8 - 8\sqrt{3}i$ iv $z^{\frac{3}{4}} = \sqrt{6} + \sqrt{2}i$

3) Solve the following giving your answer in the form $re^{\theta i}$

i) $z^5 = 3 - 3\sqrt{3}i$ iii) $z^4 - 8 + 8\sqrt{3}i = 0$

4) *Express in polar form* $\sqrt{6} + \sqrt{2}i$, Hence find the fourth roots of $\sqrt{6} + \sqrt{2}i$ leaving your answers in the form $r(\cos\theta + i\sin\theta)$

5) Solve $z^4 + 2z^2 + 17 = 0$ giving your answers in the form $re^{\theta i}$

6) By means of substitution $w = z^4$, solve the equation $z^8 - z^4 - 6 = 0$, where z is a complex number .

SOLVING POLYNOMIALS WITH REAL COEFFICIENTS

Quadratic equations

~If α *and* β are complex roots of a quadratic equation they occur as conjugate pairs

~Given any complex roots of quadratic equation you can find the equation as follows

$$z^2 - (sum\ of\ roots)z + (products\ of\ roots) = 0$$

Polynomials of higher degree of powers

~Complex roots of a polynomial equation with real coefficients occur as conjugate pairs

~The number of roots for a polynomial are determined by the highest degree of power of the polynomial , if highest degree of power is an odd number there is at least one real root

SO for the polynomial P_n

$az^n + bz^{n-1} + cz^{n-2} + \cdots \ldots \ldots + k = 0$, the number of roots are $\boldsymbol{n}$ roots

Then

$$\sim Sum\ of\ roots = -\frac{b}{a} \quad or \quad coefficients\ of\ -\frac{z^{n-1}}{z^n}$$

$$\sim Sum\ of\ product\ of\ possible\ pairs\ of\ roots = \frac{c}{a} \quad or \quad coefficients\ of\ \frac{z^{n-2}}{z^n}$$

$$\sim Product\ of\ roots = \frac{(-1)^n.k}{a}$$

Several techniques can be applied when dealing with solving of polynomial with real coefficients

Some of techniques : Long division of polynonials , equal complex number techniques, polynomial identities technique and so on

Example 1

Find the quadratic equations given that one its roots is $1 - 2i$

Solution

$1 - 2i$ *so the conjugate is a root* $1 + 2i$

$applying \quad z^2 - (sum\ of\ roots)z + (products\ of\ roots) = 0$

$\Longrightarrow z^2 - (1 - 2i + 1 + 2i)z + (1 + 2i)(1 - 2i) = 0$

$\Longrightarrow z^2 - 2z + 5 = 0$

Example 2

Given that $1 + 3i$ is one the root of the equation $z^3 + 6z + 20 = 0$, find the other roots of the equation ,and represent the roots of the equation on the same argand diagram

Solution

$1 + 3i$ $is\ a\ root$, so the conjugate is also a root $1 - 3i$

$z^3 + 6z + 20 = 0$

We have three roots check z^3

Let the third root be α

$\sim Sum\ of\ roots = -\frac{b}{a}$

$(1 + 3i) + (1 - 3i) + \alpha = \frac{0}{1}$

$2 + \alpha = 0$

$\alpha = -2$

The roots are -2 ; $1+3i$ and $1-3i$

$Take\ note\ on\ the\ above\ example\ \ z^3+6z+20=O\ \ can\ written\ as\ \ z^3+0z^2+6z+20=0$

Most students make this error of wrong siting the values of **a; b; c….. k** so check your equation properly to determine the values of $a, b\ and\ c\ \ldots upto\ \ k$

