Vikash Polytechnic, Bargarh

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Lecture Note on *Mathematics-III*

Diploma 3rd Semester

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def^n A number of the form $z = x + iy$ is called a complex number, where x, y are real numbers & $i = \sqrt{-1}$ is iota.

also, x = real part of z = $Re(z)$

y = Imaginary part of z = $Im(z)$

Set of Complex Numbers is denoted by $\mathbb{C}$

Complex Conjugate :-

A pair of complex numbers $z = x + iy$ and $x - iy$ are said to be conjugate of each other denoted by $\bar{z}$.

or If $z = x + iy$ then it's complex conjugate is given by $\bar{z} = x - iy$.

Also, if $z = x - iy$ then $\bar{z} = x + iy$

Ex

Properties

If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$

1) Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff

$$Re(z_1) = Re(z_2) \quad \text{i.e } x_1 = x_2$$

$$\text{and } Im(z_1) = Im(z_2) \quad \text{i.e } y_1 = y_2$$

2) Sum, difference, product, quotient of any two complex numbers is itself a complex number.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two given complex numbers then

(i) $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + (y_1 + y_2)i$

(ii) $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(iii) $z_1 . z_2 = (x_1 + iy_1).(x_2 + iy_2) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i + i^2 y_1 y_2$

$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$

(iv) $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \cancel{x_1 x_2 + y_1 y_2}$

$= \frac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

Every complex number $z = x + iy$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$

Put $Re(z) = x = r\cos\theta$ and ——— (i)

$Im(z) = y = r\sin\theta$ ——— (ii)

Squaring and adding. $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$ ——— (iii)

Now, dividing (ii) by (i) $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\Rightarrow \tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Thus, $z = x + iy = r(\cos\theta + i\sin\theta)$ also called Polar form

where $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Modulus of z

The number $r = \sqrt{x^2+y^2}$ is called the modulus of z. i.e of $x+iy$. It is denoted by $|z|$

$$\Rightarrow \quad |z| = \sqrt{x^2+y^2} \quad \text{for} \quad z = x+iy$$

Argument / Amplitude : (arg z)

The angle θ is called the amplitude or argument of z. It is denoted by $\text{amp}(z)$ or $\arg(z)$.

i.e $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Principal Value of Amplitude : (Arg z)

The value of θ which lies between $-\pi$ to π is called the principal value of the amplitude. (i.e $-\pi < \theta \leq \pi$)

NOTE : $\cos\theta + i\sin\theta$ is briefly written as $\text{cis}\,\theta$ (pronounced as $\text{sis}\,\theta$)

Properties :

If $\bar{z}$ is the conjugate of $z = x+iy$ then

(i) $Re(z) = \frac{1}{2}(z+\bar{z})$, $Im(z) = \frac{1}{2i}(z-\bar{z})$

(ii) $|z| = \sqrt{\{Re(z)\}^2 + \{Im(z)\}^2} = |\bar{z}|$

(iii) $z \cdot \bar{z} = |z|^2$

(iv) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $\bar{z}_2 \neq 0$

Examples:

Q1 ~~find~~ Check all algebraic properties with $z_1 = 2+3i$ and $z_2 = 1+i$.

$$z_1 + z_2 = (2+3i) + (1+i) = (2+1) + (3i+i) = 3+4i$$

$$z_1 - z_2 = (2+3i) - (1+i) = 2+3i-1-i = (2-1)+(3i-i) = 1+2i$$

$$z_1 \cdot z_2 = (2+3i)(1+i) = 2+2i+3i+3i^2 = 2+5i-3 = -2+5i$$

$$\frac{z_1}{z_2} = \frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i-3i^2}{1-i^2}$$

$$= \frac{2+i+3}{1-(-1)} = \frac{5+i}{2}$$

Equality: $3a + 5bi = 6 + 15i$

$3a = 6$, $5b = 15$

$a = 2$, $b = 3$

Q2 find conjugates: of $6+7i$, $5-4i$, $3-i$, 7, $-6i$

(i) $6-7i$, (ii) $5+4i$, (iii) $3+i$, (iv) 7 (v) $6i$

$z = -3 + ix^2y$, $\bar{z} = -3 - ix^2y$

Q1 Reduce $1 - \cos\alpha + i\sin\alpha$ to the modulus Amplitude form (Polar form).

here $z = 1 - \cos\alpha + i\sin\alpha$

$x = 1 - \cos\alpha$, $y = \sin\alpha$

we know, $r^2 = \cancel{\sqrt{x^2+y^2}} = (1-\cos\alpha)^2 + \sin^2\alpha$

$= 1+\cos^2\alpha - 2\cos\alpha + \sin^2\alpha$

$= \sin^2\alpha + \cos^2\alpha + 1 - 2\cos\alpha$

$= 1+1-2\cos\alpha$

$= 2-2\cos\alpha$

$= 2(1-\cos\alpha)$

$= 2.2\sin^2\frac{\alpha}{2} = 4\sin^2\frac{\alpha}{2}$

$\Rightarrow r^2 = 4\sin^2\frac{\alpha}{2} \Rightarrow r = 2\sin\frac{\alpha}{2}$ —— (1)

$\frac{1-\cos2\alpha}{2} = \sin^2\alpha$

$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$

Again, $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sin\alpha}{1-\cos\alpha}\right)$

$= \tan^{-1}\left(\frac{\cancel{2}.\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}}{\cancel{2}\sin^2\frac{\alpha}{2}}\right)$

$= \tan^{-1}\left(\cot\frac{\alpha}{2}\right)$

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}$

$\theta = \frac{\pi}{2}-\frac{\alpha}{2}$

(r,θ) form or modulus - Amplitude form is

$z = (1-\cos\alpha) + i\sin\alpha = r(\cos\theta + i\sin\theta)$

$= 2\sin\frac{\alpha}{2}\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]$

$=$

Q Find the complex number z if $\arg(z+1) = \frac{\pi}{6}$ and $\arg(z-1) = \frac{2\pi}{3}$

1) Let $z = a+ib$

$z+1 = (a+1)+ib$

$\arg(z+1) = \tan^{-1}\left(\frac{b}{a+1}\right)$ —— (1)

Again

$z-1 = (a-1)+ib$

$\arg(z-1) = \tan^{-1}\left(\frac{b}{a-1}\right)$ —— (2)

Given $arg(z+1) = \frac{\pi}{6}$ and $arg(z-1) = \frac{2\pi}{3}$

from ①: $\tan^{-1}\left(\frac{b}{a+1}\right) = \frac{\pi}{6}$ and $\tan^{-1}\left(\frac{b}{a-1}\right) = \frac{2\pi}{3}$ (from ②)

$\frac{b}{a+1} = \tan\frac{\pi}{6}$ and $\frac{b}{a-1} = \tan\left(\frac{2\pi}{3}\right)$

$\frac{b}{a+1} = \frac{1}{\sqrt{3}}$ and $\frac{b}{a-1} = -\sqrt{3}$

$\sqrt{3}b = a+1$ —— ③ $\qquad b = \sqrt{3} - \sqrt{3}a$ —— ④

Solving ③ & ④

$$a - \sqrt{3}b + 1 = 0 \quad \text{—— } \times \sqrt{3}$$

$$\sqrt{3}a + b - \sqrt{3} = 0$$

$$(-) \quad (-) \quad (+)$$

$$-4b + 2\sqrt{3} = 0$$

$$b = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$a = \sqrt{3}b - 1 = \sqrt{3}.\frac{\sqrt{3}}{2} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

So, $z = a + ib = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Q Find the real values of x, y so that $-3 + ix^2y$ and x^2+y+4i may represent complex conjugate numbers.

Sol^n Given both complex numbers represent complex conjugates.

If $z = -3 + x^2yi$ —— ①, then $\bar{z} = x^2+y+4i$ —— ②

but $\bar{z}$ of eqn ① is $-3 - x^2yi$ —— ③

then eqn ② and eqn ③ are equal.

$$-3 - x^2 y i = (x^2 + y) + 4i$$

$\Rightarrow \quad x^2 + y = -3$ and $-x^2 y = 4$

$x^2 = \frac{-4}{y}$ put

$$\frac{-4}{y} + y = -3$$

$-4 + y^2 = -3y \Rightarrow y^2 + 3y - 4 = 0$

$\Rightarrow (y+4)(y-1) = 0$

$y = 1, -4$

$y = 1, \; x^2 = -4 \Rightarrow x = \pm 2i$ (not feasible)

$y = -4, \; x^2 = 1 \Rightarrow x = \pm 1$

Geometrical Representation :-

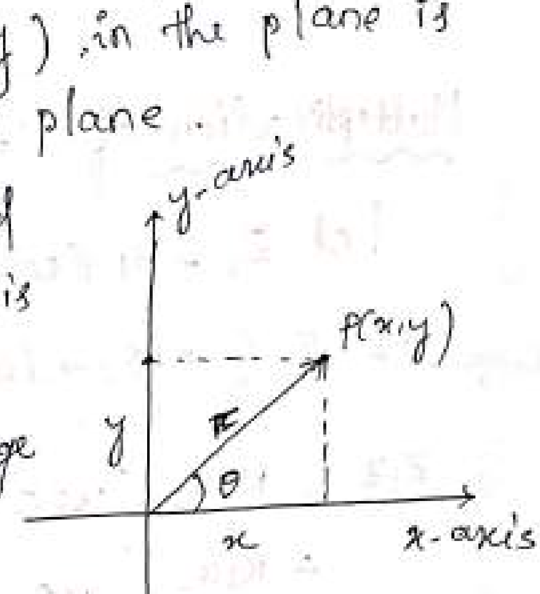
We know that complex numbers can be written as

$$z = x + iy, \; x, y \in \mathbb{R}, \text{ and } i = \sqrt{-1}$$

A complex number can be represented by a point in the two dimensional plane. We represent the complex number $x + iy$ by the point $P(x,y)$. in the plane is called complex plane or Argand plane.

The horizontal axis is called Real axis and the vertical axis is called Imaginary axis.

The point $P(x,y)$ is called image of complex number z.



