# Vikash Polytechnic, Bargarh

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# Lecture Note on *Mathematics-II*

## Diploma 2nd Semester

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# Deteremninant

definition :- It is a scalare value (constant) of a square matrix of oreder 'n'. It is denoted by det(A) or |A| where. $A = [a_{ij}]$ of order n.

## Deteraminant of oredere 1

It has one row and one column..

Ex $|A| = |2|_{1\times1}$ then $|A| = 2$

## Ordere 2 determinant :

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \quad \left\{ \begin{array}{l} \text{from equations} \\ a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array} \right.$$

Ex $\begin{vmatrix} 9 & 1 \\ 4 & 2 \end{vmatrix} = 18 - 4 = 14$, $\begin{vmatrix} 7 & 0 \\ 20 & 2 \end{vmatrix} = 14 - 0 = 14$

sign of det of oredere 2 $= \begin{vmatrix} + & - \\ - & + \end{vmatrix}$

oredere 3 $= \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

## Oredere 3 det.

If we consider
$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$
then determinant of order 3 is given by

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A| = \begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\text{or} = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

<u>or</u>

$$|A| = \begin{array}{c|ccc|c} c & a & b & c & a \\ & d & e & f & \\ i & g & h & i & g \end{array} = (c \times d \times h + a \times e \times i + b \times f \times g) - (i \times d \times b + g \times e \times c + h \times f \times a)$$

<u>Ex</u>

$$\begin{array}{c|ccc|c} 0 & 0 & 1 & 0 & 0 \\ & -1 & 2 & 3 & \\ 1 & -2 & 4 & 1 & -2 \end{array} = (0 \times (-1) \times 4 + 0 \times 2 \times 1 + 1 \times 3 \times (-2)) - (1 \times (-1) \times 1 + (-2) \times 2 \times 0 + 3 \times 4 \times 0)$$

$$= (0 + 0 - 6) - (-1 + 0 + 0)$$

$$= -6 + 1 = -5$$

<u>Minors of Determinant :- ($M_{ij}$)</u>

The minor of a given element of determinant obtained by deleting the row and column in which the given element stands.

It is denoted by $M_{ij}$ for $a_{ij}$.

Ex Find -3, 5 and -1 in the determinant $\begin{vmatrix} 1 & -3 & 2 \\ 3 & 0 & 5 \\ -1 & 1 & 7 \end{vmatrix}$

Solⁿ Minor of (-3) = $M_{12} = \begin{vmatrix} 3 & 5 \\ -1 & 7 \end{vmatrix}$

$= 3\times7 - (-1)\times5 = 21-(-5) = 21+5 = 26$

Minor of 5 = $M_{23} = \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} = 1\times1 - (-1)\times(-3)$

$= 1-3 = -2$

Minor of (-1) = $M_{31} = \begin{vmatrix} -3 & 2 \\ 0 & 5 \end{vmatrix} = (-3)\times5 - 0\times2$

$= -15-0 = -15$

## Cofactor of Determinant : ($C_{ij}$)

If $M_{ij}$ is the minor of $a_{ij}$ then cofactor of $a_{ij}$ is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Ex In above ~~equation~~ problem

cofactor of (-3) = $C_{12} = (-1)^{1+2} M_{12} = -26$

Cofactor of 5 = $C_{23} = (-1)^{2+3} M_{23} = -(-2) = 2$

cofactor of (-1) = $C_{31} = (-1)^{3+1} M_{31} = +M_{31} = -15$

# Properties of Determinants

If row and columns are interchanged then the determinant will be same.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If any two rows (or columns) are inter-chancheged then the determinant change in sign.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$(R_1 \leftrightarrow R_2)$

If all the elements of a row (or column) are zero then the value of determinant is zero.

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If any row or column are multiplied by 'k' (same number) then the determinant is multiplied by 'k'.

If all elements of a row (or column) are same (identical) to any other row (or column) then the value of determinant is zero.

6) If ~~any~~ each element of any row (or column) ~~are~~ is expressed as sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

7) <u>Row-column Operation :-</u>

If we performe operation $R_i \rightarrow R_i + KR_j$ $(j \neq i)$ or $C_i \rightarrow C_i + KC_j$ then value of determinant unchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \xrightarrow{R_1 \rightarrow R_1 + KR_2} \begin{vmatrix} a_1+Ka_2 & b_1+Kb_2 & c_1+Kc_2 \\ a_2 & b_2 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

both determinants are qual. (~~the~~ value not change).

8) <u>Factore Theorem:-</u>

If each element in a matrix A is a polynomial of $x$ and $x=a$ vanishes (zero) determinant then $(x-a)$ is a factor of $|A|$.

## Multiplication of Two determinants :

i) It can be done by Row by column.

ex $A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ , $B = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}$

$$AB = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 \times l_1 + b_1 \times l_2 & a_1 \times m_1 + b_1 \times m_2 \\ a_2 \times l_1 + b_2 \times l_2 & a_2 \times m_1 + b_2 \times m_2 \end{vmatrix}$$

Also it can be done by row by row, column by row and column by column product.

## System of Linear Equation's Solution

Let $a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$ given

Nature : (1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (unique solution)

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Infinite Solution)

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (No Solution)

## Creamer's Rule :-

Let us consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$ for $\Delta \neq 0$ is the solution of the given equation.

### Nature of the solution :-

1) $\Delta \neq 0$ & at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$. (Unique soln)

2) $\Delta \neq 0$ & $\Delta_1 = \Delta_2 = \Delta_3 = 0$ (Trivial solution) ($x = y = z = 0$)

3) $\Delta = 0$ & $\Delta_1 = \Delta_2 = \Delta_3 = 0$ (Infinite solutions)

4) $\Delta = 0$ & at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$ (No solution)

Ex Solve the system of equations by Cramer's rule. Write Consistency:

$$x+y=5$$
$$3x-2y=7$$

Sol$^n$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2-3 = -5 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 1 & 5 \\ -2 & 7 \end{vmatrix} = 7-(-10) = 7+10 = 17$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 7-15 = -8$$

$x = \frac{\Delta_1}{\Delta} = \frac{17}{-5}$, $y = \frac{\Delta_2}{\Delta} = \frac{-8}{-5} = \frac{8}{5}$ (By Cramer's Rule)

Unique solution.

Ex Solve the system of linear equations by Cramer's rule. Write consistency (nature of solution).

$$x+2y+z=7$$
$$2x+4y+5z=8$$
$$3x+y+9z=6$$

**Solⁿ** $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$

$= 1(36-5) - 2(18-15) + 1(2-12)$

$= 31 - 2\times 3 + (-10)$

$= 31-6-10 = 15$

$\Rightarrow \Delta \neq 0$ So the system has unique solution.

$\Delta_1 = \begin{vmatrix} 7 & 2 & 1 \\ 8 & 4 & 5 \\ 6 & 1 & 9 \end{vmatrix} = 7\begin{vmatrix} 4 & 5 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 8 & 5 \\ 6 & 9 \end{vmatrix} + 1\begin{vmatrix} 8 & 4 \\ 6 & 1 \end{vmatrix}$

$= 7(36-5) - 2(72-30) + 8 - 24$

$= \cancel{2}7\times 31 - 2\times 42 - 16 = 217 - 84 - 16$

$= 201 - 84 = 117$

$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 8 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 1\begin{vmatrix} 8 & 5 \\ 6 & 9 \end{vmatrix} - 7\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} + 1\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix}$

$= 72 - 30 - 7(18-15) + 1(12-24)$

$= 42 - 7\times(+3) + (-12)$

$= 42 - 21 - 12 = 42 - 33 = 9$

$\Delta_3 = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} 4 & 8 \\ 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + 7\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$

$= 24 - 8 - 2(12-24) + 7(2-12)$

$= 16 - 2(-12) + 7(-10) = 16 + 24 - 70$

$= 40 - 70 = -30$

The solution is given by.

$$x = \frac{\Delta_1}{\Delta} = \frac{117}{15}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{9}{15}$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-30}{15}$$

So, the system of linear equations ~~are~~ have uniq non-trivial solution.

<u>Homogeneous System of Linear Equations:</u>

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

R.H.S all are zero is homogeneous equation.

$\boxed{\text{Solution}}$

$\Delta \neq 0$

Unique trivial solution

$\Delta = 0$

Infinite solution (trivial + nontrivial).

Find the nature of the solution for the given system of equations.

$$x + y + 3z = 3$$
$$2x + 2y + 4z = 4$$
$$3x + 3y + 5z = 0$$

$$D = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 5 \end{vmatrix} = 0, \quad D_1 = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 4 & 4 \\ 3 & 5 & 0 \end{vmatrix} = 10.$$

Since $D = 0$ & $D_1 \neq 0$.

The system has no solution.

# Unit-1 Matrices & Determinants

## A) MATRICES :-

A matrix is a rectangular array of numbers, arranged in rows (horizontal line) and columns (vertical line).

A matrix is generally given by Capital letters

Let $A = [a_{ij}]_{m \times n}$, $i = 1, 2, \dots m$
$j = 1, 2, \dots n$

$m$ = numbers of rows
$n$ = number of columns
$m \times n$ = order of the matrix A.

We can write

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mn} \end{bmatrix}_{m \times n}$$

each $a_{ij}$ is called element of the matrix.
$mn$ is the number of elements of the matrix.

### Types of Matrices

① Square Matrix: If row (m) is equal to columns (n) then this type of matrix is a square matrix.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}_{4\times4}$

(2) Row Matrix : (Row Vector) :

If A matrix has exactly one row. $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}_{1\times n}$

Ex $A = \begin{bmatrix} a & b & c & d \end{bmatrix}_{1\times 4}$

(3) Column Matrix : (Column Vector)

If A matrix has only one column. $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m\times 1}$

Ex $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

(4) Zero / Null Matrix : $(A = O_{m\times n})$

A matrix whose all entries (elements) are zero

Ex $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1\times 3}$ , $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

(5) Rectangular Matrix :

If a matrix has $(m \neq n)$. row $\neq$ column form.

(6) Triangular Matrix :-

A square matrix where all the elements above or below the main diagonal are zero

$\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$ , $\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$

Upper triangular Matrix     Lower triangular Matrix

## (7) Trace of a Matrix :-

The sum of all diagonal elements of a square matrix. is called Trace of a matrix.

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $Tr(A) = 1+4 = 5$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

$Tr(A) = 1+5+2 = 8$

## (8) Singleton Matrix :

If a matrix has only one element is called a singleton matrix. ( $m=n=1$ )

ex $[1]$, $[-2]$

# Algebra of MATRICES

1) Addition and Subtraction :

If A & B are two matrices of same order then addition & subtraction can be done with corresponding elements.