So in this case , $a=1$; $b=0$; $c=6$; $k=20$

Method 2

$1+3i$ is a root so the conjugate is also a root $1-3i$

Finding the quadratic quadratic factor

$applying\quad z^2-(sum\ of\ roots)z+(products\ of\ roots)=0$

$\Longrightarrow z^2-[(1+3i)+(1-3i)]z+(1+3i)(1-3i)=0$

$z^2-2z+10=0$

$\Longrightarrow\ z^2-2z+10\ is\ a$ quadratic factor

Therefore, since our function is a cubic function so it has three solutions(roots)

let the factor be $(az+b)$

$(z^2-2z+10)(az+b)=z^3+6z+20$ OR you can apply long division

Comparing the coefficients of z^3 , $z\ \ and\ \ z^0(constant)$ $z^2-2z+10\,\overline{)\,z^3+6z+20}$

$a=1$ $\quad 10a-2b=6$ $\quad 10b=20$

$\Longrightarrow b=2$

$az+b\Longrightarrow z+2$

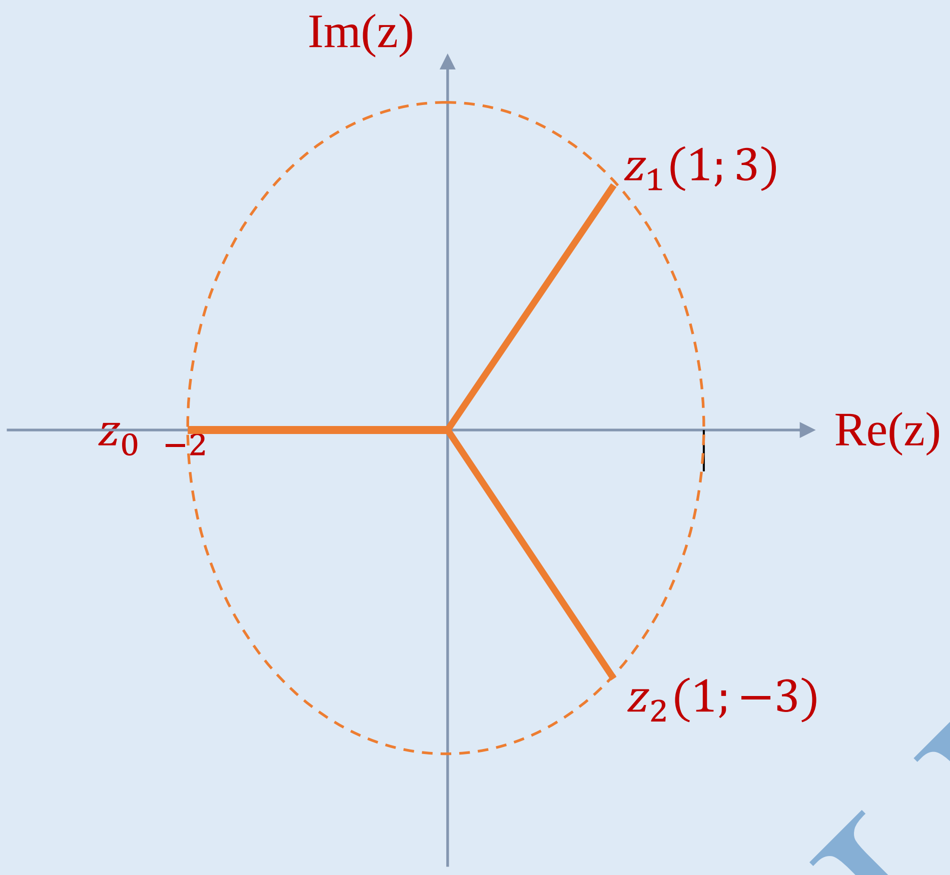
$by\ equating\ the\ factor\ to\ zero\ \ \ z+2=0$

$z=-2$

So the solution are $1+3i\ ;\ 1-3i\ ;\ and-2$

Representing solution on the argand diagram

Note: the modulus of each root is 2



Example 3

Given that $3-2i$ *is a solution of* $z^4-6z^3+19z^2-36z+78=0$

Find the other roots of the equation

Solution

Since the highest degree of power of the polynomial is four so we have four roots: check z^4