Geometrically, z is identified as a vector. The directed line segment $\vec{OP}$ joining the origin to the point $P(x,y)$, where r is the length of the vector (i.e z) with angle θ forms polar coordinates (r,θ)

$$z = x+iy$$
$$= r(\cos\theta + i\sin\theta)$$

Polar Argument of a Point :-

If $\underset{= P(x,y)}{z = x+iy} = r(\cos\theta + i\sin\theta)$ then $r = \sqrt{x^2+y^2}$

and $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$, Here Let $\theta = \operatorname{Arg} z$

1. If $P(x,y)$ lies in the first quadrant then $\theta = \tan^{-1}\left|\frac{y}{x}\right|$
2. If $P(x,y)$ lies in the 2^{nd} quadrant then $\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$
3. If $P(x,y)$ lies in the 3^{rd} quadrant then $\theta = \tan^{-1}\left|\frac{y}{x}\right| - \pi$
4. If $P(x,y)$ lies in 4^{th} quadrant then $\theta = -\tan^{-1}\left|\frac{y}{x}\right|$

Multiplication of Complex No.s in Polar form :

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ with $\arg(z_1) = \theta_1$, $|z_1| = r_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ with $\arg(z_2) = \theta_2$, $|z_2| = r_2$

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 \left[\cos\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \sin\theta_2\, i + \sin\theta_1 \cdot \cos\theta_2\, i - \sin\theta_1 \sin\theta_2\right]$$

$= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \right]$

$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ having angle $\theta_1+\theta_2$

1) $\arg(z_1 z_2) = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

2) $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

3) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

$= \theta_1 - \theta_2$, $z_2 \neq 0$

4) $\arg z + \arg \bar{z} = 2n\pi$

5) $\arg z - \arg(-z) = \pi$

Q. Express $\frac{2+3i}{5-2i}$ in the $x+iy$ form

Ans $\frac{2+3i}{5-2i} = \frac{(2+3i)(5+2i)}{(5-2i)(5+2i)} = \frac{10+4i+15i-6}{25+4} = \frac{4+19i}{29}$

Q. Reduce z to Amplitude Modulus | Polar form

$z_1 = -\sqrt{3}-i$, $z_2 = -\sqrt{3}+i$, $z_3 = \sqrt{3}-i$.

Soln $|z_1| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2 = |z_2| = |z_3|$

$\text{Arg}(z_1) = \theta - \pi = \tan^{-1}\left|\frac{y}{x}\right| - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$ (3rd quadr.

$\text{Arg}(z_2) = \pi - \theta = \pi - \tan^{-1}\left|\frac{y}{x}\right| = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\text{Arg}(z_3) = -\theta = -\tan^{-1}\left|\frac{y}{x}\right| = -\frac{\pi}{6}$

Cube Root of Unity

Obtain properties of the cube roots of unity.

i.e $x^3 = 1 \Rightarrow x = (1)^{\frac{1}{3}}$

$Z = (1)^{1/3}$ $\qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Sol^n Since $x^3 = 1$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow x = 1$ or $x^2+x+1 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1-4.1.1}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

i.e $\alpha = \frac{-1+\sqrt{3}i}{2}$; $\beta = \frac{-1-\sqrt{3}i}{2}$

Hence cube roots are $1, \alpha, \beta$.

Properties

(i) $\alpha = \overline{\beta}$, $\beta = \overline{\alpha}$

The two complex roots of the equation are conjugate of each other.

(ii) $\alpha = \beta^2$, $\beta = \alpha^2$

The square of any one complex root is the other complex root.

(iii) If 'ω' is one complex root ; then other complex root is 'ω^2'.

Hence 3 cube roots of 1 are $1, \omega, \omega^2$.

(iv) Since ω is root of $x^3 = 1$

So, $\boxed{\omega^3 = 1}$

(v) Also, ω is root of the equation $x^2+x+1=0$ so

$\boxed{\omega^2+\omega+1=0}$

The sum of the three cube roots of unity vanishes.

De-Moivre's Theorem :- (For Integral Index)

Statement :

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for any integer n.

Pf Case-1 (n = +ve integer)

We prove by induction :

Let the statement is $p(n): (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$p(1): (\cos\theta + i\sin\theta)^1 = $ LHS

$\cos\theta + i\sin\theta = $ RHS.

$\therefore$ P(1) is true.

Suppose that P(K) is true.

i.e $P(K): (\cos\theta + i\sin\theta)^K = \cos K\theta + i\sin K\theta$ —— (1)

to prove : $P(K+1)$ is true.

Let $P(K+1): (\cos\theta + i\sin\theta)^{K+1}$

$= (\cos\theta + i\sin\theta)^K (\cos\theta + i\sin\theta)$

$= (\cos K\theta + i\sin K\theta)(\cos\theta + i\sin\theta)$

$= \cos K\theta.\cos\theta + \cos K\theta.\sin\theta\, i + \sin K\theta.\cos\theta\, i - \sin K\theta.\sin\theta$

$= \cos K\theta\cos\theta - \sin K\theta.\sin\theta + i(\sin K\theta\cos\theta + \cos K\theta\sin\theta)$

$= \cos(K\theta+\theta) + i\sin(K\theta+\theta) = \cos(K+1)\theta + i\sin(K+1)\theta$

$=$

Hence by Mathematical Induction $p(k+1)$ is true.

$\therefore$ $p(n)$ is true. (proved).

Case-II (If $n = -ve$ integer)

Let $n = -m$. (where $m = +ve$ integer)

observe that $\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$ ——— (1)

$\frac{1}{3-5i} = \frac{(3+5i)}{34}$

Now, $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)} \quad \text{(by case-1)}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos n\theta + i\sin n\theta$$

$\cos(-m\theta) = \cos m\theta$

$\cos(-m)\theta = \cos n\theta$

$\sin(-m)\theta = -\sin m\theta$

$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for any integer n.

De-Moivre's Theorem (for Rational Index)

Statement:

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

i.e $n = \frac{p}{q}$ where n is +ve or negative integer/fraction.

or

Pf We know, $(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$

$$= \cos\left(q \cdot \frac{1}{q} p\theta\right) + i\sin\left(q \cdot \frac{1}{q} p\theta\right)$$

$$= \cos\left(q \cdot \frac{p\theta}{q}\right) + i\sin\left(q \cdot \frac{p\theta}{q}\right)$$

$$\Rightarrow (\cos\theta + i\sin\theta)^p = \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)^q$$

$$\Rightarrow (\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)^{q/q}$$

$$= \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}.$$

$\Rightarrow$ Due to periodicity of $\sin\theta$ & $\cos\theta$

we take $\theta = 2k\pi + \theta$, $k = 1, 2, \dots, (q-1)$

$$\Rightarrow (\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}(2k\pi+\theta) + i\sin\frac{p}{q}(2k\pi+\theta)$$

take $n = \frac{p}{q}$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos\{n(2k\pi+\theta)\} + i\sin\{n(2k\pi+\theta)\}$$

Leave

General solution of $x^n = 1$, $n = \pm$ve integer

$\boxed{x^n = 1 \Rightarrow x = 1^{1/n}}$ $\cos 2k\pi + i\sin 2k\pi = 1 = \cos 0 + i\sin 0$

$$1 = r(\cos 2k\pi + i\sin 2k\pi)$$

$\cos 0 = \cos(2k\pi + 0)$
$= \cos 2k\pi$

$$x = 1^{1/n}$$

$(r = 1)$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$

Roots of a CN $x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ $k = 0, 1, 2, \dots, (n-1)$

If $x^n = a$, then $a = r(\cos\theta + i\sin\theta)$

$z = (\cos\theta + i\sin\theta) r$
$z^{1/n} = (\cos\theta + i\sin\theta)^{1/n} r^{1/n}$

$$x^n = r(\cos\theta + i\sin\theta) = r\{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}$$

$$x = r^{1/n}(\cos(\theta + 2k\pi) + i\sin(\theta+2k\pi))^{1/n}$$

$$z = x = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

$$z = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right],$$

$n = 0, 1, 2, \dots, (n-1)$

Ex Obtain the square roots of $3+4i$.

Sol Let $x, y \in \mathbb{R}$ such that

$$z = x+iy = \sqrt{3+4i}$$

$$\Rightarrow (x+iy)^2 = 3+4i$$

$$\Rightarrow x^2+i^2y^2+2xyi = 3+4i$$

$$\Rightarrow x^2-y^2+2xyi = 3+4i$$

Equating the corresponding parts,

$$x^2-y^2 = 3 \quad \text{--- (1)}$$

$$2xy = 4 \quad \text{--- (2)}$$

We know that, from the formula $\left[\because (a+b)^2 = (a-b)^2+4ab\right]$

$$(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$$

$$= 3^2+4^2 = 9+16 = 25$$

$$x^2+y^2 = \sqrt{25} = 5$$

$$\Rightarrow x^2+y^2 = 5 \quad \text{--- (3)}$$

from (1) & (3)

$$\begin{aligned} x^2-\cancel{y^2} &= 3 \\ + \; x^2+\cancel{y^2} &= 5 \\ \hline 2x^2 &= 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \end{aligned}$$

$$y^2 = 5-x^2 = 5-4 = 1$$

$$\Rightarrow y^2 = \pm 1$$

from $x = \pm 2$, $y = \pm 1$ we have to choose correct values from equatio(2) Here the product xy is positive.

So, both x, y must have the same sign.

Thus if $x = 2$, $y = 1$ and if $x = -2$; $y = -1$

Hence the square roots of $3+4i$ is $2+i$ and $-2-i$.