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & -3 \\ 0 & 3 & -2 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -3 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+2 & 3-3 \\ 4+0 & 5+3 & 6-2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 8 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 2 & -3 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1-(-1) & 2-2 & 3-(-3) \\ 4-0 & 5-3 & 6-(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 6 \\ 4 & 2 & 8 \end{bmatrix}$$

## (2) Multiplication

If A & B are two matrices, A with order $m \times n$ and B with order $p \times q$ then multiplication is possible if and only if $\boxed{n=p}$. Multiplication done with corresponding elements of row & column.

$$[a_{ij}]_{m\times n} \times [b_{ij}]_{p\times q} = [c_{ij}]_{m\times q} \quad (n=p)$$

Ex $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 9 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 8 & 1 \\ 0 & 1 \end{bmatrix}$

Sol$^n$

$$AB = \begin{array}{c} R_1 \\ R_2 \end{array} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 9 & 2 \end{bmatrix}_{2\times 3} \times \begin{array}{cc} C_1 & C_2 \\ \end{array} \begin{bmatrix} 3 & 4 \\ 8 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

$$= \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 1\times 3 + 3\times 8 + (-1)\times 0 & 1\times 4 + 3\times 1 + (-1)\times 1 \\ 0\times 3 + 9\times 8 + 2\times 0 & 0\times 4 + 9\times 1 + 2\times 1 \end{bmatrix} \end{array}$$

$$AB = \begin{bmatrix} 3+24+0 & 4+3-1 \\ 0+72+0 & 0+9+2 \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 72 & 11 \end{bmatrix}.$$

## Properties of Matrix Multiplication:

1) In general $AB \neq BA$ (not commutative) where A & B are matrices.

2) Matrix multiplication is Associative. If A, B, C are matrices (conformable) then

> $(AB)C = A(BC)$
> $A(B+C) = AB + AC$
> $(A+B)C = AC + BC$

3) If A is a square matrix then $A^n = A \times A \times \cdots \times A$ (n-times).

## Transpose of a Matrix: (changing rows & columns)

If $A = [a_{ij}]_{m\times n}$ then transpose of A is

$A'$ or $A^T = [a_{ji}]_{n\times m}$.

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}_{2\times 3}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 2 \end{bmatrix}_{3\times 2}$

## Orthogonal Matrix :-

A square matrix $A$ is said to be orthogonal matrix if $AA^T = I$, $I$ = identity matrix.

> If $A$ & $B$ are orthogonal then $AB$ is also orthogonal.

> If $AA^T = I$ then $A^{-1} = A^T$

> If $A$ is orthogonal then $A^T$ & $A^{-1}$ are orthogonal.

> Determinant of orthogonal matrix is either 1 or -1.

## Symmetric & Skew Symmetric Matrix :-

A square matrix $A = [a_{ij}]$ is symmetric if

$\boxed{A = A^T}$ or $a_{ij} = a_{ji}$, $\forall$ $i$ & $j$.

format

$$A = A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

A square matrix $A = [a_{ij}]$ is skew-symmetric if

$\boxed{A = -A^T}$ or $a_{ij} = -a_{ji}$ $\forall$ $i$ & $j$.

format $$A = -A^T = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

Theorem :- Every square matrix can be uniquely expressed as a sum or difference of a symmetric and skew-symmetric matrix.

i.e $A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{symmmetric}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{skew-symmetric}}$

$$A = \frac{1}{2}(A^T+A) - \frac{1}{2}(A^T-A)$$

Ex Express $A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

Sol$^n$ Given $A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$

$$A+A^T = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1+1 & 4+3 \\ 3+4 & -2-2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 7 & -4 \end{bmatrix}$$

$$\frac{1}{2}(A+A^T) = \frac{1}{2}\begin{bmatrix} 2 & 7 \\ 7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix}$$

$$A-A^T = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1-1 & 4-3 \\ 3-4 & -2-(-2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A-A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$$

$$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) \Rightarrow \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$$

## Singular and Non-Singular Matrices :-

A square matrix A is singular if $|A| = 0$ . and Non-singular if $|A| \neq 0$ ( $|A|$ = determinant of A)

**Ex** $|A| = \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 6 - 6 = 0 \Rightarrow |A| = 0$ (Singular)

$|A| = \begin{vmatrix} 5 & 3 \\ 4 & 5 \end{vmatrix} = 5 \times 5 - 4 \times 3 = 25 - 12 = 13 \neq 0$. (Non-singular)

## Adjoint of a Square Matrix :- (or Adjugate)

If $A = [a_{ij}]_{n \times n}$ a square matrix ~~and~~ then adjoint of A is given by adj A.

$$\cancel{\text{adj } A =} \begin{bmatrix} \cancel{C_{11} \quad C_{12} \quad C_{13}} \\ C_{21} \quad C_{22} \quad C_{23} \\ C \end{bmatrix}$$

$\boxed{\text{adj} A = [C_{ij}]^T_{n \times n}}$ where $C_{ij}$ is cofactor of $a_{ij}$ in $|A|$.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

**Note :**

(i) $|\text{adj} A| = |A|^{n-1}$

(ii) $A(\text{adj} A) = (\text{adj} A) . A = |A| . I$

Ex Find adjoint of $A = \begin{bmatrix} -2 & 2 & 3 \\ -1 & 5 & 10 \\ 4 & 4 & 2 \end{bmatrix}$.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 10 \\ 4 & 2 \end{vmatrix} = 5 \times 2 - 4 \times 10 = 10 - 40 = -30$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 10 \\ 4 & 2 \end{vmatrix} = -((-1) \times 2 - 4 \times 10) = -(-2-40) = 42$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 5 \\ 4 & 4 \end{vmatrix} = (-1) \times 4 - 4 \times 5 = -4 - 20 = -24$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = -(2 \times 2 - 4 \times 3) = -(4-12) = -(-8) = 8$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} -2 & 3 \\ 4 & 2 \end{vmatrix} = (-2) \times 2 - 4 \times 3 = -4 - 12 = -16$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} -2 & 2 \\ 4 & 4 \end{vmatrix} = -((-2) \times 4 - 4 \times 2) = -(-8-8) = 16$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 5 & 10 \end{vmatrix} = 2 \times 10 - 5 \times 3 = 20 - 15 = 5$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} -2 & 3 \\ -1 & 10 \end{vmatrix} = -((-2) \times 10 - (-1) \times 3) = -(-20+3) = 17$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} -2 & 2 \\ -1 & 5 \end{vmatrix} = (-2) \times 5 - (-1) \times 2 = -10 + 2 = -8$$

$$\text{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -30 & 42 & -24 \\ 8 & -16 & 16 \\ 5 & 17 & -8 \end{bmatrix}^T$$

$$\Rightarrow \cancel{\phi}\ \text{adj} A = \begin{bmatrix} -30 & 8 & 5 \\ 42 & -16 & 17 \\ -24 & 16 & -8 \end{bmatrix}$$

## Inverse of a Matrix (Reciprocal / Determinant Method)

A square matrix $A$ (non singular) of order '$n$' is invertible (inverse exist) if $\exists$ a matrix $B$

s.t $\quad AB = I_n = BA$

It is denoted by $A^{-1}$

We know $\quad A(\text{adj} A) = |A| I_n$

$$A^{-1} A (\text{adj} A) = A^{-1} |A| I_n$$

$$I_n (\text{adj} A) = A^{-1} |A| I_n$$

$$\Rightarrow \boxed{A^{-1} = \frac{\text{adj} A}{|A|}}, \quad |A| \neq 0$$

Properties: (If $A$ & $B$ are invertible)

1) $(AB)^{-1} = B^{-1} A^{-1}$

2) $(A^T)^{-1} = (A^{-1})^T$

3) $(A^{-1})^{-1} = A$

4) $(A^k)^{-1} = (A^{-1})^k$ , $k \in N$

5) If $|A| \neq 0$ , then $|A^{-1}| = |A|^{-1}$

6) Orthogonal Matrix is always invertible.

example find inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Given $|A| \neq 0$, $|A| = 2$.

Solⁿ $C_{ij}$ is the cofactor of $a_{ij}$.

$C_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1$

$C_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1$

$C_{22} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$C_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(0-1) = 1$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$C_{32} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(0-1) = 1$

$C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

H.W $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$, Ans. $|A| = 2$,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

## Matrix Method (for finding Inverse)

Consider the system of '$n$' linear equations in '$n$' unknowns $x_1, x_2, \ldots, x_n$

i.e $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$

$i = 1, 2, \ldots, n$

These equations can be written in Matrix form

$$\boxed{AX = B}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If $|A| \neq 0$ (non-singular matrix) then for the solution of the system of equation

$$\boxed{X = A^{-1}B}$$

Consistent : A system has at least one solution otherwise inconsistent for no solution.

**System of linear equation**

- $AX = B$ (non-homogeneous)
  - → $|A| \neq 0$ (unique sol$^n$ $X = A^{-1}B$)
  - → $|A| = 0$ & $\text{adj}A \cdot B \neq 0$ (No solution)
  - → $|A| = 0$ & $\text{adj}A \cdot B = 0$ (Infinite solution)
- $AX = 0$ (Homogeneous)
  - → $|A| \neq 0$ (one solution; trivial) (zero solution)
  - → $|A| = 0$ (Infinite sol$^n$)
  - → Equations < unknowns (Non trivial solution)

## Creamer's Rule :-

Let us consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$ for $\Delta \neq 0$ is the solution of the given equation.

Nature of the solution :-

1) $\Delta \neq 0$ & at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$ (Unique solⁿ)

2) $\Delta \neq 0$ & $\Delta_1 = \Delta_2 = \Delta_3 = 0$ (Trivial solution) ($x=y=z=0$)

3) $\Delta = 0$ & $\Delta_1 = \Delta_2 = \Delta_3 = 0$ (Infinite solutions)

4) $\Delta = 0$ & at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$ (No solution)

Ex Solve the system of equations by Cramer's rule. Write Consistency:

$$x+y=5$$
$$3x-2y=7$$

Sol$^n$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2-3 = -5 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 1 & 5 \\ -2 & 7 \end{vmatrix} = 7-(-10) = 7+10 = 17$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 7-15 = -8$$

$x = \frac{\Delta_1}{\Delta} = \frac{17}{-5}$, $y = \frac{\Delta_2}{\Delta} = \frac{-8}{-5} = \frac{8}{5}$ (By Cramer's Rule)

Unique solution.

Ex Solve the system of linear equations by Cramer's rule. Write Consistency (nature of solution).

$$x+2y+z=7$$
$$2x+4y+5z=8$$
$$3x+y+9z=6$$

Sol$^n$ $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$

$= 1(36-5) - 2(18-15) + 1(2-12)$

$= 31 - 2\times 3 + (-10)$

$= 31-6-10 = 15$

$\Rightarrow \Delta \neq 0$ So the system has unique solution.

$\Delta_1 = \begin{vmatrix} 7 & 2 & 1 \\ 8 & 4 & 5 \\ 6 & 1 & 9 \end{vmatrix} = 7\begin{vmatrix} 4 & 5 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 8 & 5 \\ 6 & 9 \end{vmatrix} + 1\begin{vmatrix} 8 & 4 \\ 6 & 1 \end{vmatrix}$

$= 7(36-5) - 2(72-30) + 8-24$

$= 7\times 31 - 2\times 42 - 16 = 217-84-16$

$= 201-84 = 117$

$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 8 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 1\begin{vmatrix} 8 & 5 \\ 6 & 9 \end{vmatrix} - 7\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} + 1\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix}$

$= 72-30 - 7(18-15) + 1(12-24)$

$= 42 - 7\times(+3) + (-12)$

$= 42 - 21 - 12 = 42 - 33 = 9$

$\Delta_3 = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} 4 & 8 \\ 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + 7\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$

$= 24-8 - 2(12-24) + 7(2-12)$

$= 16 - 2(-12) + 7(-10) = 16+24-70$

$= 40-70 = -30$

The solution is given by,

$$x = \frac{\Delta_1}{\Delta} = \frac{117}{15}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{9}{15}$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-30}{15}$$

So, the system of linear equations ~~are~~ have uniqu
non-trivial solution.