$3-2i$ *is root , conjugate is also a root* $3+2i$ and let the remaining two roots be x *and* y

$\sim Sum\ of\ roots\ -\frac{b}{a}$

$(3-2i)+(3+2i)+x+y=-\left(\frac{-6}{1}\right)$

$6+x+y=6$

$y=-x$

Also

$$\sim Products\ of\ roots\ = \frac{(-1)^n.k}{a}$$

$$(3-2i)(3+2i)(x)(y) = \left(\frac{(-1)^4(78)}{1}\right)$$

$$(3^2+2^2)xy = 78$$

$$but\ y = -x$$

$$13xy = 78$$

$$-13x^2 = 78$$

$$x^2 = -6$$

$$x = \pm\sqrt{6}i\ \rightarrow y = \sqrt{6}i\ \ , \qquad x = -\sqrt{6}i$$

So the roots are $-\sqrt{6}i$; $\sqrt{6}i$; $3-2i$ *and* $3+2i$

***NOTE:* WE CAN SOLVE THE ABOVE AS FOLLOWS**

~ FORMING A QUADRATIC FACTOR USING OUR CONJUGATE COMPLEX PAIR ROOTS

~DIVIDE QUADRATIC FACTOR INTO THE ORIGINAL POLYNOMIAL AND HAVE ANOTHER NEW FACTOR

~SOLVE THE NEW FACTOR BY EQUATING IT TO ZERO

Example 3

Given that $1-i$ is a root of $z^3+pz^2+qz+12=0$, find the real numbers $p\ and\ q$