Prove: $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

Pf $\Rightarrow (1+\omega^2-\omega)^5 + (1+\omega-\omega^2)^5$

$\Rightarrow (-\omega-\omega)^5 + (-\omega^2-\omega^2)^5$

$\Rightarrow (-2\omega)^5 + (-2\omega^2)^5$

$\Rightarrow (-2)^5 \omega^5 + (-2)^5 (\omega^2)^5$

$\Rightarrow -32\,\omega^3.\omega^2 - 32\,\omega^9.\omega$

$\Rightarrow -32(\omega^2+\omega)$

$\Rightarrow -32(-1) = 32$

$$\left\{\begin{aligned} 1 &= \cos 0 + i\sin 0 \\ -1 &= \cos\pi + i\sin\pi \\ i &= \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} \\ -i &= \cos\tfrac{3\pi}{2} - i\sin\tfrac{3\pi}{2} \end{aligned}\right.$$

Q $(1+\omega^2)^4 = \omega$

(ii) $(1-\omega+\omega^2)(1+\omega-\omega^2) = 4$

(iii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$

(iv) $(2+5\omega+2\omega^2)^6 = (2+2\omega+5\omega^2)^6 = 729$

(v) $(1-\omega+\omega^2)(1-\omega+\omega^4)(1-\omega^4+\omega^2)\ldots$ to $2n$ factors $= 2^{2n}$

Q Find square roots of (Find)

(i) $-5+12\sqrt{-1}$

(ii) $-8+\sqrt{-1}$

(iii) $a^2-1+2a\sqrt{-1}$

(iv) $4ab-2(a^2-b^2)\sqrt{-1}$

$-8+\sqrt{-1} = -8+i$

Let $z = x+iy = \sqrt{-8+i}$

$(x+iy)^2 = -8+i$

$\Rightarrow x^2+i^2y^2+2xyi = -8+i$

$\Rightarrow x^2-y^2+2xyi = -8+i$

$x^2-y^2 = -8$ ——— ①

$2xy = 1$ ——— ②

we know

$(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$

$= (-8)^2+1^2 = 64+1 = 65$

$(x^2+y^2)^2 = 65 \Rightarrow x^2+y^2 = \sqrt{65}$ ——— ③

solving ① & ③

$$\begin{array}{r} x^2-y^2 = -8 \\ + \quad x^2+y^2 = \sqrt{65} \\ \hline 2x^2 = \sqrt{65}-8 \end{array}$$

$x^2 = \frac{1}{2}(\sqrt{65}-8)$

$x = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{65}-8}$

$y^2 = \sqrt{65}-x^2$

$= \sqrt{65}-\frac{1}{2}(\sqrt{65}-8) = \frac{2\sqrt{65}-\sqrt{65}+8}{2}$

$= \frac{\sqrt{65}+8}{2}$

$y = \pm\left(\frac{\sqrt{65}+8}{2}\right)^{1/2}$

By eqn (2) x & y both have same sign.

$$x = \left(\frac{\sqrt{65}-8}{2}\right)^{1/2} \; \& \; y = \left(\frac{\sqrt{65}+8}{2}\right)^{1/2}$$

and

$$x = -\left(\frac{\sqrt{65}-8}{2}\right)^{1/2}, \; y = -\left(\frac{\sqrt{65}+8}{2}\right)^{1/2}$$

---

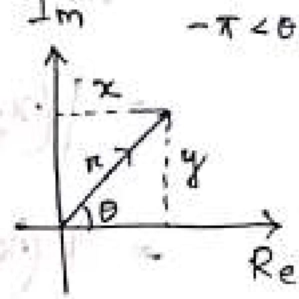
we know $z = x + iy$

$$= r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

$r = |z|$

$-\pi < \theta \le \pi$



Prove $\cos\theta + i\sin\theta = e^{i\theta}$

Pf we know that

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{—— (3)}$$

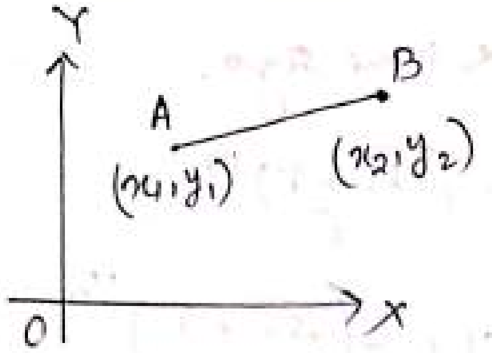
Now, put $x = i\theta$ in eqn (3)

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3 \cdot \theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{\theta^5 i^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3 i}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!}i + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta.$$



cartesian plane/ coordinate plane

We know distance formula

$$|AB| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= |(x_2-x_1) + i(y_2-y_1)|$$

$$= |(x_2+y_2 i) - (x_1+y_1 i)|$$

also know that

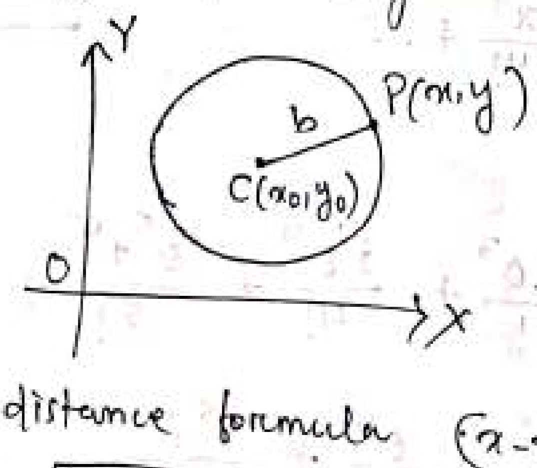
$$z = x+iy$$

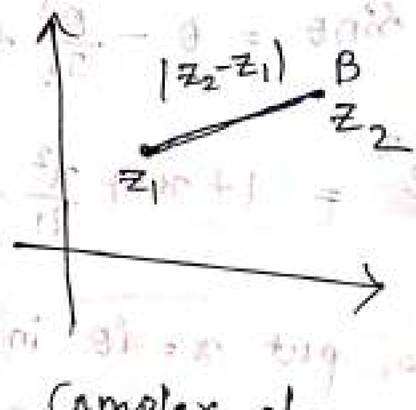
$$|z| = \sqrt{x^2+y^2}$$

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$\Rightarrow |AB| = |z_2 - z_1|$

Let take another diagram





Complex plane

By distance formula $(x-x_0)^2 + (y-y_0)^2 = b^2$

or $\sqrt{(x-x_0)^2 + (y-y_0)^2} = b$

Similarly, $|(x+iy) - (x_0+iy_0)| = b$

take $x+iy = z$, $x_0+iy_0 = z_0$

(Module) Matrix

① **Matrix** (definition)

A matrix is a rectangular array of numbers (any system of numbers), arranged in rows and columns.

If there m rows & n columns in a matrix then it is called as 'm by n' ~~or~~ matrix or a matrix of order $m \times n$.

Matrix is given by $A = [a_{ij}]_{m \times n}$

i.e

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Ex $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ unit matrix.

If $m = n$ then it is a square matrix.

Linear Equation

An expression of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, a_2, \ldots, a_n$ are constants.

Ex 1) A line of the form ~~$Ax+B$~~ $y = mx + c$ is a linear equation

2) $x_1 + 2x_2 - 4x_3 + x_4 = 3$ is linear

3) $x_1 + 3x_2 - x_3 = \sqrt{2}$ is linear

4) $x_1 + x_2 + 2x_1x_2 = 1$ is not linear.

System of linear Equation

A system of linear equations is a collection of one or more linear equations involving the same variables. this is called as a linear system of equations.

A system of m linear equations with n unknown variables $x_1, x_2, \cdots, x_n$ is of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - - - - + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + - - - - - - + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + - - - - - - + a_{mn}x_n &= b_m \end{aligned}\right\} \text{---} (1)$$

where a_{ij}, b_j are constants.

Example

$$2x_1 + x_2 - 2x_3 = 10$$
$$3x_1 + 2x_2 + 2x_3 = 1$$
$$5x_1 + 4x_2 + 3x_3 = 4$$

this is the system of 3-linear equations with variables x_1, x_2, x_3.

Matrix form of linear Equations:

Equation (1) can be also written as.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e $AX = B$ is matrix form

where $\left.\begin{aligned} A &= \text{coefficient matrix} \\ X &= \text{Variable/var matrix} \end{aligned}\right\}$ $B =$

and $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$ is called augmented matrix of the system.

which of order $m \times (n+1)$.
definitions of homogeneous & Non-homogeneous LE and also consistent and inconsistent LE in Solution of LE.

(3) RANK OF A MATRIX

The rank of a matrix A is the number of non-zero rows in the row echlon form of A.

OR It is also defined as the number of non zero column in the column echlon form of the matrix.

The rank of matrix A is denoted by $\rho(A)$ or rank(A)
Rank can not be more than its number of rows & columns

(1) (2)

2.
$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \quad \begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix}$$

All arrows / numbers each are entries

(*) Pivot Entry / Leading Entry :- It is the first non-zero entry in ~~each row~~ row echlon form / Reduced Echlon form.

Row Echlon form : A rectangular matrix is in Echlon form if

1) All the zero rows of A are at the bottom

2) for non-zero row, The first non zero entry (after first row) is to right of the first non-zero entry in the preceeding row.

The posn of the leading entry in a row echlon / reduced row echlon form matrix are pivot positions. A non-zero entry in a pivot position is pivot

B

Examples :-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 & 0 & 4 \\ 0 & 0 & 7 & 1 & -3 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$ are in Echlon form.

Including the properties of Echlon form, Reduced Echlon form of a matrix A is defined if

3) The leading entry in each non-zero row is 1.

4) Each leading entry 1 is the only non-zero entry in its column.

Ex $$\begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ are in Reduced Echlon form.

But, $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ are not reduced Echlon form.

2a) Properties If A is a matrix then

(i) $\text{rank}(A) = \text{rank}(A^T)$

(ii) If A is of order $m \times n$ then rank(A) is at most $\min\{m, n\}$.

(iii) If B is sub matrix of A then rank(B) is less than or equal to rank(A).

SOLUTION OF LE

Consistent : A system of eqns is consistent if it has at least one solution. Inconsistent : A system is inconsistent if it has no solution.

Homogeneous : A system is of the form $AX = 0$, where 0 is zero column is homogeneous.

~~Inconsistent~~ : A system is of the form $AX = B$, is called non homogeneous.

Sol^n : A list of values $(k_1, k_2, \dots, k_n)$ is a solution of (1) i.e $a_1k_1 + a_2k_2 + \dots + a_nk_n = b$. We denote the solution by $u = (k_1, k_2, \dots, k_n)$. The set of all possible solutions is called solution set of the linear system.