## Homogeneous System of Linear Equations:

$$a_1 x + b_1 y + c_1 z = 0$$

$$a_2 x + b_2 y + c_2 z = 0$$

$$a_3 x + b_3 y + c_3 z = 0$$

R.H.S all are zero is homogeneous equation.

$\boxed{\text{Solution}}$

$\Delta \neq 0$ — Unique trivial solution

$\Delta = 0$ — Infinite solution (trivial + nontrivial).

**Ex** Find the nature of the solution for the given system of equations.

$$x + y + 3z = 3$$
$$2x + 2y + 4z = 4$$
$$3x + 3y + 5z = 0$$

**Sol$^n$**

$$D = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 5 \end{vmatrix} = 0, \quad D_1 = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 4 & 4 \\ 3 & 5 & 0 \end{vmatrix} = 10$$

Since $D = 0$ & $D_1 \neq 0$.

The system has no solution.

# INTEGRAL CALCULUS

Integral Calculus can further be studied under two parts.

1) Indefinite Integrals
2) Definite Integrals

## 1. INDEFINITE INTEGRALS

If $F'(x) = f(x)$ then $F(x)$ is called integral or antiderivative or primitive of $f(x)$ with respect to $x$. It is given by

$$\int f(x)\,dx = F(x) + c$$

### Properties of Indefinite Integrals :-

1. $\int k f(x)\,dx = k\int f(x)\,dx$ ($k$ = constant)

2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

3. $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$, $a \neq 0$

### Formulae

1. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$, $n \neq -1$

2. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log(ax+b) + c$

3. $\int e^{ax+b}\,dx = \frac{1}{a} e^{ax+b} + c$

4. $\int a^{px+q}\,dx = \frac{1}{p} \cdot \frac{a^{px+q}}{\log a} + c \quad (a > 0)$

5. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

6. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

7. $\int \tan(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b)| + c$

8. $\int \cot(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b)| + c$

9. $\cancel{\int \sec^2(ax+b)\,dx}$ $\int \sec x\,dx = \log|\sec x + \tan x| + c$

10. $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c$

11. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

12. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x + \sqrt{x^2+a^2}\right| + c$

13. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

14. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x + \sqrt{x^2-a^2}\right| + c$

15. $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

16. $\int \frac{1}{a^2 + x^2}\, dx = \log \left| x + \sqrt{x^2 + a^2} \right| + c$

17. $\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

18. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

19. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log \left( x + \sqrt{a^2 + x^2} \right) + c$

20. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right) + c$

21. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} ( a \sin bx - b \cos bx ) + c$

22. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} ( a \cos bx + b \sin bx ) + c$

Examples :

1) $\int x^4 dx = \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$

2) $\int (3x^2 - 10x + 7)\, dx = 3\int x^2 dx - 10\int x\,dx + 7\int dx$

$= 3\frac{x^3}{3} - 10\frac{x^2}{2} + 7x = x^3 - 5x^2 + 7x + c$

3) $\int \frac{1}{x+7} dx = \log(x+7) + c$

4) $\int \frac{1}{9x+7} dx = \frac{1}{9}\int \frac{dt}{t} = \frac{1}{9}\log(t) + c = \frac{1}{9}\log(9x-7) + c$

Put $t = 9x - 7$

$dt = 9\,dx$

5) $\int e^{5x} dx = \frac{1}{5}\int e^{5x} \times 5\,dx = \frac{1}{5}e^{5x} + c$

6) $\int a^{9x} dx = \frac{1}{9} a^{9x} \cdot \frac{1}{\log a} + c$

7) $\int \frac{1}{\sqrt{9 - x^2}} dx = \sin^{-1}\frac{x}{3} + c$

8) $\int \frac{1}{x\sqrt{9x^2 - 25}} dx = \int \frac{3\,dx}{3x\sqrt{(3x)^2 - 5^2}} = \frac{1}{5}\sec^{-1}\frac{3x}{5} + c$

9) $\int \sqrt{x^2 - 9}\, dx = \frac{x}{2}\sqrt{x^2 - 9} - \frac{9}{2}\log\left|x + \sqrt{x^2 - 9}\right| + c$

## Important Methods of Integration:

If the integrand is not a derivative of a known function, then there are 3-methods to use.

1) Integration by substitution
2) Integration by parts
3) Integration by partial fraction.

### 1) Integration by Substitution:

If $f(x)$ is continuous differentiable function then for $\int \phi[f(x)]\, f'(x)\,dx$ we will put $f(x)=t$

so that it becomes $\int \phi(t)\,dt$ for easy integration,

or, for $\int [Q(x)]^n Q'(x)\,dx$ or $\int \frac{Q'(x)}{\sqrt{Q(x)}}\,dx$ or

$\int \frac{Q'(x)}{[Q(x)]^n}\,dx$ we will put $Q(x)=t$

So that above term becomes $\int t^n\,dt$, $\int \frac{dt}{\sqrt{t}}$, $\int \frac{dt}{t^n}$

Examples

1) $\int \frac{\cos(\log x)}{x}\,dx$

put $\log x = t$

differentiate both sides

$\frac{1}{x}\,dx = dt$

$= \int \cos(t)\,dt$

$= \sin t + c = \sin(\log x) + c$

2) $\int \frac{(1+\log x)^3}{x}\,dx$

$= \int (1+\log x)^3 \frac{1}{x}\,dx$

put $1+\log x = t$

$\frac{1}{x}\,dx = dt$

$= \int (t)^3\,dt = \frac{t^{3+1}}{3+1} + c$

$= \frac{t^4}{4} + c = \frac{(1+\log x)^4}{4} + c$

3) $\int e^x \cos e^x\,dx$

put $e^x = t$

$e^x dx = dt$

$= \int \cos e^x \cdot e^x dx$

$= \int \cos t \cdot dt$

$= \sin t + c$

$= \sin e^x + c$

2) Integration by parts:

If $u$ & $v$ are differentiable functions of $x$ then,

$$\int u \cdot v \, dx = u \int v \, dx - \int \left[ u' \cdot \int v \, dx \right] dx.$$

where $u' = \frac{du}{dx}$

The integral must be divided into two parts $u$ & $v$ in the priority manner of ILATE.

I = Inverse function ($\sin^{-1} x$, $\cos^{-1} x$ etc).

L = Logarithmic function ($\log x$, $\frac{1}{\log x}$ etc)

A = Algebraic function ($x$, $x^2+x+1$, $x^5$ etc)

T = Trigonometric function ($\sin x$, $\cos x$, $\sec x$ etc)

E = Exponential function ($e^x$, $e^{5x}$, $e^{-7x}$ etc)

Examples:

1) $I = \int \sec^3\theta \, d\theta$

$$I = \int \underset{u}{\underline{\sec\theta}} \cdot \underset{v}{\underline{\sec^2\theta}} \, d\theta$$

$$= \sec\theta \int \sec^2\theta \, d\theta - \int \tan\theta (\sec\theta \cdot \tan\theta) \, d\theta$$

$$= \sec\theta \cdot \tan\theta - \int (\sec^2\theta - 1) \sec\theta \, d\theta$$

$$= \sec\theta \cdot \tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$$I + \int \sec^3\theta \, d\theta = \sec\theta \tan\theta + \log(\sec x + \tan x) + c$$

$\Rightarrow 2I = \sec\theta \cdot \tan\theta + \log(\sec x + \tan x) + c$

2) $\int \log x \, dx$ (As only one function is there we take unity as '1' as the second function).

$\Rightarrow I = \int 1 \cdot \log x \, dx$ $\quad u = 1, \; v = \log x$

$= \log x \int 1 \cdot dx - \int \left[ \frac{d}{dx}(\log x) \int 1 \cdot dx \right] dx + c$

$= \log x \cdot x - \int \left( \frac{1}{x} \cdot x \right) dx + c$

$= x \log x - \int 1 \cdot dx + c = x \log x - x + c$

3) $\int e^x \sin x \, dx$

take $I = \int e^x \cdot \sin x \, dx$ $\quad , u = \sin x \quad v = e^x$

$= \sin x \int e^x dx - \int e^x (\cos x) dx + c$

$= e^x \sin x - \int \underset{u}{\cos x} \cdot \underset{v}{e^x} dx + c$

$= e^x \sin x - \left[ \cos x \int e^x - \int \left( \frac{d}{dx}\cos x \cdot \int e^x dx \right) dx \right] + c$

$= e^x \sin x - e^x \cos x + \int (-\sin x) e^x dx + c$

$I + \int \sin x \cdot e^x dx = e^x (\sin x - \cos x) + c$

$I + I = e^x (\sin x - \cos x) + c$

$2I = e^x (\sin x - \cos x) + c$

$I = \frac{e^x}{2} (\sin x - \cos x) + c$

## 3) Integration By Partial Fractions :

If $f(x)$, $g(x)$ are polynomials then $\frac{f(x)}{g(x)}$ is called rational fraction. (also called fraction).

$\frac{f(x)}{g(x)}$ is called proper rational fraction if degree of $f(x) <$ degree of $g(x)$. Otherwise called as an Improper rational fraction.

Every proper rational fraction can be expressed as a sum of simpler fractions, called partial fractions.

| Proper rational fraction | Partial fraction |
|---|---|
| 1. $\frac{px+q}{(ax+b)(cx+d)}$ | $\frac{A_1}{ax+b} + \frac{A_2}{cx+d}$ |
| 2. $\frac{px+q}{(ax+b)^2(cx+d)}$ | $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{cx+d}$ |
| 3. $\frac{px^2+qx+r}{(ax+b)(cx+d)(ex+f)}$ | $\frac{A_1}{ax+b} + \frac{A_2}{cx+d} + \frac{A_3}{ex+f}$ |
| 4. $\frac{px^2+qx+r}{(ax+b)^2(cx+d)}$ | $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{cx+d}$ |

Examples :

$$I = \int \frac{(3x+1)}{(x+1)(x-2)}\,dx$$

It is a proper ~~integral~~ rational fraction.

$$\frac{3x+1}{(x+1)(x-2)} = \frac{A_1}{x+1} + \frac{A_2}{x-2} \quad \text{——— (1)}$$

$$\frac{3x+1}{(x+1)(x-2)} = \frac{A_1(x-2) + A_2(x+1)}{(x+1)(x-2)}$$

$$\Rightarrow \quad 3x+1 = A_1(x-2) + A_2(x+1)$$

$$= A_1x + A_2x - 2A_1 + A_2$$

$$3x+1 = (A_1+A_2)x + (-2A_1+A_2)$$

comparing & equating

| | |
|---|---|
| $A_1 + A_2 = 3$ | $-2A_1 + A_2 = 1$ |
| $A_1 + 1 + 2A_1 = 3$ | $A_2 = 1 + 2A_1$ |
| $3A_1 = 3 - 1$ | $A_2 = 1 + 2A_1$ |
| $A_1 = \frac{2}{3}$ | $A_2 = 1 + 2 \times \frac{2}{3} = 1 + \frac{4}{3} = \frac{7}{3}$ |

Put in equation (1) value of $A_1$ & $A_2$

$$\frac{3x+1}{(x+1)(x-2)} = \frac{2}{3}\frac{1}{x+1} + \frac{7}{3}\cdot\frac{1}{x-2}$$