Hence for this value of $p\ and\ q$, find the other roots

Solution

$If\ \ 1-i\ \ is\ a\ root$

$\Longrightarrow f(1-i)=0$

$(1-i)^3+p(1-i)^2+q(1-i)+12=0,$

$1^3+3(1)^2i+3(1)\,i^2+\ i^3+p(1-2i+i^2)+q-qi=0$

$1+3i-3-i+p-2pi-p+q-qi=0$

$-2+q+2i-2pi-qi=0$

$-2+q+(2-2p-q)i=0$

Equal complex real parts are equal and imaginary parts are equal

Now

$-2+q=0\ldots\ldots i$

$2-2p-q=0\ldots..ii$

$\Longrightarrow q=2$

$2-2p-2=0$

$\Longrightarrow p=0$

$z^3+pz^2+qz+12=0\quad when\ p=0\ and\ q=2$

$\Longrightarrow z^3+2z+12=0$

$1-i$ is a root ,so the conjugate is also a root $1+i$

$z^3+2z+12=0$

We have three roots check z^3

Let the third root be α

$$\sim Sum\ of\ roots = -\frac{b}{a}$$

$$(1+i)+(1-i)+\alpha = \frac{0}{1}$$

$$2+\alpha = 0$$

$$\alpha = -2$$

The roots are -2 ; $1+i$ and $1-i$