3b) **Example** find rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank. 1

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2}$$

rank . 3

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$ rank = 1

2b) Elementary Row / Column Operations / transformation -

1) Interchange of two rows (or columns) $R_1 \leftrightarrow R_2$

2) Addition of scalar multiple of one row (or column) to another row (or column). $R_2 \to R_2 - 2R_1$

3) Multiplication of row (column) by a non-zero scalar. $R_3 \to 3R_3$

3b2) find rank of $\begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1}$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$ (rank = 2)

$$\begin{bmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix} \text{(rank = 3)} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 & 2 & 5 & 2 \\ 2 & -2 & 3 & 4 & 1 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & 7 & 14 & 5 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 4 & 8 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 7 & 14 & 5 \\ 0 & 0 & 4 & 8 & 2 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + 4R_2]{R_3 \to R_3 + 7R_2} \begin{bmatrix} -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 26 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix} \xrightarrow[R_4 \to \frac{R_4}{(14)}]{R_3 \to \frac{R_3}{26}} \begin{bmatrix} -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank is 3

Linear Differential Equations

1) What is LDE

2) Homogeneous & Non-homogeneous LDE with constant coefficients & Examples

3) G.S of LDE in terms of C.F and P.I

4) Derive rules for C.F, P.I by operator D excluding $\frac{1}{F(D)} x^n$

5) Define Partial differential Equation (P.D.E)

6) form PDE by eliminating arbitrary constants & functions

7) Solve PDE of the form $Pp + Qq = R$

Differential Equation:

A DE is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

NOTE:

Derivative indicates a change in a dependent variable with respect to an independent variable.

Ex

$$\frac{dy}{dt} + 5y = 3t$$

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 3y = 0$$

Here $y \rightarrow$ dependent variable

$x, t \rightarrow$ independent variable

Types of DE

① Ordinary differential Equation

② Partial differential Equation

① ODE:

This equation involves only one independent variable.

(2) PDE :

This equation involves more than one independent variable.

(1) $\frac{dy}{dx} + xy = x^2$

(2) $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$

(3) $\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 4$, (4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(1) & (2) are ODE & (3) & (4) are PDE.

Order of ~~Linear~~ Differential Equation

A DE is linear if

(a) Every dependent variable.

Order of DE :

It is the order of highest derivative in the equation.

Degree of DE :

It is the power of the highest order derivative in the equation (which is a positive integer value).

ex $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 + 4y = 0$

order = 2 , degree = 1

(2) $$\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^6 + 2y = x^2$$

order = 2

degree = 1

(3) $$\left(1 + \frac{d^2y}{dx^2}\right)^{3/2} = a\frac{d^2y}{dx^2}$$

Square both sides

$$\left(1 + \frac{d^2y}{dx^2}\right)^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

order = 2 , degree = 3

Linear Differential Equation

A DE is called linear if

(a) every dependent variable and derivatives are of first order only

(b) Products of derivatives and/or dependent variables do not occur.

Otherwise it is called Non-linear DE.

Ex $$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^2 \quad \text{(Non-linear)}$$

$\frac{d^2y}{dx^2} + y^2 = 0$ (Non linear) ~~product~~ degree of DV is 2

$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$ (Non-linear) ~~product~~ degree of $\frac{dy}{dx}$ is 3

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} \quad (\text{linear})$$

$$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = 0 \quad (\text{linear})$$

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = xy \quad (\text{linear})$$

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} = y\cos x \quad (\text{linear})$$

$$x\frac{dy}{dx} + 2y = (x^2 - x + 1) \quad (\text{linear})$$

Linear Differential equation with Constant coefficient :

A linear DE with constant coefficient is a differential equation where only constant functi appear as coefficients in the equation.

General form

The most general linear equation of order n is of the form

$$\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x)} \quad \text{--- (1)}$$

where

$a_0, a_1, \ldots, a_n$ are constants

$f(x)$ is function of x.

D denotes the differential operator $\frac{d}{dx}$

i.e $\boxed{D = \frac{d}{dx}}$ $\qquad$ $\boxed{\therefore \frac{1}{D} = \int dx}$

eqn (1) becomes

$$\left[a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right] y = f(x)$$

Non-Homogeneous LDE

If right hand side of the equation (1) is a function $f(x)$ or constant then the equation is called Non-homogeneous LDE.

Homogeneous LDE with constant Coefficient :

When the right side of the equation

$$\left[a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right] y = f(x)$$

is equal to zero then the equation is called homogeneous linear DE. (i.e if $f(x) = 0$)

If $a_0 \neq 0$ then

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0$$

where $P_i = \frac{a_i}{a_0}$, $i = 0, 1, 2, \ldots, n$

i.e $\left[D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n\right] y = 0$

where $P_1, P_2, \ldots, P_n$ are constants.

__Examples__

$\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} + 4y = 0$ (homogeneous)

$\frac{d^2 x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ (homogeneous)

$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = e^{-x}$

Auxiliary equation / Characteristic equation:

The equation of homogeneous differential equation with constant coefficient is

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = 0 \quad (3)$$

If $y = e^{mx}$ is a solution of eqn (3) then

$$\frac{dy}{dx} = \frac{d}{dx}(e^{mx}) = me^{mx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(me^{mx}) = m\frac{d}{dx}e^{mx} = m^2e^{mx}$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}(m^{n-1}e^{mx}) = m^{n-1}\cdot me^{mx} = m^ne^{mx}$$

Putting those values in eqn (3)

$$m^ne^{mx} + P_1m^{n-1}e^{mx} + \cdots + P_ne^{mx} = 0$$

$$(m^n + P_1m^{n-1} + \cdots + P_n)e^{mx} = 0$$

Here $\boxed{m^n + P_1m^{n-1} + \cdots + P_n = 0}$

is called Auxiliary equation of the given differential equation.

Solution of Homogeneous LDE

To find the solution of Homogeneous linear differential equation we have to find first auxiliary equation and solve it for 'm'.

There are 3-different cases for 'm' values

1) Distinct and Real Roots
2) Repeated Roots
3) Conjugate complex Roots.

1) For case-1 the roots of auxiliary equation are distinct real numbers $m_1, m_2, \ldots, m_n$. Then the general solution of homogeneous equation is

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \ldots, c_n$ are distinct arbitrary constants.

2) For case-2, the roots of auxiliary equation are $m, m, m, \ldots, m$ (n-times repeated) then general solution is

$$y(x) = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \cdots + c_n x^{n-1} e^{mx}$$

or,

$$y(x) = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$$

3) If the auxiliary equation has complex roots $\alpha + i\beta$ and $\alpha - i\beta$ then the general solution is given by

$$y = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

Examples :

① Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6 = 0$

Ans The auxiliary equation is

$$m^2 - m - 6 = 0$$

$$\Rightarrow m^2 - 3m + 2m - 6 = 0$$

$$\Rightarrow m(m-3) + 2(m-3) = 0$$

$$\Rightarrow (m-3)(m+2) = 0$$

$$m - 3 = 0 \quad \text{or} \quad m + 2 = 0$$

$$m = 3 \quad \text{or} \quad m = -2$$

Roots are 3 and -2 real and distinct.

Hence the general solution is given by

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

(2) Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$

Ans The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow (m+1)(m+2)(m+2) = 0$$

$$\Rightarrow m = -1, 2, 2$$ (2 roots are repeated and 1 root is real)

Hence the general solution is

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-x}$$

(3) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$, $y(0) = 1$, $y'(0) = 0$

Ans The auxiliary equation is

$$m^2 + 4m + 5 = 0$$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16 - 4.1.5}}{2.1} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$ (complex conjugate Roots)

Hence the general solution is

$$y(x) = e^{-2x}\left[A\cos x + B\sin x\right] \quad \text{———} (*)$$

where $\alpha = -2$, $\beta = 1$

initial condition $y(0) = 1$

i.e $x=0, y=1$ put in $y(x)$

and $y'(0) = 0$., i.e $x=0, y=0$ put this in $y'(x)$

we have $y = e^{-2x}(A\cos x + B\sin x)$

$$y = Ae^{-2x}\cos x + Be^{-2x}\sin x \quad \text{—— (1)}$$

$$y'(x) = -Ae^{-2x}\sin x - 2Ae^{-2x}\cos x +$$

$$Be^{-2x}\cos x - 2Be^{-2x}\sin x \quad \text{—— (2)}$$

Put $x=0, y=1$ in eqn (1)

$$1 = A\cdot 1\cdot 1 + 0 \Rightarrow \boxed{A=1}$$

put $x=0, y=0$ in eqn (2)

$$0 = 0 - 2\cdot A\cdot 1\cdot 1 + B\cdot 1\cdot 1 - 0$$

$$0 = -2A + B$$

$$\Rightarrow B - 2\cdot 1 = 0 \Rightarrow \boxed{B=2}$$

Put values of A & B in eqn(*)

$$y(x) = e^{-2x}[A\cos x + B\sin x]$$

$$y(x) = e^{-2x}(\cos x + 2\sin x)$$

(soln)

Solution of Non-Homogeneous LDE with constant coefficients :

Non-homogeneous LDE is given by

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = f(x)$$

where $f(x) \neq 0$

Theorem :

The general solution of the non-homogeneous equation is represented as the sum of some particular solution of the equation y_p and the general solution y_c of the corresponding homogeneous equation.

The solution y_c is called Complementary function (C.F) and the solution y_p is called the Particular integral (P.I).

OPERATOR METHOD

The non-homogeneous linear equation with constant coefficients in terms of the differential operator is written as

$$F(D)y = f(x)$$

where $F(D) = D^n + P_1 D^{n-1} + \cdots\cdots + P_{n-1}D + P_n$

Some Important Results :

1) $F(D)e^{ax} = F(a)e^{ax}$

2) $F(D)\{e^{ax}V(x)\} = e^{ax}F(D+a)V(x)$

3) $F(D^2)\sin(ax+b) = F(-a^2)\sin(ax+b)$

4) $F(D^2)\cancel{\sin}\cos(ax+b) = F(-a^2)\cos(ax+b)$

$\frac{1}{F(D)}$ is called inverse operator.

5) $\frac{1}{F(D)}\cdot e^{ax} = \frac{e^{ax}}{F(a)}$, provided $F(a) \neq 0$

6) $\frac{1}{F(D)}\{e^{ax}V(x)\} = e^{ax}\frac{1}{F(D+a)}V(x)$

7) $\frac{1}{F(D^2)}\sin ax = \frac{\sin ax}{F(-a^2)}$

8) $\frac{1}{F(D^2)}\cos ax = \frac{\cos ax}{F(-a^2)}$

$F(-a^2) \neq 0$

9) For $F(-a^2)=0$

$$\frac{1}{F(D^2)}\sin ax = \frac{x}{2a}\sin ax$$

$$\frac{1}{F(D^2)}\cos ax = -\frac{x}{2a}\cos ax$$

Examples

① Solve $(D^2 - 2D + 1)y = e^{-x}$.