Integrating both side's $\displaystyle\int \frac{3x+1}{(x+1)(x-2)}\,dx = \frac{2}{3}\int \frac{1}{x+1}\,dx + \frac{7}{3}\int \frac{1}{x-2}\,dx$

$$I = \frac{2}{3}\log(x+1) + \frac{7}{3}\log(x-2) + C \quad \text{(Ans)}$$

8) $\int \frac{3x-4}{(x-1)^2(x+1)}\,dx$

Let $I = \int \frac{3x-4}{(x-1)^2(x+1)}\,dx$ (proper rational fraction)

Now, $\frac{3x-4}{(x-1)^2(x+1)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1}$ —— (*)

$$\frac{3x-4}{(x-1)^2(x+1)} = \frac{A_1(x-1)(x+1) + A_2(x+1) + A_3(x-1)^2}{(x-1)^2(x+1)}$$

$\Rightarrow 3x-4 = A_1(x^2-1) + A_2(x+1) + A_3(x^2+1-2x)$

$= A_1x^2 - A_1 + A_2x + A_2 + A_3x^2 + A_3 - 2A_3x$

$\Rightarrow 3x-4 = (A_1+A_3)x^2 + (A_2-2A_3)x + (-A_1+A_2+A_3)$

Equating both sides terms of corresponding degree

$A_1 + A_3 = 0$ —— (1)

$A_2 - 2A_3 = 3$ —— (2)

$-A_1 + A_2 + A_3 = -4$ —— (3)

from (1) $A_1 = -A_3$ & from (2) $A_2 = 2A_3 + 3$

put in (3) $-(-A_3) + 2A_3 + 3 + A_3 = -4$

$\Rightarrow A_3 + 2A_3 + A_3 = -4 - 3$

$\Rightarrow 4A_3 = -7 \Rightarrow A_3 = \frac{-7}{4}$

$A_1 = -A_3 = -\left(-\frac{7}{4}\right) = \frac{7}{4}$

$A_2 = 2A_3 + 3 = 2 \times \left(\frac{-7}{4}\right) + 3 = -\frac{7}{2} + 3 = \frac{-7+6}{2} = -\frac{1}{2}$

$\therefore A_1 = \frac{7}{4},\ A_2 = -\frac{1}{2},\ A_3 = -\frac{7}{4}$ put in eqn (*)

$$I = \int \frac{7}{4}\left(\frac{1}{x-1}\right)dx + \int \left(-\frac{1}{2}\right)\frac{1}{(x-1)^2}dx + \int \left(-\frac{7}{4}\right)\frac{1}{(x+1)}dx$$

$$= \frac{7}{4}\log(x-1) - \frac{1}{2}\frac{1}{(x-1)} + \left(-\frac{7}{4}\right)\log(x+1) + C$$

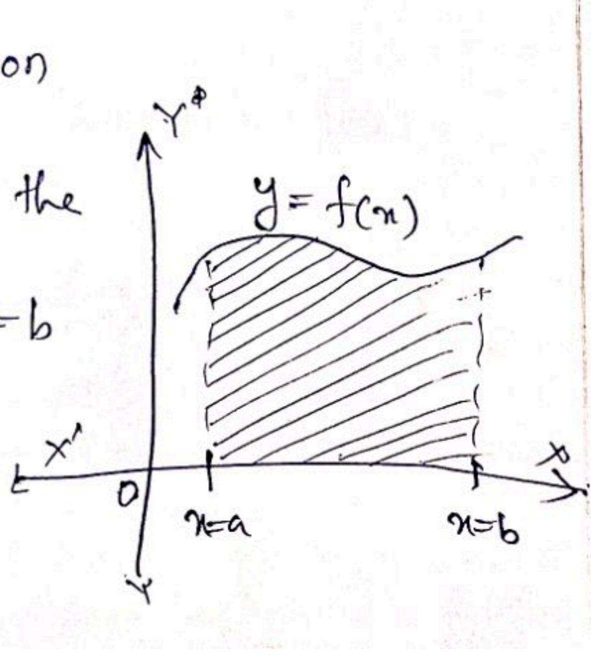
## Definite Integrals :-

A is defined as .

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $f(x)$ is continuous function

Geometrically, $\int_a^b f(x)\,dx$ is for the curve $y = f(x)$ between $x = a$ ; $x = b$ and x-axis.



Note : $\int_a^b f(x)dx = 0 \Rightarrow f(x)=0$ has at least 1-root in $(a,b)$, provided $f(x)$ is continuous function in $(a,b)$

$\int_a^b \int_a^a f(x)\,dx = 0 \quad (a=b)$

Examples :

i) $\int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right]_0^2 = \frac{1}{4}(2^4 - 0^4) = \frac{16-0}{4} = \frac{16}{4} = 4$

ii) $\int_0^{\pi/4} \sin^4 2t \cdot \cos 2t\,dt$

put $\sin 2t = z$

$\cos 2t(2dt) = dz$

$\cos 2t\,dt = \frac{1}{2}dz$

$= \int \cdot z^4 \frac{1}{2}dz = \frac{1}{2}\frac{z^5}{5} = \frac{1}{10}(\sin 2t)^5$.

~~For~~ Here $\int_0^{\pi/4} \sin^4 2t \cos 2t\,dt = F\left(\frac{\pi}{4}\right) - F(0)$

$= \frac{1}{10}\left[\left(\sin 2\times\frac{\pi}{4}\right)^5 - (\sin 0)^5\right]$

$= \frac{1}{10}\left[\left(\sin\frac{\pi}{2}\right)^5 - \sin 0^5\right]$

$I = \frac{1}{10}(1-0) = \frac{1}{10}$

# Common Properties of Definite Integral :

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ provided f is same.

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ ; $c \in (a,b)$

4. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, & f(x) \text{ is even.} \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$$= \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

7. $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$, $f(T+x) = f(x)$; T = period of function

Examples:

1) $\int_0^2 f(x)\,dx$ if $f(x) = \begin{cases} x^3, & 0 < x < 1 \\ 2x+1, & 1 \le x \le 2 \end{cases}$

Sol

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 x^3\,dx + \int_1^2 (2x+1)\,dx$$

$$= \left.\frac{x^4}{4}\right]_0^1 + \left[2\frac{x^2}{2} + x\right]_1^2$$

$$= \frac{1}{4}(1-0) + \left[(2^2-1^2) + (2-1)\right]$$

$$= \frac{1}{4} + [4-\cancel{1}+\cancel{1}] = \frac{1}{4} + 4 = \frac{1+16}{4} = \frac{17}{4}$$

2) $$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \text{——— (1)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos(0+\frac{\pi}{2}-x)}}{\sqrt{\cos(0+\frac{\pi}{2}-x)} + \sqrt{\sin(0+\frac{\pi}{2}-x)}}\,dx$$ using formula (5)

$$= \int_0^{\pi/2} \frac{\sqrt{\cos(\frac{\pi}{2}-x)}}{\sqrt{\cos(\frac{\pi}{2}-x)} + \sqrt{\sin(\frac{\pi}{2}-x)}}\,dx$$

$\cos(\frac{\pi}{2}-x) = \sin x$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{——— (2)}$$

from (1) & (2) (adding)

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$2I = \int_0^{\pi/2} dx$$

$$I = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4} \text{ (Ans)}.$$

## WALLI'S INTEGRAL FORMULA $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$

1) Case-1 (m, n are positive)

$$\int_0^{\pi/2} \sin^m x . \cos^n x\, dx \quad \text{or} \quad \int_0^{\pi/2} \sin^n x \cos^m x\, dx$$

$$= \begin{cases} \dfrac{(m-1)(m-3)\ldots\ldots \times 1 \text{ or } (n-1)(n-3)\ldots \times 1}{(m+n)(m+n-2)(m+n-4)\ldots} \cdot \dfrac{\pi}{2}, & \text{if } m, n = \text{even (both)} \\[2ex] \dfrac{(m-1)(m-3)\ldots \text{ or } (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \cdot 1, & \text{otherwise} \end{cases}$$

2) Case-2 (n is a positive integer)

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 3 \cdot 1}{n(n-2)(n-4)\ldots 4.2} \times \dfrac{\pi}{2}, & n = \text{even} \\[2ex] \dfrac{(n-1)(n-3)(n-5)\ldots 4.2}{n(n-2)(n-4)\ldots 5.3} \cdot 1, & n = \text{odd} \end{cases}$$

Examples:

1) $\int_0^{\pi/2} \sin^6 x \cos^2 dx$ ( $m=6$, $n=2$ both even )

$$= \frac{(6-1)(6-3)(6-5)(2-1)}{(6+2)(6+2-2)(6+2-4)(6+2-6)} \cdot \frac{\pi}{2}$$

$$= \frac{5\cdot3\cdot1\cdot1}{8\cdot6\cdot4\cdot2\cdot2}\frac{\pi}{2} = \frac{15}{384}\frac{\pi}{4} = \frac{15}{768}\pi$$

2) $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x\, dx$ ( $\sin^4 x \cdot \cos^6 x$ both even function. )

$$I = 2\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$$

$$= 2\,\frac{(3\cdot1)(5\cdot3\cdot1)}{10\cdot8\cdot6\cdot4\cdot2}\frac{\pi}{2} = \frac{3\pi}{256}$$

3) $I = \int_0^{\pi/2} \cos^7 x\, dx$

$$= \frac{6\times4\times2}{7\times5\times3} = \frac{16}{35}$$ (By walli's formula)

H.W $= \int_0^{\pi/2} \sin^8 x\, dx$

$$I = \frac{35}{256}\pi$$

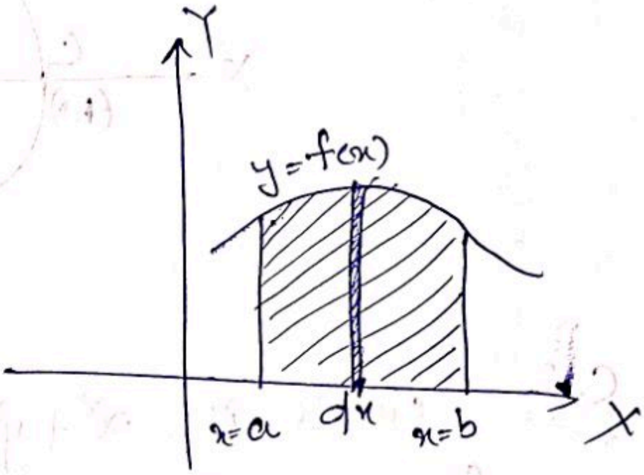
# Application of Integration :