LOCI OF COMPLEX NUMBERS

A locus of points refers to a set of points subject to certain conditions

Types of loci

Loci involving modulus

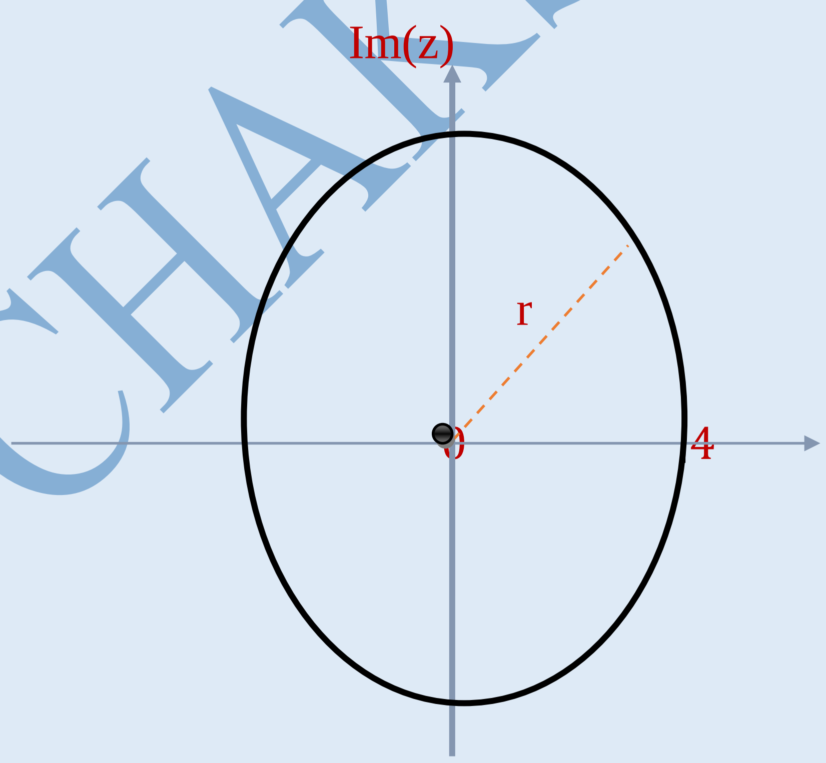
CASE 1

$|z| = k$ *represents a circle with centre at origin (0;0) and radius* k

$|z - z_1| = k$ *represents a circle with centre* z_1 *and radius* k

Examples

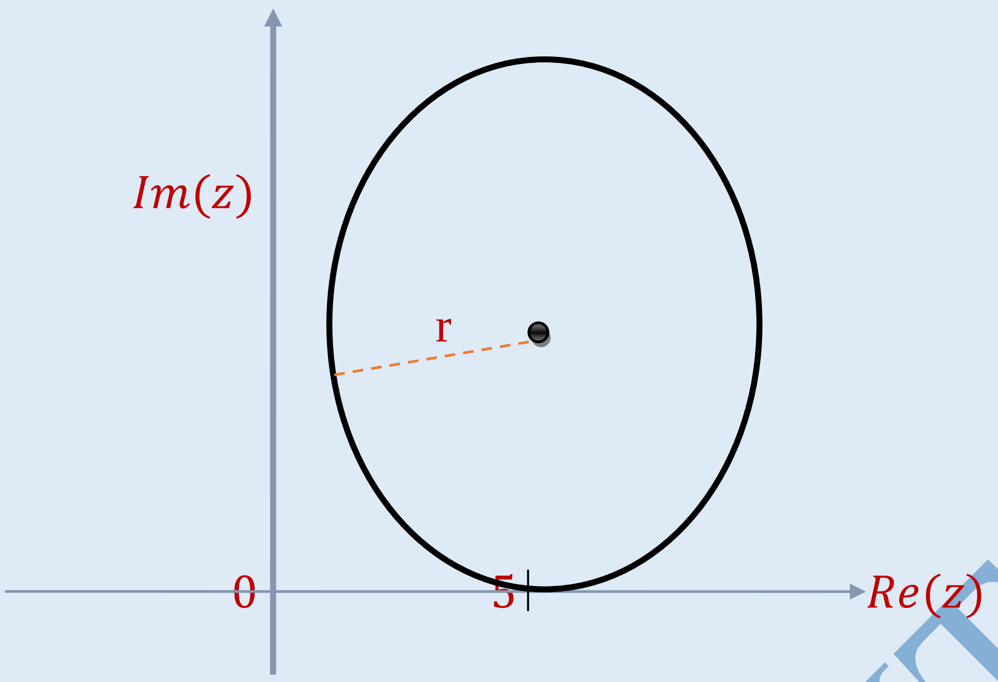
Sketch the locus of $|z| = 4$



$centre(0;0) \quad r = 4$

Sketch the locus of $|z-5-3i|=3$

$|z-(5+3i)|=3$ $\quad$ *centre* $(5;3)$, *radius* $r=3$



$|\boldsymbol{z}| \leq \boldsymbol{k}$ *represents part which lies* ***inside*** *and* **on** *the circle with centre at origin (0;0) and radius* ***k*** ***and for*** $|\boldsymbol{z}| \geq \boldsymbol{k}$ ***vice versa is also true***