Sol^n The homogeneous equation of the given equation is

$$(D^2 - 2D + 1)y = 0$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1, 1$$

Hence the Complementary function (C.F) is

$$y_c = (c_1 + c_2 x)e^{x}$$

Now, to find particular integral (P.I).

$$y_p = \frac{1}{D^2 - 2D + 1} e^{-x}$$

$$= \frac{1}{(D-1)^2} e^{-x} \quad \text{, here } a = -1$$

$$y_p = \frac{e^{-x}}{(-1-1)^2} = \frac{1}{4} e^{-x}.$$

∴ The general solution is $y = y_c + y_p$

i.e $y = (c_1 + c_2 x)e^{x} + \frac{1}{4} e^{-x}$

(Ans)

(2) Solve $(D^2+2)y = x^2 e^{3x}$ (not included in syllabus)

Soln The homogeneous equation of the given equation is

$$(D^2+2)y = 0$$

The auxiliary equation is

$$m^2+2=0$$

$$m = \pm\sqrt{-2} = \pm\sqrt{2}\,i \quad ; \quad \alpha = 0,\ \beta = \sqrt{2}$$

Hence the complementary function (C.F) is

$$y_c = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

$$= e^{0x}\left[A\cos\sqrt{2}x + B\sin\sqrt{2}x\right]$$

$$y_c = A\cos\sqrt{2}x + B\sin\sqrt{2}x$$

For, particular integral

$$y_p = \frac{1}{D^2+2}\, e^{3x}.x^2$$

$$= \frac{e^{3x}}{(D+3)^2+2}\, x^2$$

$$= e^{3x}\,\frac{1}{D^2+6D+9+2}\, x^2$$

$$= e^{3x}\,\frac{1}{D^2+6D+11}\, x^2$$

$$= e^{3x}(11 + D^2 + 6D)^{-1} x^2 = e^{3x} \cdot 11^{-1}\left(1 + \frac{D^2+6D}{11}\right)^{-1} x^2$$

$$= e^{3x} \cdot \frac{1}{11}\left(1 + \frac{D^2+6D}{11}\right)^{-1} x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{D^2+6D}{11} + \frac{(D^2+6D)^2}{11^2} - \cdots\right] x^2$$

$$= \frac{e^{3x}}{11}\left[x^2 - \frac{2 + 6.2x}{11} + \frac{36}{11^2} D^2 x^2\right]$$

$$= \frac{e^{3x}}{11}\left(x^2 + \frac{12x - 2}{11} + \frac{36}{11^2} \times 2\right)$$

$$= \frac{e^{3x}}{11} \quad 11x^2 + 12x$$

③ Find General Solution of $(D^3+1)y = e^x \cos x + \sin 3x$

Sol^n The homogeneous equation of given DE is

$$(D^3+1)y = 0$$

The Auxiliary equation is

$$m^3+1 = 0$$

$$\Rightarrow (m+1)(m^2-m+1) = 0$$

$$\Rightarrow m = -1, \frac{-(-1) \pm \sqrt{1^2-4.1.(-1)}}{2\times 1}$$

$$\Rightarrow m = -1, \frac{1 \pm \sqrt{5}i}{2}$$

i.e $m = -1, \frac{1}{2} \pm \frac{\sqrt{5}}{2} i$

The complementary function is

$$y_c = C_1 e^{-x} + e^{x/2}\left[A \cos \frac{\sqrt{5}}{2}x + B \sin \frac{\sqrt{5}}{2}x\right]$$

Now, for Particular integral

$$y_p = \frac{e^x \cos x + \sin 3x}{(D^3+1)}$$

$$= \frac{1}{D^3+1} e^x \cos x + \frac{1}{D^3+1} \sin 3x$$

$$= e^x \frac{1}{(D+1)^3+1} \cos x + \frac{1}{D^2.D+1} \sin 3x$$

$$= e^{x} \frac{1}{D^3+1^3+3D^2+3D+1} \cos x + \frac{1}{(-3^2).D+1} \sin 3x$$

$$= e^{x} \frac{1}{D.D^2+3D^2+3D+2} \cos x + \frac{1}{-9D+1} \sin 3x$$

$$= e^{x} \frac{1}{D(-1^2)+3(-1^2)+3D+2} \cos x + \frac{1}{1-9D} \sin 3x$$

$$= e^{x} \frac{1}{2D-1} \cos x + \frac{1}{1-9D} \sin 3x$$

$$= e^{x} \frac{2D+1}{4D^2-1} \cos x + \frac{1+9D}{1-81D^2} \sin 3x$$

$$= e^{x} \frac{2D+1}{4(-1^2)-1} \cos x + \frac{1+9D}{1-81(-3^2)} \sin 3x$$

$$= \frac{e^{x}}{-5}(-2\sin x + \cos x) + \frac{1}{730}(\sin 3x + 27\cos 3x)$$

$$y_p = \frac{e^{x}}{5}(2\sin x - \cos x) + \frac{1}{730}(\sin 3x + 27\cos 3x)$$

Hence the General Solution is $y = y_c + y_p$

$$y = c_1 e^{-x} + e^{x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] +$$
$$\frac{e^{x}}{5}(2\sin x - \cos x) + \frac{1}{730}(\sin 3x + 27\cos 3x)$$

(4) find the P.I of $(D^2+1)y = \cos x$

$\frac{1}{D^2-3D+2} \sin 2x$

$= \frac{1}{40}(6\cos 2x - 2\sin 2x$

Sol^n for $y_p = \frac{1}{D^2+1} \cos x$

By $\frac{1}{F(D^2)} \cos ax = \frac{1}{F(-a^2)} \cos ax$

By above formula $F(-1^2) = 0$ ·So this cannot be applied.

So, $\frac{1}{F(D^2)} \cos ax = \frac{x}{2a} \sin ax$

$\Rightarrow \frac{1}{D^2+1} \cos x = \frac{x}{2.1} \sin x$

$= \frac{x}{2} \sin x$

(5) Solve the Initial Value Problem

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^x, \quad y(0)=1, \quad y'(0)=1$$

Sol^n The homogeneous equation of given DE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$\Rightarrow m = 2, 3$$

The Complementary function (C.F) is

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

Particular Integral (P.I) is

$$y_p = \frac{2}{D^2 - 5D + 6} e^x$$

$$= \frac{2}{(D-3)(D-2)} e^x = \frac{2e^x}{(1-3)(1-2)}$$

$$= \frac{2e^x}{2} = e^x$$

Hence the general solution is

$$y = y_c + y_p$$

$$y = c_1 e^{2x} + c_2 e^{3x} + e^x \quad \text{——— (1)}$$

$$y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x} + e^x \quad \text{——— (2)}$$

By $y(0) = 1$, i.e $x = 0$, $y = 1$ put in eqn (1)

$$1 = c_1 + c_2 + 1 \Rightarrow c_1 + c_2 = 0 \quad \text{——— (3)}$$

Again $y'(0) = 1$, i.e $x = 0$, $y = 1$

$$1 = 2c_1 + 3c_2 + 1 \Rightarrow 2c_1 + 3c_2 = 0 \quad \text{——— (4)}$$

Solving (3) & (4) we get $c_1 = 0$, $c_2 = 0$

Thus the required general solution is

$$y = e^x$$

(previous one was $\frac{1}{F(D)} e^{ax} V$)

Theorem

If V is a function of x then

$$\frac{1}{F(D)} xV = x \frac{1}{F(D)} V + \left\{ \frac{d}{dD} \frac{1}{F(D)} \right\} V$$

Ex find P.I of $\cancel{\frac{1}{D^2+4} x \sin x}$ $(D^2+4)y = x \sin x$

Sol^n

$$y_p = \frac{1}{D^2+4} x \sin x$$

$$= x \frac{1}{D^2+4} \sin x + \left\{ \frac{d}{dD} \frac{1}{D^2+4} \right\} \sin x$$

$$= x \frac{\sin x}{(-1^2)+4} + \frac{-2D}{(D^2+4)^2} \sin x$$

$$= \frac{x}{3} \sin x - \frac{2D}{(-1^2+4)^2} \sin x$$

$$= \frac{x}{3} \sin x - \frac{2}{9} \cos x$$

Partial Differential Equation

A PDE is one which involves more than one independent variable so that the derivatives occurring in it are partial derivatives.

Ex $x\frac{\partial\theta}{\partial x} + y\frac{\partial\theta}{\partial y} + \theta = 0$ (1st order)

$$\frac{\partial^2\theta}{\partial x^2} = a^2\frac{\partial^2\theta}{\partial y^2}$$ (2nd order)

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{\partial\theta}{\partial y} = 0$$ (1st order)

NOTE:

There are two independent variables we shall denote them by x and y whereas the dependent variable will be denoted by z.

We write,

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x\partial y},\ t = \frac{\partial^2 z}{\partial y^2}$$

The partial differential equation of first order is denoted by

$$f(x,y,z,p,q) = 0$$

* Recall

Let $z = f(x,y)$, a funⁿ of x & y

Partial derivative of z w.r.t x is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x(x,y),\ D_x f$$

Similarly, Partial derivative of z w.r.t y is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y(x,y)$, $D_y f$

If z is a function of two or more variables say $x_1, x_2, \ldots$ then then the partial derivative of z is done by differentiating z w.r.t x_1 while other variable are treated as constant.

Proceeding in this process all the z is variables to be differentiate w.r.t other variable at a time one while other are treated as constant.

Ex find first and second partial derivative of $z = x^3+y^3-3axy$

Soln $\frac{\partial z}{\partial x} = 3x^2 - 3ay$, $\frac{\partial z}{\partial y} = 3y^2 - 3ax$

$\frac{\partial^2 z}{\partial x^2} = 6x$, $\frac{\partial^2 z}{\partial y^2} = 6y$

$\frac{\partial^2 z}{\partial x \partial y} = -3a$, $\frac{\partial^2 z}{\partial y \partial x} = -3a$

(2) Find P.derivative $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ of

(i) $z = x^2 y - x \sin xy$

$\frac{\partial z}{\partial x} = 2xy - (\sin xy - xy \cos xy)$

$\frac{\partial z}{\partial y} = x^2 - x^2 \cos xy = x^2(1 - \cos xy)$

(ii) $z = \log(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}\cdot(2x) \quad , \quad \frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}(2y)$$

(iii) $x+y+z = \log z$

We know pde ~~of~~ is of the form $f(x,y,p,q,z)=0$

Ex

$yp - xq = 0$

$z = pq$

$x^2 r + 2xys + y^2 t = 2z$

$p^2+q^2 = 4z$

$px+qy = q^2$

$xp - yq = x-y$

Formation of 1st order PDE

We can make PDE by ~~two~~ considering two cases

① By Eliminating arbitrary constants

② By Eliminating arbitrary function.