## 1) Area bounded by a Curve and Axes :

### (a) Curve $y = f(x)$ & x-axis

Given $y = f(x)$ and areas between $x = a$ to $x = b$ is given by

$$A = \int_a^b dA = \int_{x=a}^{x=b} f(x)\,dx$$

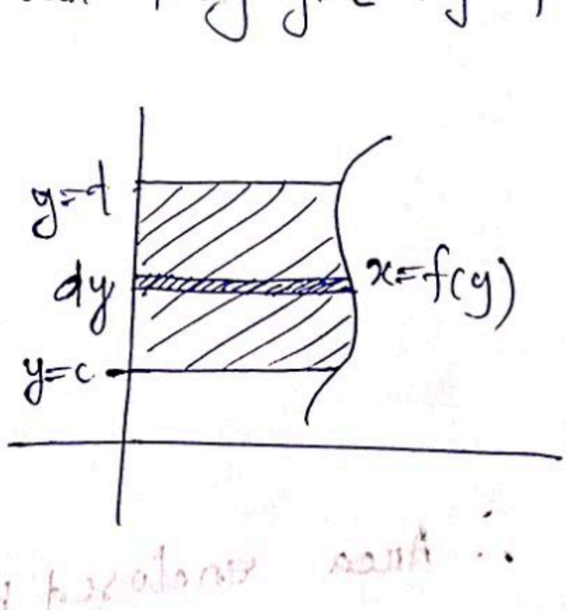



$$\Rightarrow \boxed{A = \int_a^b f(x)\,dx}$$

### (b) Curve $x = f(y)$, Y-axis :

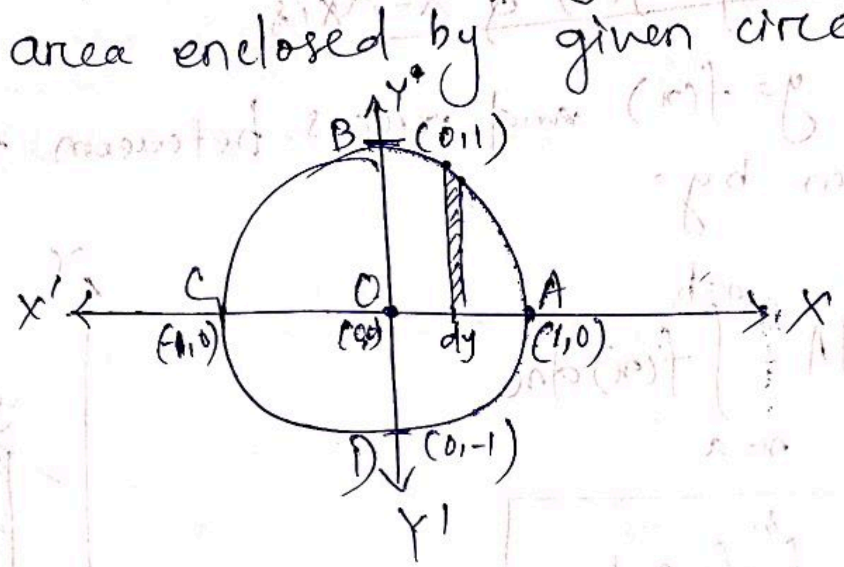
When $x = f(y)$ and area is bounded by $y = c$, $y = d$ is given by

$$A = \int_c^d dA = \int_{y=c}^{y=d} f(y)\,dy$$



Example:-

Evaluate the area enclosed by the one quadrant of the circle $x^2+y^2=1$. Hence find the total area enclosed by given circle




Sol$^n$ circle is $x^2+y^2=1 \Rightarrow y^2=1-x^2 \Rightarrow y=\sqrt{1-x^2}$

finding $\widehat{AOB}$ $y=\sqrt{1-x^2}$ and $x=0$ to $x=1$

$$A_1 = \int_a^b f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right]_0^1$$

$$= \frac{1}{2}\left[\sin^{-1}(1)\right]$$

$$= \frac{\pi}{4}$$

$\therefore$ Area enclosed by the circle $= 4 \times A_1$

$$\Rightarrow A = 4 \times \frac{\pi}{4}$$

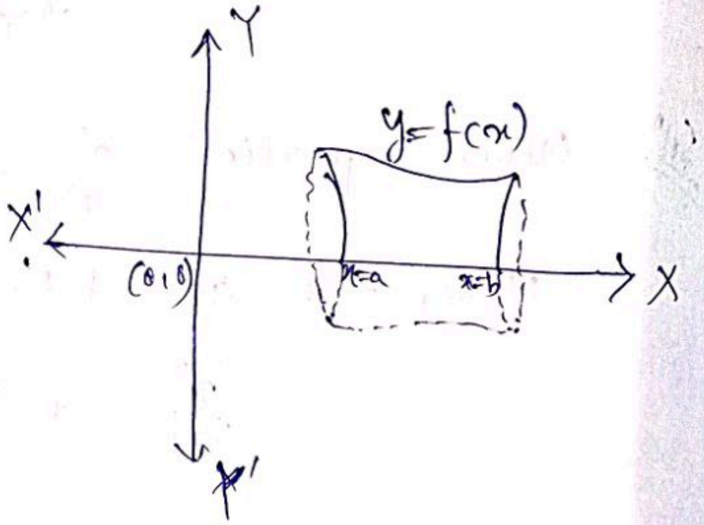
$$A = \pi \text{ sq. units.}$$

2. Volume of a Solid by Revolution of area & about axes

(a) Revolution about $x$-axis:

Volume of a solid formed by revolution of area $y=f(x)$ and ordinates $x=a$ & $x=b$ and the x-axis.

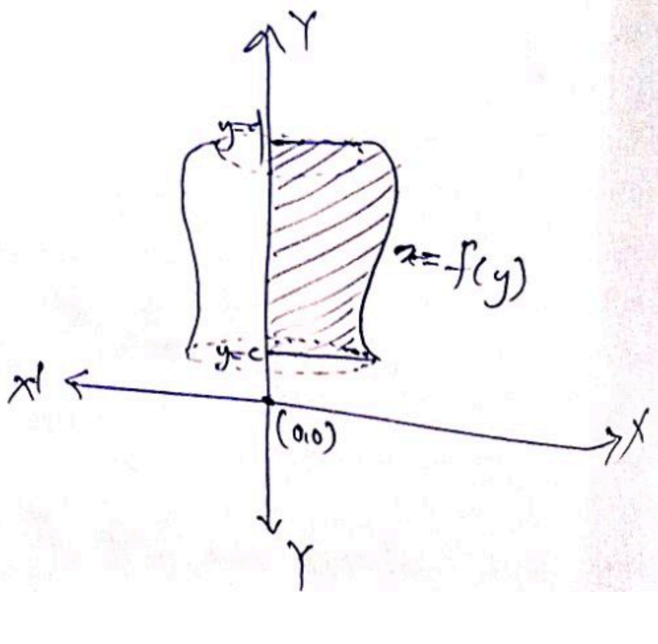
$$\boxed{V = \int_{x=a}^{x=b} \pi [f(x)]^2 dx}$$




(b) Revolution about $y$-axis:

Volume of a solid formed by revolution of area $x=f(y)$ and ordinates $y=c$, $y=d$ and the y-axis.

$$\boxed{V = \int_{y=c}^{y=d} \pi [f(y)]^2 dy}$$



## Examples

1) Let $y = f(x) = x^2$ be the curve given on $[0,2]$. Find the volume of solid generated by revolving the region between the curve in the interval around x-axis.

Sol$^n$ Given $y = x^2$ on $[0,2]$ and x-axis.

$$V = \int_a^b \pi y^2 dx = \int_0^2 \pi (x^2)^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{\pi}{5}(2^5 - 0^5) = \frac{32\pi}{5} \text{ Ans.}$$

2) Let $y = f(x) = x^2$ be on interval $[0,2]$. Find the volume of solid generated by revolving the region between the curves around y-axis.

Sol$^n$ Given $y = x^2$ but axis is 'y' on $[0,2]$

$$V = \int_a^b \pi x^2 dy = \int_0^2 \pi y \, dy = \pi \int_0^2 y \, dy = \pi \left[\frac{y^2}{2}\right]_0^2 = \frac{\pi}{2}(2^2 - 0^2)$$

$$= \frac{\pi}{2}(4 - 0) = 2\pi \text{ Ans.}$$

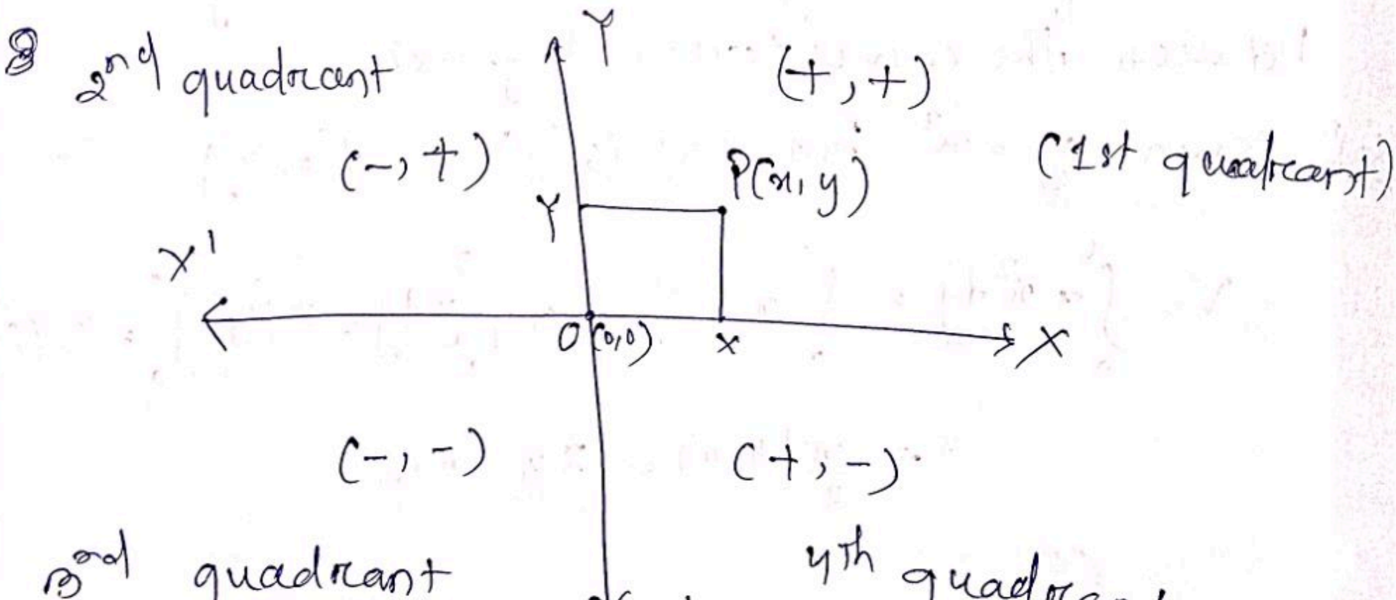
# Unit-3

## Co-ordinate Geometry

### Cartesian Co-ordinates System : (2D)

In this system we consider two axes X & Y. which makes a plane called cartesian plane or XY-plane. The horizontal line is called X-axis and vertical line is Y-axis. X & Y axes are perpendicular to each other.

In this system the coordintes of a point is written as $(x, y)$. where 1st place is always for 'x' & 2nd place is for ~~x~~ y only. The common point (0,0) is called origin.