$|\boldsymbol{z}-\boldsymbol{z_1}| \leq \boldsymbol{k}$ *represents part which lies* ***inside*** *and* **on** *the circle with centre at* $\boldsymbol{z_1}$ *and radius* ***k*** ***and for*** $|\boldsymbol{z}-\boldsymbol{z_1}| \geq \boldsymbol{k}$ ***vice versa is also true***

ALSO

$|\boldsymbol{z}| < \boldsymbol{k}$ *represents part which lies* ***inside*** *the circle with centre at origin (0;0) and radius* ***k***

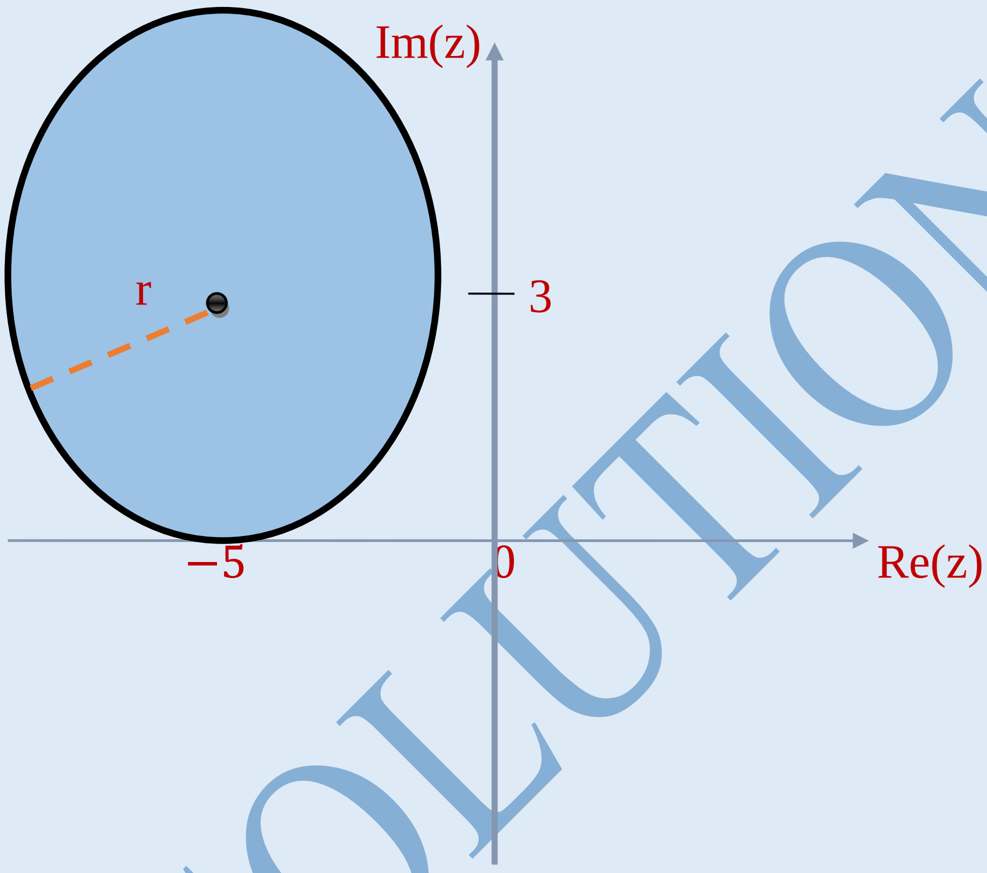
and for ***and for*** $|\boldsymbol{z}| > \boldsymbol{k}$ ***vice versa is also true***