We can eliminate both arbitrary constant and arbitrary functions by partial derivat and simplification.

Ex form PDE of (i) $z = (x^2+a)(y^2+b)$ —— (i)

Sol^n $\dfrac{\partial z}{\partial x} = 2x(y^2+b) = p$

$\dfrac{\partial z}{\partial y} = 2y(x^2+a) = q$

Multiply p & q

$pq = 4xy(x^2+a)(y^2+b)$

from (1)

$\boxed{pq = 4xyz}$

This is the required PDE.

(ii) $x^2+y^2+(z-c)^2 = a^2$ —— (1)

differentiating partially w.r.t x bothsides of (1)

$2x + 2(z-c)\dfrac{\partial z}{\partial x} = 0 \Rightarrow x + (z-c)p = 0$

Similarly, ~~be~~ w.r.t y

$2y + 2(z-c)\dfrac{\partial z}{\partial y} = 0 \Rightarrow y + (z-c)q = 0$

$$xq + (z-c)pq = 0$$

$$- \quad yp + (z-c)pq = 0$$

$$xq - yp = 0$$

(iii) $z = ax^2 + bxy + cy^2$

$p = \frac{\partial z}{\partial x} = 2ax + by \quad , \quad \frac{\partial^2 z}{\partial x^2} = 2a = r$

$q = \frac{\partial z}{\partial y} = bx + 2cy \quad , \quad \frac{\partial^2 z}{\partial y^2} = 2c = t$

$\frac{\partial^2 z}{\partial x \partial y} = b, \quad \frac{\partial^2 z}{\partial y \partial x} = b = s$

$z = r$

(2) Elimination of arbitrary function:

Ex Eliminate the function 'f' from

$$z = e^{mx} f(x+y) \quad \text{--- (1)}$$

Sol Differentiating partially eqn (1) w.r.t x and y respectively we get

$$p = \frac{\partial z}{\partial x} = m e^{mx} f(x+y) + e^{mx} f'(x+y) \quad \text{--- (2)}$$

$$q = \frac{\partial z}{\partial y} = e^{mx} f'(x+y) \quad \text{--- (3)}$$

Subtracting eqn (3) from (2)

$$p - q = m e^{mx} f(x+y)$$

$$\boxed{p - q = mz} \rightarrow \text{Required PDE.}$$

Linear PDE:

A PDE of first order is linear if it is of first degree in p and q otherwise it is non-linear.

Ex

$$\left.\begin{array}{l} z(xp - yq) = y^2 - x^2 \\ x^2 p + y^2 q = z^3 \end{array}\right\} \text{linear}$$

$$\left.\begin{array}{l} z^2(1 + p^2 + q^2) = 1 \\ xpq + yq^2 = 1 \end{array}\right\} \text{Non-linear}$$

Solution of LPDE :

def^n: A solution of pde is a complete solution or a complete integral if it contains as many arbitrary constants as there are independent variables.

General solution: A G.S or General Integral of a PDE is a relation involving functions which provides a solution to that equation.

Theorem :-

The General solution of the Linear partial DE. $Pp + Qq = R$ is $F(u,v) = 0$ where F is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Working Rule :

In order to solve $Pp + Qq = R$, first construct auxiliary (or subsidiary) equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and find its two integrals $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$. Then $F(u,v) = 0$ is the solution of the given equation.

The solution also can be written as

$$u = \phi(v)$$

where ϕ is an arbitrary function.

Q. find the general solution of

$$zxp - zyq = y^2 - x^2$$

Sol^n This given equation is in the form of . $Pp + Qq = R$

So, the Subsidiary / Auxiliary equation is

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$$

from two ratios

$$\frac{1}{zx} dx = \frac{1}{-zy} dy$$

Integrating both sides

$$\int \frac{1}{x} dx = -\int \frac{1}{y} dy$$

$$\log x = -\log y + C$$

$$\Rightarrow \log x + \log y = C$$

$$\Rightarrow \log xy = \log C \Rightarrow xy = C_1 \quad \text{--- (1)}$$

Using multipliers x, y, z we get

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2} = \frac{x\,dx + y\,dy + z\,dz}{zx^2 + (-zy^2) + zy^2 - zx^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$

Integrating, we get $x^2 + y^2 + z^2 = c_2$ —— (2)

Therefore the solution is

$$F(xy, x^2+y^2+z^2) = 0$$

or

$$xy = \phi(x^2+y^2+z^2) \qquad \text{(Ans)}$$

To solve $Pp + Qq = R$

Method - I

Q2 $$p - 2q = 3x^2 \sin(y+2x)$$

Sol^n The auxiliary equations are $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y+2x)}$

from first two ratios $\frac{dx}{1} = \frac{dy}{-2} \Rightarrow 2x + y = c_1$ —— (1)

Taking first & 3rd ratio $dx = \frac{dz}{3x^2 \sin(y+2x)} = \frac{dz}{3x^2 \sin c_1}$

from (1)

$$3x^2 dx = \frac{1}{\sin c_1} dz$$

$\Rightarrow$

$$\Rightarrow \frac{3x^3}{3} \sin c_1 = z + c_2$$

$$\Rightarrow x^3 \sin c_1 - z = c_2 \quad \text{—— (2)}$$

Hence our solution is

$$F(2x+y,\ x^3 \sin c_1 - z) = 0$$

or $$\phi(2x+y) = x^3 \sin c_1 - z$$

(Ans)

1

Laplace Transform

We need Gamma function

Usually
$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad (n>0$$

Also, (In factorial)

$$\boxed{\Gamma(n+1) = n!}, \quad n = +ve \text{ integer}$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi} = 1.772$$

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Definition:

Let $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ is

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

or
$$\bar{F}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

where s is a parameter which may be real or complex number.

Also $f(t) = L^{-1}\{\bar{F}(s)\}$ then $f(t)$ is called as the Inverse Laplace Transform of $\bar{F}(s)$.

where L = Laplace transformation operator

②

Conditions for the existence :-

The Laplace transform of $f(t)$

i.e $\int_0^\infty e^{-st} f(t)\,dt$ exists for $s > a$, if

(i) $f(t)$ is continuous

(ii) $\lim_{t \to \infty} \{ e^{-st} f(t) \}$ is finite.

These conditions are sufficient, not necessary.

~~Some~~ formulae

1) $L(1) = \frac{1}{s}$

2) $L(t^n) = \frac{n!}{s^{n+1}}$, $n = 0, 1 \ldots$

3) $L(e^{at}) = \frac{1}{s-a}$

4) $L(\sin at) = \frac{a}{s^2+a^2}$

5) $L(\cos at) = \frac{s}{s^2+a^2}$

6) $L(\sinh at) = \frac{a}{s^2-a^2}$

7) $L(\cosh at) = \frac{s}{s^2-a^2}$

$L^{-1}\left(\frac{1}{s}\right) = 1$

$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}$

$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$

$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

→ Explanation of existence:

(3)

$$L(1) = \int_0^\infty e^{-st}.1\,dt = \frac{e^{-st}}{(-s)}\Big]_0^\infty = \frac{e^{-\infty}-e^0}{(-s)} = \frac{-1}{(-s)} = \frac{1}{s}$$

<u>ex</u>

$$L(t^9) = \frac{\sqrt{10}}{s^{10}} \quad , \quad L(e^{2t}) = \frac{1}{s-2}$$

$$L(e^{7t}) = \frac{1}{s-7} \quad , \quad L^{-1}\left(\frac{1}{s-7}\right) = e^{7t}$$

$$L(\sin 4t) = \frac{4}{s^2+4} \quad , \quad L^{-1}\left(\frac{1}{s^2+49}\right) = \frac{\sin 7t}{7}$$

$$L(\cos 4t) = \frac{s}{s^2+4} \quad , \quad L^{-1}\left(\frac{s}{s^2+49}\right) = \cos 7t$$

$$L(\sinh 7t) = \frac{7}{s^2-49} \quad ; \quad L^{-1}\left(\frac{s}{s^2-9}\right) = \cosh 3t$$

$$L(\cosh 7t) = \frac{s}{s^2-49}$$

1) <u>Linearity Property</u> :—

If a, b, c be any constants & f, g, h fun's of t then

$$L[af(t) + bg(t) - ch(t)]$$

$$= aL\{f(t)\} + bL(g(t)) - cL(h(t))$$

2) <u>1st Shifting Property</u> :

If $L\{f(t)\} = \bar{F}(s)$ then

$$L\{e^{at} f(t)\} = f(s-a)$$

(4)

Q1 Find Laplace Transform of . $255 = C-C$

(i) $\sin 2t \sin 3t$

$$= \frac{1}{2}(2\sin 2t \cdot \sin 3t) = \frac{1}{2}\left[\cos\left(\frac{2t-3t}{2}\right) - \cos\left(\frac{2t+3t}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos(-t) - \cos 5t\right] = \frac{1}{2}(\cos t - \cos 5t)$$

$$L(\sin 2t \cdot \sin 3t) = \frac{1}{2}L(\cos t - \cos 5t)$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+25}\right]$$

$$= \frac{s\{(\cancel{s^2}+25) - \cancel{s^2} - 1^2\}}{2(s^2+5^2)(1^2+s^2)}$$

$$= \frac{24s}{2(5^2+s^2)(s^2+1)}$$

(ii) $\cos^2 2t = \frac{1}{2}(1+\cos 4t)$

$$L(\cos^2 2t) = \frac{1}{2}L(1+\cos 4t) = \frac{1}{2}\left[L(1) + L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right] = \frac{1}{2}\left[\frac{s^2+16+s^2}{s(s^2+16)}\right]$$

$$= \frac{\cancel{2}(s^2+8)}{\cancel{2}(s^2+16)\times s}$$

By shifting property, we get

$$L(e^{2t}\cos^2 t) = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right\}$$

(5)

Q3 $f(t) = \sqrt{t}\, e^{3t}$

$$L(\sqrt{t}) = L(t^{1/2}) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{1/2+1}} = \frac{\left(\frac{1}{2}\right)\sqrt{\pi}}{s^{3/2}}$$