### (1) STRAIGHT LINE :-

The shortest distance between the two vertices P and Q is called straight line.

for points/vertices $P(x_1, y_1)$ and $Q(\cancel{y}x_2, y_2)$ the distance between P & Q is given by

$$\boxed{\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

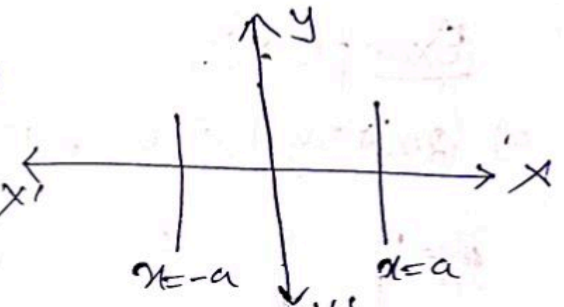
Q. The no. of vertices on x-axis, which are at a distance $a\,(a<4)$ from another vertex $Q(3,4)$ is

(Ans. 4)

Equation of Vertical Lines:

(1) Equation of y-axis is $x = 0$

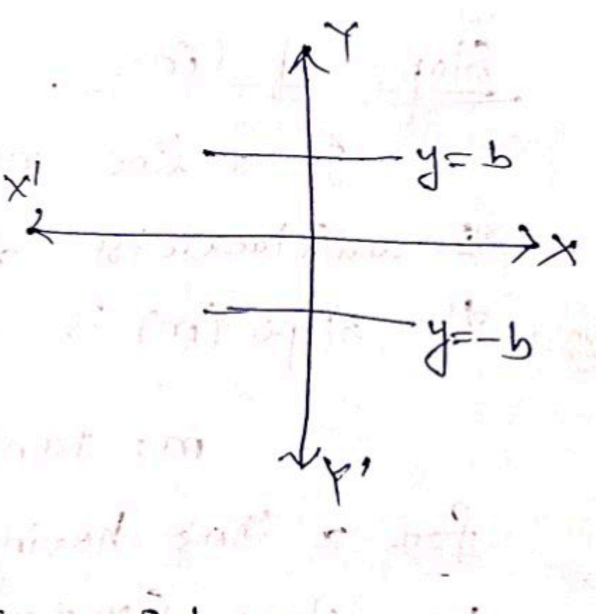
2) Eq$^n$ of line parallel to y-axis at distance of units is $x = a$ or $x = -a$.




Equation of Horizontal line:

1) Equation of X-axis is $y = 0$

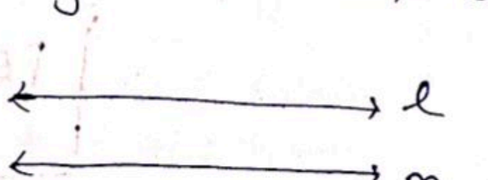
2) Eq$^n$ of line parallel to x-axis at a distance 'b' is $y = b$ or $y = -b$.




1) Parallel lines :-

If two lines do not have any intersection vertex & two lines are disjoint.

$l \parallel m$



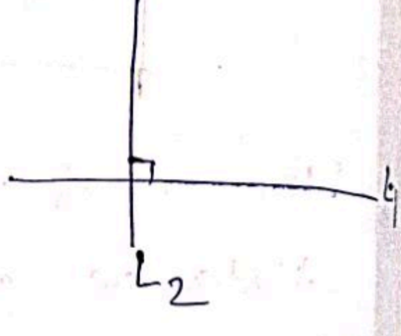
$L_1 : ax + by + c = 0$ is parallel to $L_2 : ax + by + \lambda = 0$

2). Perpendicular line:

Two lines are at $90°$ angle then it is perpendicular lines.

$L_1 : ax + by + c = 0$ is perpendular $L_2$

$L_2 : bx - ay + \lambda_1 = 0$




3) Coincident lines:

If one line overlaps the other lines.

Examples:-

a) $2x + 5y + c = 0$, $4x + 10y + 2c_1 = 0$

$m_1 = y = -\frac{2}{5}x - \frac{c}{5}$ and $y = -\frac{4}{10}x - \frac{2c}{10} = -\frac{2}{5}x - \frac{c_1}{5} = m_2$

$m_1 = \frac{-2}{5} = m_2$ (lines are parallel.

Slope of line

If a line makes an angle ($0 \leq \theta \leq 180°$) in anticlockwise direction from x-axis (initial line) the slope (m) is given by

$$m = \tan\theta$$

for a line having two points $P(x_1, y_1)$ & $Q(x_2, y_2)$ have slope. ($x_1 \neq x_2$)

$$\boxed{PQ = \frac{y_2 - y_1}{x_2 - x_1}}$$

# Equations of Straight lines

① Slope intercept form:

$$\boxed{y = mx + c}$$, m = slope of the line
c = intercept on y-axis.

② Point-slope form:

for point $(x_1, y_1)$ with slope 'm'

$$\boxed{y - y_1 = m(x - x_1)}$$

③ Intercept form:

$c_1$ is the intercept of x-axis and $c_2$ is the intercept of y-axis of one line then it is given by

$$\boxed{\frac{x}{c_1} + \frac{y}{c_2} = 1}$$

④ Two-point form:

If $(x_1, y_1)$ & $(x_2, y_2)$ are two points then eqn of line

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

⑤ Normal form:

p = perpendicular distance from origin to line
$\theta$ = angle betn positive x-axis & perpendicular

then eqn of line

$$\boxed{x\cos\theta + y\sin\theta = p}$$

⑥ General form.

$$\boxed{ax + by + c = 0}$$

Slope of the line $= -\frac{a}{b}$

intercept of line on x-axis $= -\frac{c}{a}$.

intercept of line on y-axis $= -\frac{c}{b}$

Angle between Two lines:-

If $y = a_1x + c_1$ & $y = a_2x + c_2$ are lines then angle is given by

$$\tan\theta = \pm\left(\frac{a_1 - a_2}{1 + a_1a_2}\right)$$

Nature of straight lines

If two lines are $a_1x + b_1y + c_1 = 0$ — $L_1$
$a_2x + b_2y + c_2 = 0$ — $L_2$

then $L_1 \parallel L_2 \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$L_1 \perp L_2 \Leftrightarrow a_1a_2 + b_1b_2 = 0$ $(\because a_1a_2 = -1)$

$L_1, L_2$ coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

## Distance of perpendicular from a point on a line

If the line is $ax + by + c = 0$ & the perpendicular distance from a point $P(x_1, y_1)$ is given by

$$P = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

## Distance between two parallel lines

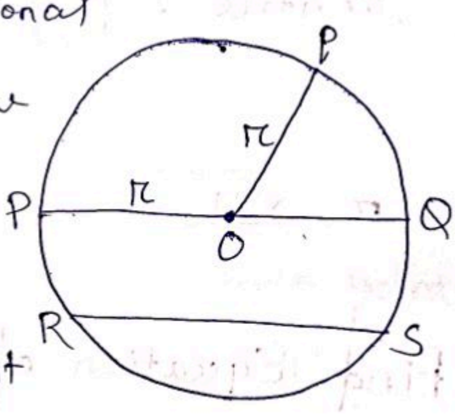
$L_1 : ax + by + c_1 = 0$

$L_2 : ax + by + c_2 = 0$

distance is $D = \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

## Concept of Circle :-

A circle is a closed two-dimensional figure in which the set of all the points in the plane is equidistant from a given fixed point is called the centre of the circle and constant distant is called radius.




Diameter $(PQ) = 2 \times r = 2r$

Circumference $= C = 2\pi r$

Area $= \pi r^2$ $\qquad \pi = \frac{22}{7} = 3.14$

Note: A circle is a special case of Ellipse.

## General Equation of Circle :-

It is given by

$$\boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$$

- centre is $(-g, -f)$, radius $= \sqrt{g^2 + f^2 - c}$

$$C = (-g, -f) = \left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$$

examples

1) $2x^2 + 2y^2 - 6x - 8y + 2 = 0$

$x^2 + y^2 - 3x - 4y + 1 = 0$

$g = -\frac{3}{2}, f = -\frac{4}{2}, c = 1$

centre $= \left(\frac{3}{2}, 2\right)$, $r = \sqrt{g^2 + f^2 - c}$

$= \sqrt{\left(\frac{3}{2}\right)^2 + 2^2 - 1} = \frac{\sqrt{21}}{2}$

$r = \frac{\sqrt{21}}{2}$

(*) Find Equation of Circle if Given

1. Centre & radius
2. Three points lying on it
3. Coordinates of end points of a diameter.

① Centre & Radius (given)

If $(-g, -f)$ is the centre & r is the radius then eqn is

$$\boxed{(x-(-g))^2 + (y-(-f))^2 = r^2}$$

Ex find equation of diameter of the circle $x^2+y^2-4x+2y-10=0$ which passes through origin.

Sol centre $= (-g, -f) = \left(-\frac{1}{2}(-4), -\frac{1}{2}(2)\right) = (2, -1)$ & passes through $(0, 0)$ diameter is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{-1-0}{2-0}(x - 0)$$

$$y = \left(\frac{-1}{2}\right)(x) \Rightarrow 2y = -x \Rightarrow x + 2y = 0$$

② Three points on Circle (given)

If $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are given points of circle then put it on general equation of circle.

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Q find the equation of the circle which passes through the points $(0,1)$, $(1,0)$, $(3,2)$. Also find the value of 'r' & centre.

Sol^n General equation of the circle ——— (1)

$$x^2+y^2+2gx+2fy+c=0$$

point (1) (0,1) put in (1)

$$0^2+1^2+2g\times 0+2f(1)+c=0 \Rightarrow 2f+c+1=0 \quad \text{——— (2)}$$

(1,0) put in (1)

$$1^2+0^2+2g(1)+2f(0)+c=0 \Rightarrow 2g+c+1=0 \quad \text{——— (3)}$$

(3,2) put in (1)

$$3^2+2^2+2g(3)+2f(2)+c=0$$

$$9+4+6g+4f+c=0$$

$$6g+4f+c+13=0 \quad \text{——— (4)}$$

consider (2) & (3)

$$2f+c+1=0$$

$$2g+c+1=0$$

$$\Rightarrow 2f=2g \Rightarrow \boxed{f=g} \quad \text{——— (5)}$$

from (3) & (4) we get

$$6g+4g+c+13=0 \Rightarrow 10g+c+13=0 \quad \text{——— (6)}$$

from (3) & (6)

$$8g=-12,\ g=-\frac{12}{8}=-1.5$$

from (5) we get $f=g=-1.5$

from (2) $2(-1.5)+c+1=0 \Rightarrow c=3.0-1=2.0$

Now $f = g = -1.5$, $c = 2$.

Hence general eqn of circle becomes

$$x^2 + y^2 - 2(1.5)x - 2(1.5)y + 2.0 = 0$$

$$x^2 + y^2 - 3x - 3y + 2 = 0$$

$$C = (-g, -f) = (1.5, 1.5), \quad r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(1.5)^2 + (1.5)^2 - 2}$$

$$= 1.6$$

③ <u>End points of a diameter (given)</u> :

If $(x_1, y_1)$ & $(x_2, y_2)$ two end points of a diameter

then $\left(\frac{y - y_1}{x - x_1}\right) \cdot \left(\frac{y - y_2}{x - x_2}\right) = 1$ is the eqn of circle.