$|\boldsymbol{z}-\boldsymbol{z_1}| < \boldsymbol{k}$ *represents part which lies* ***inside*** *the circle with centre at* $\boldsymbol{z_1}$ *and radius* ***k***

and for $|\boldsymbol{z}-\boldsymbol{z_1}| > \boldsymbol{k}$ ***vice versa is also true***

Example1

Sketch locus of $|z + 5 - 3i| \leq 3$



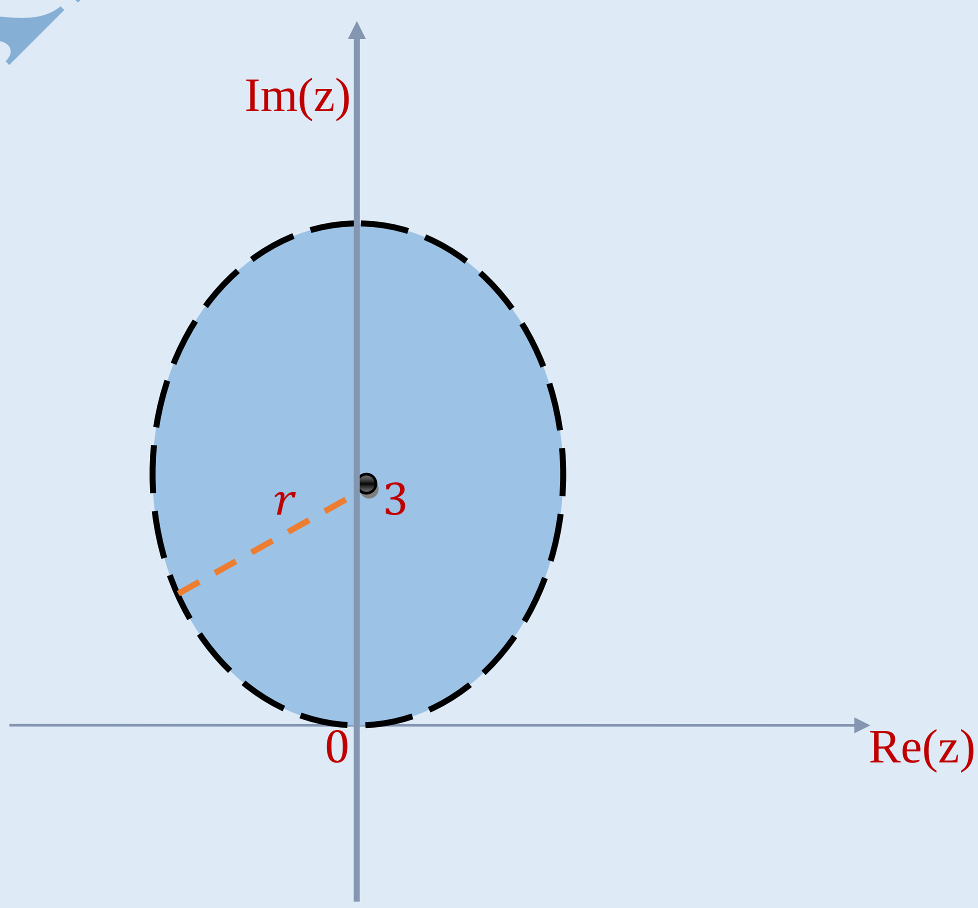
$|z - (-5 + 3i)| \leq 3$

$centre(-5; 3) \quad r = 3$

Shade inside the circle

Example 2

Sketch locus of $|z - 3i| < 3$



$|z - (3i)| < 3$

$centre(0; 3) \quad r = 3$

Shade inside the circle

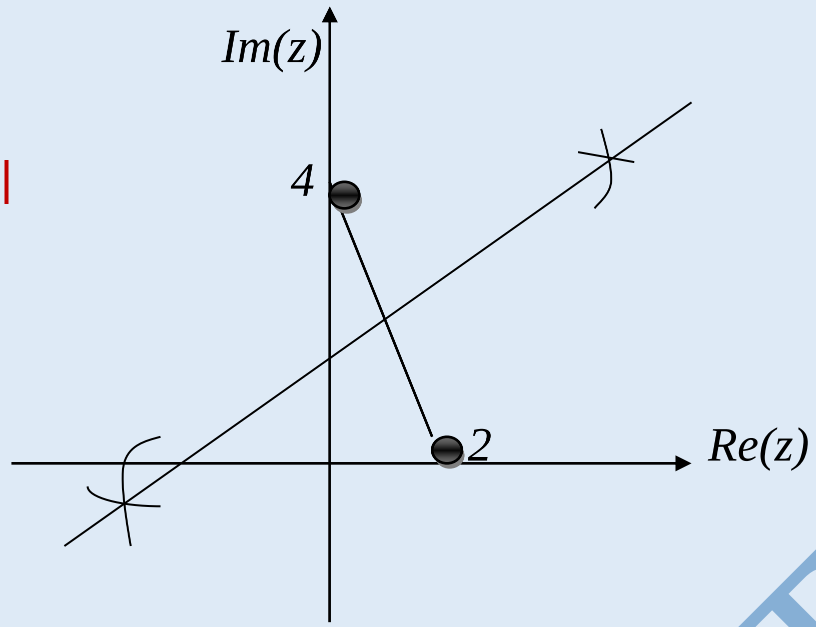
CASE 2

$|z - z_1| = |z - z_2|$ *represents a perperndicular bisector of line segment joining* z_1 *and* z_2

Example 1

$|z - 2| = |z - 4i|$

$|z - (2)| = |z - (4i)|$



Note:

We *can find the Cartesian equation of the locus above by letting* $z = x + yi$

As follows

$|z - 2| = |z - 4i|$

$|(x + yi) - 2| = |(x + yi) - 4i|$

$|(x - 2) + yi| = |x + (y - 4)i|$

$\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y-4)^2}$

$x^2 - 2x + 4 + y^2 = x^2 + y^2 - 8y + 16$

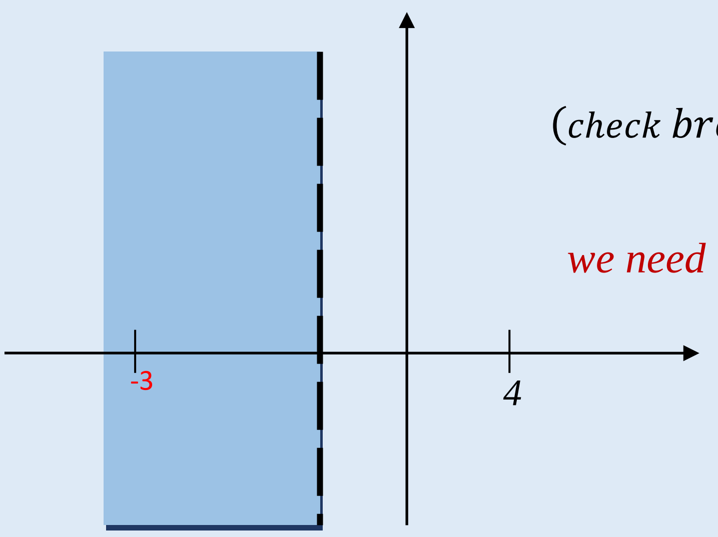
$8y - 2x = 16$

$4y - x = 8$

Example 2

Sketch the locus

$|z + 3| < |z - 4|$



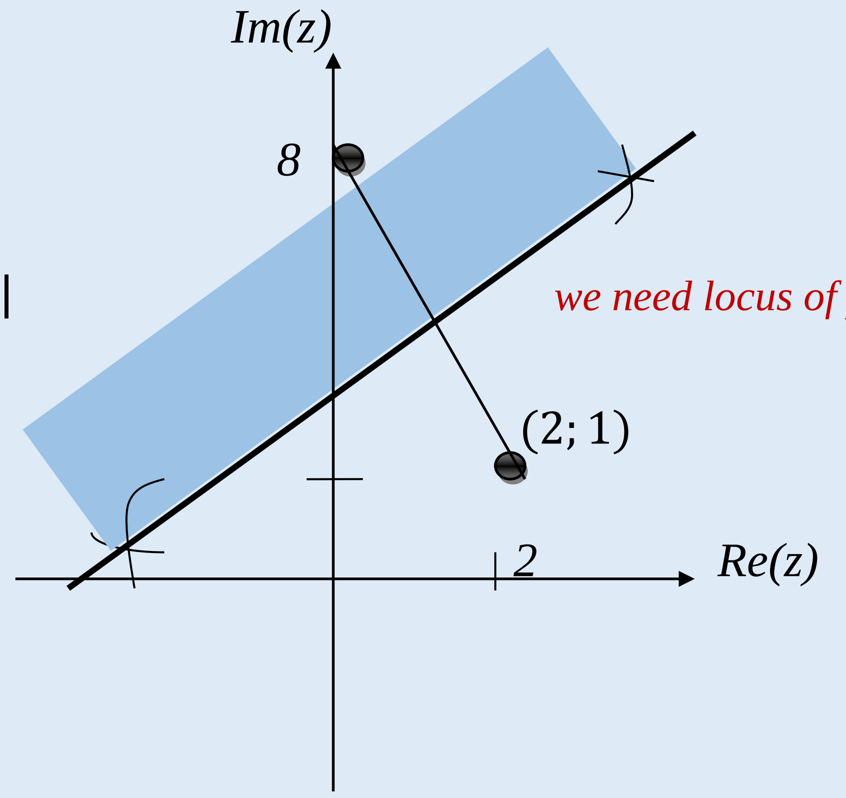
$($*check broken line for the sign*$)$,

we need region closer to $(0;3)$ *than* $(0;4)$

Example 3

$|z - 8i| \leq |z - 2 - i|$

$|z - 8i| \leq |z - 2 - i|$



we need locus of points closer to (0; 8) *than* (2,1)

CASE 3

$|z - z_1| = k|z - z_2|$ *where* $k > 0$, $k \neq 1$ ***so we need to apply the algebra method to find the equation of loci represented by letting*** $z = x + yi$

Example 1

Find the Cartesian equation of locus $|z + 4 - 2i| = 2|z - 2 - 5i|$

and describe it

$|(x + yi) + 4 - 2i| = 2|(x + yi) - 2 - 5i|$

$|(x + 4) + (y - 2)i| = 2|(x - 2)(y - 5)i|$

$\sqrt{(x + 4)^2 + (y - 2)^2} = 4\sqrt{(x - 2) + (y - 5)^2}$

$x^2 + 8x + 16 + y^2 - 4y + 4 = 4[x^2 - 4x + 4 + y^2 - 10y + 25]$

$x^2 + 8x + 16 + y^2 - 4y + 4 = 4x^2 - 16x + 16 + 4y^2 - 40y + 100$

$3x^2 + 3y^2 - 24x - 36y + 96 = 0$

$x^2 + y^2 - 8x - 12y + 32 = 0$

$x^2 - 8x + y^2 - 12y = -32$

$(x-4)^2 - (-4)^2 + (y-6)^2 - (-6)^2 = -32$ *by completing square*

$(x-4)^2 - 16 + (y-6)^2 - 36 = -32$

$(x-4)^2 + (y-6)^2 = -32 + 16 + 36$

$(x-4)^2 + (y-6)^2 = 20$

Remember equation of the circle of form $(x-a)^2 + (y-b)^2 = r^2$

The locus represented is a circle with centre $(4; 6)$ *, and* $radius = 2\sqrt{5}$

NB: YOU NEED TO ABLE TO DRAW ABOVE LOCUS

Loci involving arguments