By shifting property,

$$L(e^{3t}\sqrt{t}) = \frac{\sqrt{\pi}}{2(s-3)^{3/2}}$$

Q4 Find the Laplace Transform of the function

$$f(t) = \begin{cases} 1, & 0 < t \leq 1 \\ t, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases}$$

We know $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{f(t)\} = \int_0^1 e^{-st} \cdot 1\,dt + \int_1^2 e^{-st} \cdot t\,dt + \int_2^\infty e^{-st} \cdot 0\,dt$$

$$= \left.\frac{e^{-st}}{(-s)}\right|_0^1 + \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{(s^2)}\right]_1^2$$

$$= \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$

Transforms of Derivatives

① If $f'(t)$ be continuous and $L\{f(t)\} = F(s)$ then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$\cancel{L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt}$$

General form

$$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots f^{n-1}(0)$$

Like →

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$\vdots$$

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots f^{n-1}(0)$$

Ex (i) $f(t) = t\cos at$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(t^1 \cos at) = (-1)^1 \frac{d}{ds^1}\left(\frac{s}{s^2+a^2}\right) = -\frac{(s^2+a^2 - 2.s.s)}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

~~(iii) $f(t) = \cos^2 at$, $L\{f(t)\} = ?$~~ | ~~$L\{\cos at\} = \frac{s}{s^2+a^2}$~~

Transforms of Integrals

(I) If $L\{f(t)\} = F(s)$ then

$$\boxed{L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}F(s)}$$

Examples

Question from division by t i.e $\left(\frac{1-\cos t}{t}\right) = f(t)$

Q1 - find $L\left\{\frac{d}{dt}f(t)\right\}$ or $L\{f'(t)\}$.

$$L\left\{\frac{d}{dt}\left(\frac{1-\cos t}{t}\right)\right\} = sF(s) - f(0)$$

$$= s \log\frac{\sqrt{s^2+1}}{s} - 0$$

Ex2 $L\left\{\int_0^t u \cdot \sin^2 u\,du\right\} = ?$ $\qquad F(s) = ?$

$L(\sin^2 u\,du) =$ found $= F_1(s)$

$L(u \cdot \sin^2 u) = (-1)^n \frac{d^n}{ds^n} F(s) = F_2(s)$

Now $L\left\{\int_0^t u \sin^2 u\,du\right\} = \frac{1}{s}F_2(s) = F(s)$

6

Multiplication by t^n

If $L\{f(t)\} = F(s)$ then

$$L\{t^n . f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \quad , n = 1, 2, 3, \ldots$$

ex (i) $f(t) = t e^{-t} \cosh 2t$ t^1

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$L\{e^{-t} \cosh 2t\} = L\left\{e^{-t}\left(\frac{e^{2t} + e^{-2t}}{2}\right)\right\}$$

$$= L\left\{\frac{1}{2}(e^{t} + e^{-3t})\right\}$$

$$= \frac{1}{2} L(e^{t} + e^{-3t}) = \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right] = F(s)$$

$$= \frac{1}{2}\left(\frac{s+3+s-1}{(s-1)(s+3)}\right) = \frac{2(s+1)}{2(s-1)(s+3)}$$

$$= \frac{s+1}{(s-1)(s+3)} = F(s)$$

Now, $$L\{t \, e^{-t} \cosh 2t\} = (-1)^1 \frac{d}{ds}\left\{\frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+3}\right)\right\}$$

$$= \frac{-1}{2}\left[\frac{d}{ds}\left(\frac{1}{s-1}\right) + \frac{d}{ds}\left(\frac{1}{s+3}\right)\right]$$

Ans ✓ $$= \frac{-1}{2} \cancel{\log(s+1)} \left[\frac{-1}{(s-1)^2} - \frac{1}{(s+3)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s+3)^2 + (s-1)^2}{(s-1)^2(s+3)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+9+6s+s^2+1-2s}{(s-1)^2(s+3)^2}\right]$$

(7)

Q2 $L(t \cdot \sin^3 t) = ?$

First find $L(\sin^3 t) = L\left[\frac{1}{4}(3\sin t - \sin 3t)\right]$

$$= \frac{1}{4}\left[L(3\sin t) - L(\sin 3t)\right]$$

$$= \frac{1}{4}\left[3 \times \frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$= \cancel{\frac{3}{4} \cdot \frac{s^2+9-s^2-1}{}}$$

Now,

$$L(t \cdot \sin^3 t) = (-1)^1 \frac{d}{ds^1}\left[\frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)\right]$$

$$= \frac{-3}{4}\left[\frac{-1}{(s^2+1)^2}(2s) - \frac{(-1)}{(s^2+9)^2}(2s)\right]$$

$$= \frac{-3}{4}(-2s)\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

$$= \frac{3}{2}s\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

Division by t

If $L\{f(t)\} = F(S)$ then

$$L\left[\frac{1}{t} f(t)\right] = \int_S^\infty F(s)\,ds,$$

provided integral exists.

Ex ① $f(t) = \frac{1}{t}(1-\cos t)$

$$L\left\{\frac{1}{t}(1-\cos t)\right\} = ?$$

$$L(1-\cos t) = L(1) - L(\cos t)$$

$$= \frac{1}{s} - \frac{s}{s^2+1} = F(s)$$

Now, $L\left\{\frac{1}{t}(1-\cos t)\right\} = \int_s^\infty F(s)\,ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty = \frac{1}{2}\left(0 - \log\frac{s^2}{s^2+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right).$$

Q2 $L\left[\frac{\sin^2 2t}{t}\right] = ?$ (9)

$f(t) = \sin^2 2t = \frac{1-\cos 4t}{2}$

$L\{f(t)\} = \frac{1}{2}L(1-\cos 4t) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$

$L\left[\frac{1}{t}\sin^2 2t\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+16}\right) ds$

$= \frac{1}{2}\int_s^\infty \left[\cancel{\log}\frac{1}{s} - \frac{1}{2}\left(\frac{2s}{s^2+16}\right)\right] ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+16)\right]_s^\infty$

$= \frac{1}{4}\left[\log s^2 - \log(s^2+16)\right]_s^\infty$

$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+16}\right)\right]_s^\infty$

$= \frac{1}{4}\left(-\log\frac{s^2}{s^2+16}\right) = \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)$

Q $L\left[\frac{1}{t^2}(1-\cos t)\right] = ?$

we have already found $L\left[\frac{1}{t}(1-\cos t)\right] = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

then again $L\left[\frac{1}{t}\left[\frac{1}{t}(1-\cos t)\right]\right]$

$= \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right) ds$

(10)

$$= \frac{1}{2}\int_s^\infty \log\left(\frac{s^2+1}{s^2}\right)\cdot 1\, ds \qquad \int uv = u\int v - \int u'\int v\, d.$$

$$= \frac{1}{2}\left[\left\{\log\left(\frac{s^2+1}{s^2}\right)\int_s^\infty 1\cdot ds\right\}_s^\infty - \int_s^\infty\left\{\frac{1}{\left(\frac{s^2+1}{s^2}\right)}\left(\frac{s^2\cdot 2s - s^2\cdot 2s}{s^2}\right)\int 1\cdot ds\right\}ds\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right)\cdot s\right]_s^\infty - \frac{1}{\cancel{2}}\int_s^\infty \frac{\cancel{s^2}}{s^2+1}\left(\frac{\cancel{2}s^2}{\cancel{s^2}\cdot\cancel{s}}\right)\cdot\cancel{s}\; ds$$

$$= \cancel{\frac{1}{2}(-s)\log\left(\frac{s^2+1}{s}\right)}$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+1}{s^2}\right)\cdot s\right] - \int_s^\infty\left(\frac{1}{s^2+1}\right)ds$$

$$= \frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)\cdot s -$$

Inverse Transforms (Method of Partial Fractions)

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

(2) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

3) $L^{-1}\left(\frac{1}{s^{(n+1)}}\right) = \frac{t^n}{n!}, \; n = 0, 1, 2, \ldots$

4) $L^{-1}\left(\frac{1}{(s^2+a^2)}\right) = \frac{1}{a}\sin at$

5) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

6) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

7) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

8) $L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$

9) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$

10) $L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$

11) $L^{-1}\left(\frac{(s-a)}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

Ex (1) $F(s) = \frac{s^2+3s+4}{s^3} = \frac{1}{s} + \frac{3}{s^2} + \frac{4}{s^3}$

$$L^{-1}\{F(s)\} = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{3}{s^2}\right) + L^{-1}\left(\frac{4}{s^3}\right)$$

$$= 1 + 3 \times \frac{t^1}{1} + 4 \times \frac{t^2}{2!} = 1 - 3t + 2t^2$$

(ii) $F(s) = \dfrac{s+2}{s^2-4s+13}$

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left\{\frac{s+2}{s^2-4s+4-4+13}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\}$$

$$= L^{-1}\left[\frac{(s-(+2))+4}{(s-2)^2+9}\right]$$

$$= L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+9}\right]$$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$$

$$= e^{2t}\cos 3t + 4 \times \frac{1}{3} e^{2t} \cos 3t$$

Partial fraction method

(1) $\dfrac{1}{(s-a)(s-b)(s-c)} = \dfrac{A}{s-a} + \dfrac{B}{s-b} + \dfrac{C}{s-c}$

(2) $\dfrac{1}{(s-a)(s-b)^2} = \dfrac{A}{(s-a)} + \dfrac{B}{(s-b)} + \dfrac{C}{(s-b)^2}$

(3) $\dfrac{1}{(s-a)(s^2+b)} = \dfrac{A}{s-a} + \dfrac{Bs+C}{s^2+b}$

Q) find the Inverse of LT

$$L^{-1}\left[\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right] = ?$$

Let $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)} = \dfrac{A}{s+1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$ —— (1)

$\Rightarrow \dfrac{2s^2-4}{(s+1)\cancel{(s-2)}(s-3)} = \dfrac{A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)}{(s+1)\cancel{(s-2)}(s-3)}$