# CONIC SECTION

Conic section (conics) well defined curves obtained by the intersection of the surface of a cone with a plane. There are 3-types of conics (basic)

(i) parabola

(ii) Hyperbola

(iii) Ellipse

## General Equation of a Conic :-

The general equation of a conic with focus $(s, t)$ and directrix $lx + my + n = 0$ is

$$(l^2 + m^2)\left[(x-s)^2 + (y-t)^2\right] = e^2(lx + my + n)$$

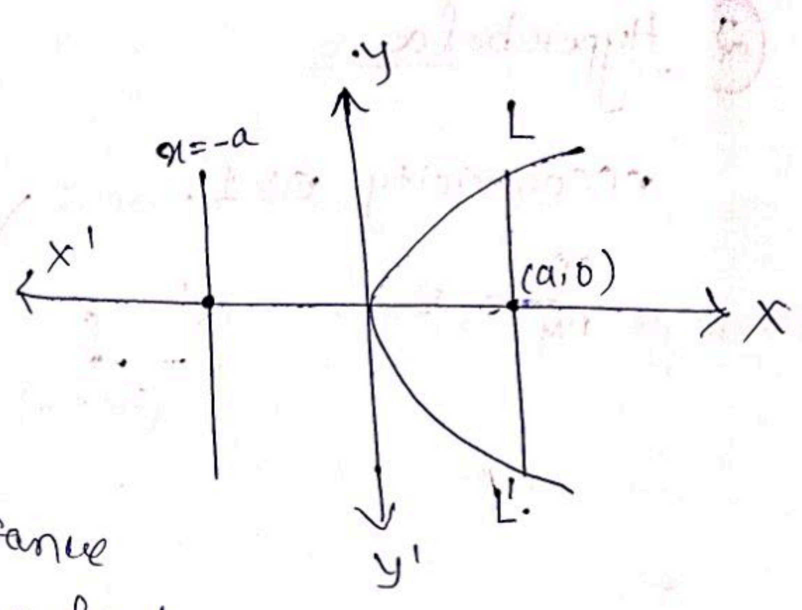
$$\Rightarrow \boxed{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$$

If $e > 1$, the conic is hyperbola

$e = 1$, conic is parabola

$e < 1$, conic is Ellipse.

(1) Parabola:




A parabola is a structure which is the locus of a point such as its distance from a fixed point is always equal to its distance from a fixed ~~point~~ straight line.

equation is $\boxed{y^2 = 4ax}$

Vertex = (0,0) ; Focus (a,0)

Axis is $y=0$ , Directrix $x+a=0$ ($x=-a$)

Important terms

1> Focal distance : The length of a point on the parabola from its focus is called the focal distance of the point.

2> Focal chord : A chord of the parabola which moves through its focus is called a focal chord.

3> Latus rectum : The chord ~~of the~~ passes through focus and perpendicular to the directrix called axis of the parabola is called latus rectum (LR).

for $y^2 = 4ax$ (parabola)

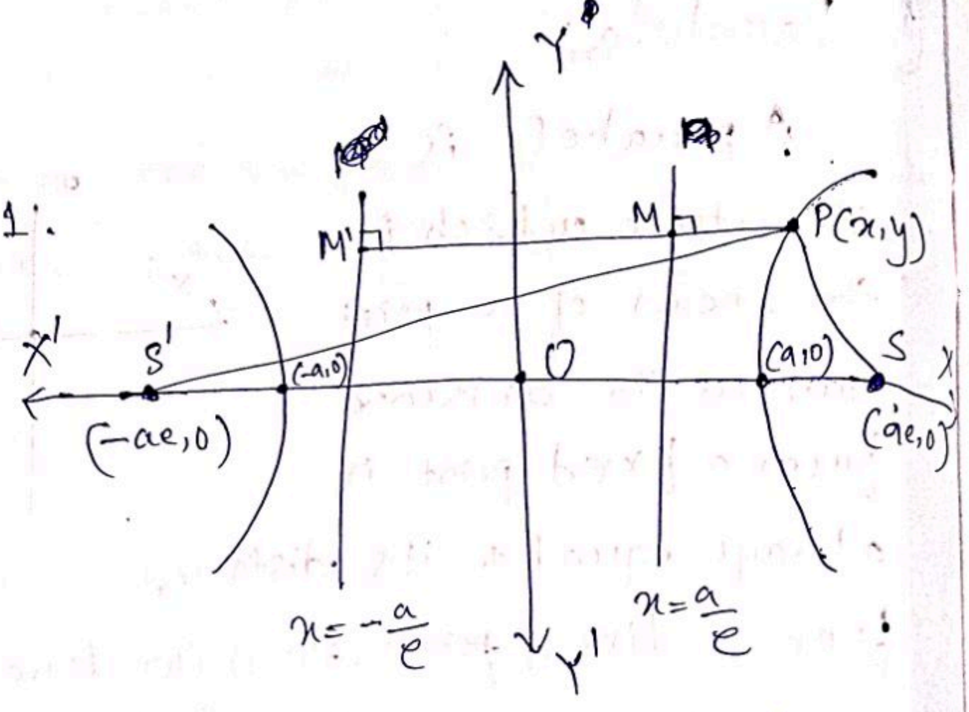
LR length = 4a

End points of LR = $(a, 2a)$ & $(a, -2a)$

## (2) Hyperbola.

eccentricity $e > 1$.

$$\frac{SP}{PM} = e$$




The locus of a point which moves such that its length from a fixed point (focus) is 'e' times its length from a fixed straight line is defiend as hyperbola.

Focus $(ae, 0)$

directrix $x = \frac{a}{e}$

Standard equation of hyperbola

$$\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 \times PM^2$$

$$\Rightarrow (x-ae)^2 + (y-0)^2 = e^2\left[x - \left(\frac{a}{e}\right)\right]^2$$

$$\Rightarrow (x-ae)^2 + y^2 = (ex-a)^2$$

$$\Rightarrow x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{or} \quad \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

where $b^2 = a^2(e^2-1)$, since $e > 1$, $b^2 > 0$

1) only even powers of $x$ & $y$, so hyperbola is symmetrical about both axes.

2) This hyperbola do not cut y-axis, whereas it cuts x-axis at $(a,0)$ & $(-a,0)$

3) for $-a \le x \le a$, the curve does not exist

4) $x$ increases then $y$ also increases. i.e curve extends to infinity.

5) 'c' length of foci from centre then

$\boxed{c^2 = a^2 + b^2}$ for hyperbola and $\boxed{e = \frac{c}{a}}$

## Important Terms

1) Foci and Directrices : Since the curve is symmetrical about y-axis, So there exist another focus $S'(-ae, 0)$. So the foci are $(-ae, 0)$, $(ae, 0)$ and directrices $x = \frac{a}{e}$, $x = -\frac{a}{e}$.

2) Centre : Any chord of the hyperbola through 'O', the mid point of $AA'$ will bisected at 'O', so 'O' is called centre.

## (3) Ellipse :-

Standard equation of ellipse referred to its principal axes along the coordinate axes is :

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

where $a > b$, $b^2 = a^2(1 - e^2)$

eccentricity $(0 < e < 1)$

Foci $= (ae, 0), (-ae, 0)$

Vertices : $(-a, 0), (a, 0)$

directrices : $\frac{a}{e}, -\frac{a}{e}$

Major axis :

The line in which both foci lies is of length $2a$ and is called major axis. $(a > b)$ of the Ellipse.

Minor axis :

The line perpendicular to Major axis ~~along~~ and intersect at $(0, b)$, $(0, -b)$ is Minor axis of length $2b$.

Principal axes : The major and minor axes together are called principal axes of the Ellipse.

Centre: The point which ~~by~~ bisects every chord of the conic drawn through it. The origin is the centre of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

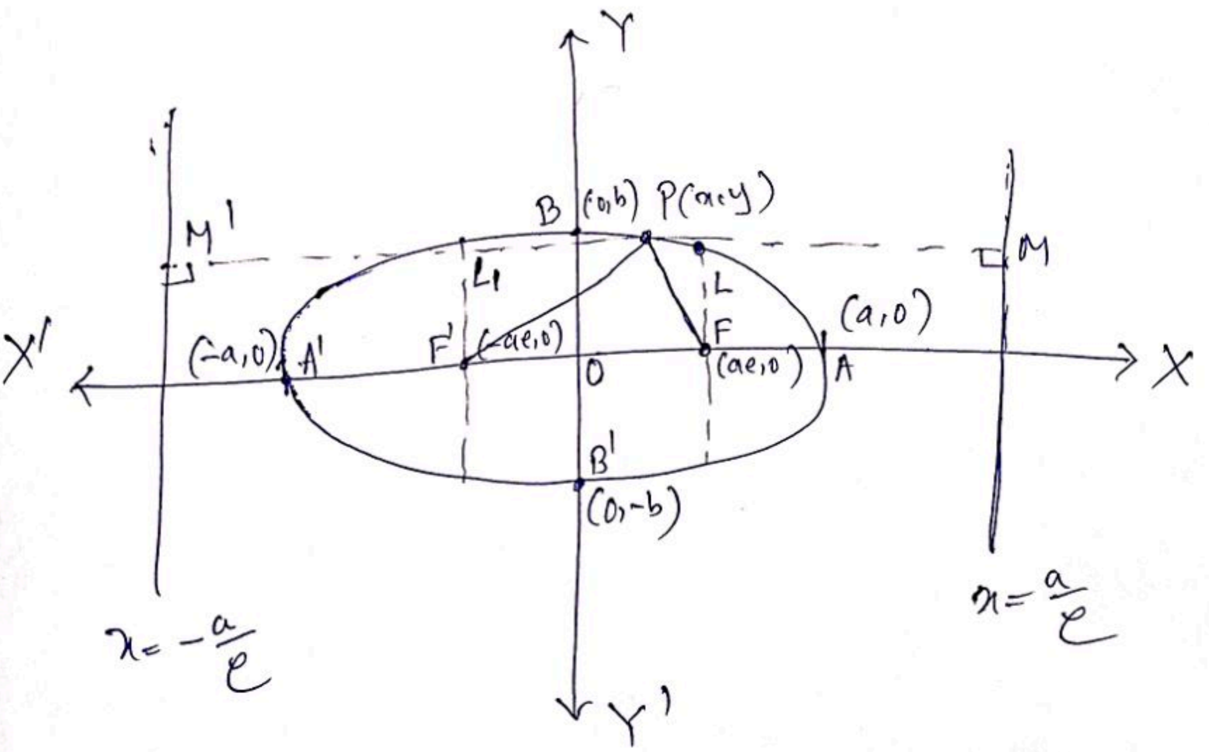
Latus Rectum: The focal chord perpendicular to the major and minor axis is called the latus rectum (LR)

length of LR $= \frac{2b^2}{a}$, equation of LR $x = \pm ae$

Focal Radii: $SP = a - ex$ & $S'P = a + ex$

$SP + SP' = 2a =$ Major axis

Eccentricity $\boxed{(e) = \sqrt{1 - \frac{b^2}{a^2}}}$



# Unit-4

## Vector Algebra

1) Vector Quantities: They have both magnitude and direction.

   ex displacement, velocity, weight, force, angular velocity, moment etc.

2) Scalar Quantities: They have only magnitude.

   ex Mass, Volume, Work, temperature etc.

### Representation of a Vector:

A directed line segment is called vector, with intial point (A) & terminal point (B). is given by $\overrightarrow{AB}$.

1) Length: The length of $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$

2) Support: The line of unlimited length of which $\overrightarrow{AB}$ vector is a part is called its support.

3) Sense: It is from initial pt to terminal point.

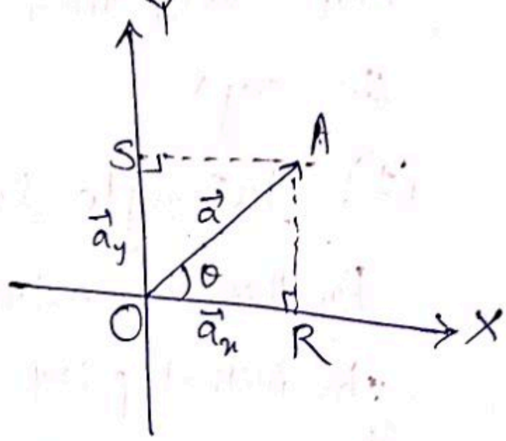
$$\overrightarrow{AB} = \underset{\text{(initial point)}}{A} \longrightarrow \underset{\text{(terminal point)}}{B}$$

### Rectangular Resolution of a vector:

The effect of a vector in a particular direction and the split vectors so obtained are called as components.

If these components are ~~gr~~ perpendicular to each other they are called rectangular components.

The rectangular resolution of a vector $\vec{a}$ given by $\vec{OA}$.




$$\cos\theta = \frac{OR}{OA}$$

$$\Rightarrow \quad \vec{a}_x = \vec{a}\cos\theta$$

Similarly, $\vec{a}_y = \vec{a}\sin\theta$

$\vec{a}$ vector have resolved into two rectangular components $\vec{a}_x$ & $\vec{a}_y$ along X & Y axes respectively.

$$\boxed{|\vec{a}| = a_x^2 + a_y^2}$$

> Note: $\hat{i}$ & $\hat{j}$ are denoted as vectors of unit magnitudes along OX & OY axes respectively.
>
> So that, $\vec{a}_x = a\cos\theta\,\hat{i}$ , $\vec{a}_y = a\sin\theta\,\hat{j}$
>
> Similarly, $\hat{k}$ is for OZ axis.

$\therefore \vec{a} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}$

<u>Unit vector</u>: It is a vector in the direction of given vector $\vec{a}$ given by

$$\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$$ It has length 1.

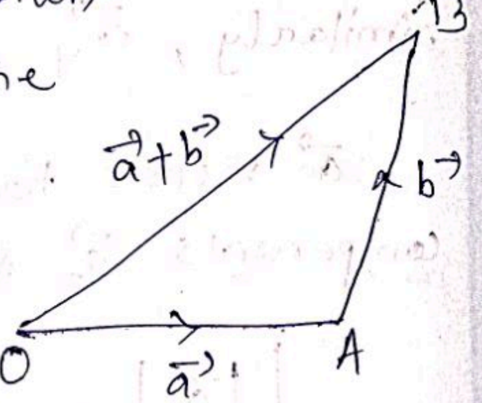
## Algebra of Vectors :-

(a) Addition of two vectors : If $\vec{a}$ & $\vec{b}$ are two vectors in plane represented by $\vec{OA}$ and $\vec{AB}$ respectively then their addition can be done in 2 ways.

### (1) Triangle law of Vector Addition :

By this law addition can be done when the initial point $\vec{b}$ coincides with the terminal point of $\vec{a}$.

The vector joining the initial point $\vec{a}$ with the terminal point of $\vec{b}$ is the vector sum of $\vec{a}$ & $\vec{b}$.




$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{OB}$$

Ex $\vec{a} = a_1\hat{i} + a_2\hat{j}$ , $\vec{b} = b_1\hat{i} + b_2\hat{j}$

$$\vec{a} + \vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j}$$

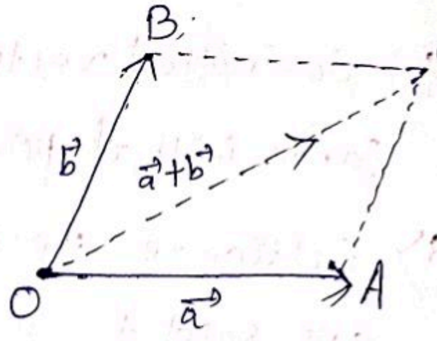
### (2) Parallelogram law of Vector addition :

By this law, we draw $\vec{a}$ & $\vec{b}$ with both the initial points coinciding. Then we consider these two vectors as adjacent sides of a parallelogram. We complete the drawing of parallelogram.

The diagonal so obtained from this parallelogram through the common initial point gives the sum of two vectors shown in figure.

$$\vec{OA} + \vec{OB} = \vec{OC}$$

$$\vec{a} + \vec{b} = \vec{OC}$$




## Properties of Vector Addition:

1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)

2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative)

3) $\vec{a} + \vec{0} = \vec{a} + \vec{0} = \vec{a}$ ( Additive identity)

4) $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$ (additive inverse)

## ⓑ Multiplication of a vector by a scalar:

If $\vec{a} = a_1\hat{i} + a_2\hat{j}$ then

$$m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j}$$ , where $m$ = scalar.

## ⓒ Subtraction:

If $\vec{a} = a_1\hat{i} + a_2\hat{j}$ , $\vec{b} = b_1\hat{i} + b_2\hat{j}$ then

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j}$$

Note: for a vector in 3D , $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors parallel to $x, y, z$ axes respectively. $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

## Types of Vectors :

1) Zero | Null vector : A vector whose magnitude is zero is called a zero | null vector and is given by $\vec{0}$.

2) Co-initial vectors : Two or more vectors having the same initial point are called co-initial vectors.

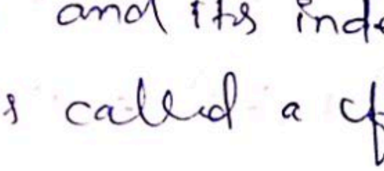
3) Collinear vectors : Two vectors or more vectors are said to be collinear if they are parallel to the same line ~~irrespecti~~ irrespective of their magnitude & directions.

4) Free Vectors : If the value of a vector depends only on its length and direction and its independent of its position in the space, it is called a free vector.

5. Coterminous Vectors : Vectors having the same terminal points are called coterminous vectors.

## Position Vector :-

It is a vector from the origin for any vector $\vec{a}$. given by



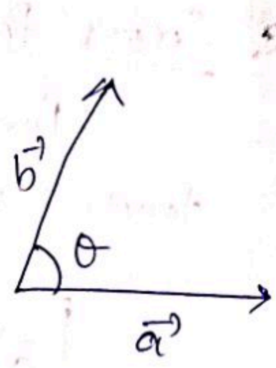

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

# Product of Two Vectors:

## 1. Dot product / Scalar product:

This product of two vectors $\vec{a}$ & $\vec{b}$ is given by $\vec{a}\cdot\vec{b}$




i.e $\vec{a}\cdot\vec{b} = |a||b|\cos\theta = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

### Properties

1) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

2) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)

3) angle between $\vec{a}$ & $\vec{b}$ is $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

### Application of Dot Product: (work done)

A force acting on a particle is said to do work, if the particle is displaced in a direction which is not perpendicular to force, It is a scalar quantity.

work done $= \vec{F}\cdot\vec{a} = Fa\cos\theta$

$=$ Force x displacement along force (dirn)

2. <u>Cross Product / Vector Product :</u>

The cross product of two vectors $\vec{a}$ & $\vec{b}$ denoted by $\vec{a} \times \vec{b}$. & defined by

$$\boxed{\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \; \hat{n}}$$

$\hat{n}$ = unit vector perpendicular to both $\vec{a}$ & $\vec{b}$

for $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Properties :</u>

1) $\vec{a} \times \vec{b} = 0$ iff $\theta = 0$ ($\vec{a}$ & $\vec{b}$ parallel).

2) If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \,\hat{n}$

3) In vector product angle is $\theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}\right)$

4) $\boxed{\text{Vector product is not commutative}}$

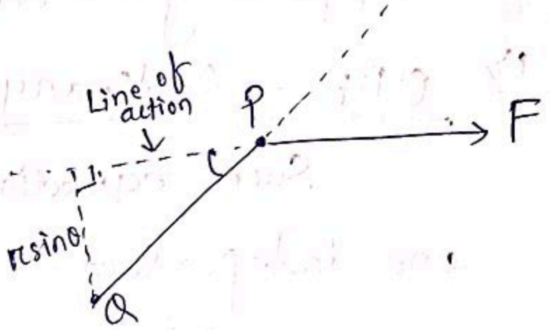
5) If $\vec{a}$ & $\vec{b}$ represent adjacent sides of a triangle then its area $= \frac{1}{2} |\vec{a} \times \vec{b}|$

6) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a parallelogram then its area is $|\vec{a} \times \vec{b}|$

<u>Moment of force ($\vec{M}$)</u> : (Application of vector product)

It is given by

$$\vec{M} = \vec{r} \times \vec{F} = \vec{\tau}$$




where $\vec{F}$ = force applied

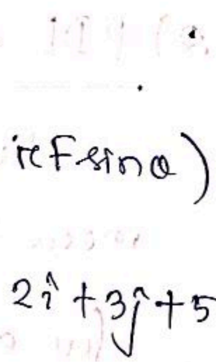
P = point of application of force

Q = point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point $(|\vec{\tau}| = rF\sin\theta)$

Q Find the moment about (1, 0, 1) of the force $2\hat{i}+3\hat{j}+5\hat{k}$ acting at (2, 1, −1).

Sol$^n$ Let $O = (1, 0, 1)$, $A = (2, 1, -1)$, $\vec{F} = 2\hat{i}+3\hat{j}+5\hat{k}$




Moment about O is $\vec{OA} \times \vec{F}$

here $\vec{OA} = (2\hat{i}+\hat{j}-\hat{k}) - (\hat{i}+\hat{k})$

$= \hat{i}+\hat{j}-2\hat{k}$

$$\vec{OA} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & 3 & 5 \end{vmatrix} = \hat{i}(5+6) - \hat{j}(5+4) + k(3-2)$$

$$= 11\hat{i} - 9\hat{j} + \hat{k}.$$

## Differential Equation :

A DE is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

**NOTE:**

Derivative indicates a change in a dependent variable with respect to an independent variable.

Ex

$$\frac{dy}{dt} + 5y = 3t$$

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 3y = 0$$

Here $y \rightarrow$ dependent variable

$x, t \rightarrow$ independent variable

### Types of DE

1. Ordinary differential Equation
2. Partial differential Equation

### 1 ODE :

This equation involves only one independent variable.

(2) PDE:

This equation involves more than one independent variable.

(1) $\frac{dy}{dx} + xy = x^2$

(2) $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$

(3) $\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 4$ , (4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(1) & (2) are ODE & (3) & (4) are PDE.

Order of ~~Linear~~ Differential Equation

A DE is linear if

(a) Every dependent variable.

Order of DE:

It is the order of highest derivative in the equation.

Degree of DE:

It is the power of the highest order derivative in the equation (which is a positive integer value).

ex $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 + 4y = 0$

order = 2, degree = 1

(2) $\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 + 2y = x^2$

order = 2

degree = 1

(3) $\left(1 + \frac{d^2y}{dx^2}\right)^{3/2} = a\frac{d^2y}{dx^2}$

Square both sides

$\left(1 + \frac{d^2y}{dx^2}\right)^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

order = 2 , degree = 3

## Linear Differential Equation

A DE is called linear if

(a) every dependent variable and derivatives are of first order only

(b) Products of derivatives and/or dependent variables do not occur.

Otherwise it is called Non-linear DE.

Ex $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^2$ ( Non-linear )

$\frac{d^2y}{dx^2} + y^2 = 0$ (Non linear) ~~product~~ degree of DV is 2

$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$ (Non-linear) ~~product~~ degree of $\frac{dy}{dx}$ is 3

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} \quad (\text{linear})$$

$$\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + xy = 0 \quad (\text{linear})$$

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = xy \quad (\text{linear})$$

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} = y\cos x \quad (\text{linear})$$

$$x\frac{dy}{dx} + 2y = (x^2 - x + 1) \quad (\text{linear})$$