$\Rightarrow 2s^2-4 = A(s^2-3s-2s+6) + B(s^2-3s+s-3) + C(s^2-2s+s-2)$

$= \cancel{s^2(A+B)}$

$= A(s^2-5s+6) + B(s^2-2s-3) + C(s^2-s-2)$

$2s^2-4 = s^2(A+B+C) + s(-5A-2B-C) + (6A-3B-2C)$

$A+B+C=2$

$-5A-2B-C=0$

$6A-3B-2C=-4$

$$\sim\begin{bmatrix}1 & 1 & 1 & 2\\ -5 & -2 & -1 & 0\\ 6 & -3 & -2 & -4\end{bmatrix}$$

we found $A=\frac{-1}{6}, B=-\frac{4}{3}, C=\frac{7}{2}$

$$\sim\begin{bmatrix}1 & 1 & 1 & 2\\ 0 & 3 & 4 & 10\\ 0 & -9 & -8 & -16\end{bmatrix}$$

$-12-4$

30
-12

Now,

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)}=\frac{\left(-\frac{1}{6}\right)}{s+1}+\frac{\left(-\frac{4}{3}\right)}{s-2}+\frac{\left(\frac{7}{2}\right)}{s-3}$$

$$\begin{bmatrix}1 & 1 & 1 & 2\\ 0 & 3 & 4 & 10\\ 0 & 0 & 4 & 14\end{bmatrix}$$

$$L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}=-\frac{1}{6}L^{-1}\left(\frac{1}{s+1}\right)-\frac{4}{3}L^{-1}\left(\frac{1}{s-2}\right)+\frac{7}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$=-\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$$

(Ans)

$4C=14$

$C=\frac{14}{4}=\frac{7}{2}$

$3B=10-4\times\frac{7}{2}$

$=\frac{20-28}{2}=\frac{-8}{2}=-4$

$\boxed{B=\frac{-4}{3}}$

$A=2-B-C$

$=2-\frac{7}{2}+\frac{4}{3}=2+\frac{-21+8}{6}$

$=2-\frac{13}{6}=\frac{12-13}{6}$

$=\cancel{\frac{12-21+8}{12}}=\cancel{\frac{-1}{12}}=\frac{-1}{6}$

Ex $L(te^{-2t})=?$ here $f(t)=t^1$

$$L\{f(t)\}=L(t^1)=\frac{1!}{s^2}=\frac{1}{s^2}=F(s)$$

$$L\{e^{-2t}t\}=F(s-a)$$

$$=F(s+2)=\frac{1}{(s+2)^2}$$

No 2 find inverse L.T of $L^{-1}\left\{\frac{s^2}{(s+1)^2(s-2)}\right\}$

Soln

Let $\frac{s^2}{(s+1)^2(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-2)}$

$\Rightarrow \frac{s^2}{(s+1)^2\cancel{(s-2)}} = \frac{A(s+1)(s-2) + B(s-2) + C(s+1)^2}{(s+1)^2\cancel{(s-2)}}$

$\Rightarrow s^2 = A(s^2-2s+s-2) + B(s-2) + C(s^2+2s+1)$

$= (A+C)s^2 + (-A+B+2C)s + (-2A-2B+C)$

Here $A+C = 1 \Rightarrow A = 1-C$ —— ①

$-A+B+2C = 0 \Rightarrow C-1+B+2C = 0$

$\Rightarrow B+3C = 1$ —— ②

$-2A-2B+C = 0$

$\hookrightarrow -2(1-C)-2B+C = 0$

$-2+2C-2B+C = 0$

$-2B+3C = 2$ —— ③

$B+3C = 1$ .. (eqn ②)

$-3B = 1$

$B = \frac{-1}{3}$

$3C = 1-B$

$C = \frac{1}{3}\left(1+\frac{1}{3}\right)$

$= \frac{1}{3} \times \frac{4}{3} = \frac{4}{9}$, $A = 1-C = 1-\frac{4}{9} = \frac{5}{9}$

so, $\frac{s^2}{(s+1)^2(s-2)} = \frac{\left(\frac{5}{9}\right)}{(s+1)} + \frac{\left(-\frac{1}{3}\right)}{(s+1)^2} + \frac{\left(\frac{4}{9}\right)}{(s-2)}$

$$L^{-1}\left[\frac{s^2}{(s+1)^2(s-2)}\right] = \frac{5}{9} L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{4}{9} L^{-1}\left(\frac{1}{s-2}\right)$$

$$= \frac{5}{9} e^{-t} - \frac{1}{3} t e^{-t} + \frac{4}{9} e^{2t}$$

Explanation:

$L(e^{-t}.t) = ?$, here $f(t) = t$

$L\{f(t)\} = L(t) = \frac{1!}{s^{1+1}} = \frac{1}{s^2} = F(s)$

Now, $L\{te^{-t}\} = F(s-a)$, $a = -1$

$L^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}$ $\leftarrow$ $\boxed{L(te^{-t}) = \frac{1}{(s+1)^2}}$

Laplace Derivative Problems :-

Q1 find $L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = ?$

Here $f(t) = \frac{\sin t}{t}$

to find $L\{f(t)\} = L\left(\frac{\sin t}{t}\right)$ ——— (1)

Again, here $g(t) = \sin t$

$$L\{g(t)\} = L(\sin t) = \frac{1}{s^2+1} = F_1(s)$$

Now, $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds$

$$= \left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty \quad \text{here } a = 1$$

$$= \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$= \cot^{-1}s = F(s)$$

We have to find $L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = sF(s) - f(0)$

here $f(t) = \frac{\sin t}{t}$, $F(s) = \cot^{-1}s$

find $\underset{t\to 0}{f(0)} = \frac{\sin t}{t}$ ($\frac{0}{0}$ form)

$$= \frac{\cos t}{1} \quad \text{(L' Hopital Rule)}$$

$f(0) = 1$

Now, $L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = sF(s) - f(0)$

$= s\cot^{-1}s - 1$ (Ans)

Q2 $L\left\{\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right\} = ?$

here $f(t) = \frac{1-\cos 2t}{t}$, $L\{f(t)\} = F(s)$

$L\left\{\frac{1-\cos 2t}{t}\right\}$ is in division form of t.

Let $g(t) = 1-\cos 2t$

$L\{1-\cos 2t) = L(1) - L(\cos 2t)$

$= \frac{1}{s} - \frac{s}{s^2+4}$

$L\left\{\frac{1}{t}(1-\cos 2t)\right\} = \int_s^\infty \left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} ds$

$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$

$= \left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$

$= \log(1) - \log\frac{s}{\sqrt{s^2+4}} = 0 - \log\frac{s}{\sqrt{s^2+4}}$

$= \log\frac{\sqrt{s^2+4}}{s} = \log\left(\frac{s^2+4}{s^2}\right)^{1/2}$

$F(s) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$ (~~Ans~~)

We get $f(t) = \frac{1-\cos 2t}{t}$, $F(s) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$

Now, $f(t=0) = \frac{1-\cos 2t}{t}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \frac{+2\sin 2t}{1}$$

$$f(0) = 0$$

So, $L\left\{\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right\} = sF(s) - f(0)$

$$= s\left[\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)\right] - 0$$

Laplace Integration Problem :-

Q1 Find $L\left\{\int_0^t u.\cos^2 u\,du\right\} = ?$

$$\boxed{L\left[\int_0^t F(u)du\right] = \frac{1}{s}F(s) \qquad L(f(t)) = F(s)}$$

here $F(u) = u\cos^2 u$

$L(F(u)) = L(u'\cos^2 u) = L\left(u.\left(\frac{1+\cos 2u}{2}\right)\right)$

$$= \frac{1}{2}L\{u'.(1+\cos 2u)\}$$

$$= \frac{1}{2}(-1)'\frac{d}{ds}(1+\cos 2u)$$

$$= \frac{-1}{2}(-2\sin 2u) = \sin 2u = F(s)$$

$L\left\{\int_0^t u \cdot \cos^2 u\, du\right\} = ?$

Ans

$L(\cos^2 u) = L\left(\frac{1+\cos 2t}{2}\right) = \frac{1}{2}\left\{L(1) + L(\cos 2t)\right\}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = F(s)$

Now

$L\{u \cdot \cos^2 u\} = (-1)^n \frac{d}{ds} F(s)$

$= (-1)^1 \frac{d}{ds}\left\{\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)\right\}$

$= -\frac{1}{2}\left[\cancel{\log s}\left(-\frac{1}{s^2}\right) + \frac{s^2+4-2s.s}{(s^2+4)^2}\right]$

$= -\frac{1}{2}\left[-\frac{1}{s^2} - \frac{4-s^2}{(s^2+4)^2}\right]$

$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right] = F(s)$

$L\left\{\int_0^t u \cos^2 u\, du\right\} = \frac{1}{s} F(s)$

$= \frac{1}{2s}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right]$

Q2 $L\left\{\int_0^t e^{-2u}\cos^2 u\, du\right\} = ?$

for integration to find $F(s)$ of $f(t)$

Now.

$$L\left(e^{-2u}\cos^2 u\, du\right) ; F(s-a), \quad a=-2$$

$$L(\cos^2 u) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$L\left(e^{-2u}\cos^2 u\right) = \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4}\right] = F(s)$$

(First Shifting Property)

$$L\left\{\int_0^t e^{-2u}\cos^2 u\, du\right\} = \frac{1}{s}F(s)$$

$$= \frac{1}{2s}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4}\right]$$

Q3 $L\left\{e^{2t}\int_0^t u.\sin 3u\, du\right\} = ?$

We have to find $\int_0^t u\sin 3u\, du$ first.

$L(u^1 \sin 3u)$ need $L(\sin 3u)$ first.

So, $L(\sin 3u) = \dfrac{3}{s^2+u^2} = f_1(s)$

$$L(u^1 \sin 3u) = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+u^2}\right)$$

$$= -3\left(\frac{-1}{(s^2+u^2)^2}\right)(2s) = \frac{6s}{(s^2+u^2)^2}$$

$$\therefore L(u\sin 3u) = \frac{6s}{(s^2+u^2)^2} = F(s)$$

Now, $$L\left[\int_0^t u\sin 3u\,du\right] = \frac{1}{s}F(s) = \frac{6\cancel{s}}{(s^2+u^2)^2\cancel{s}}$$

Practice : Examples of $L(e^{at}\sin bt)$, $L(e^{at}\cos bt)$

: Fourier Series :

Periodic Function:

A function $f(x)$ is said to be periodic if

$$f(x+T) = f(x) \quad , \forall x$$

T - positive number

also T is the interval betn two successive repetitions. is called period of $f(x)$.

eg [$y = \sin x$, $T = 2\pi$], [$y = \cos x$, $T =$].

$$\sin x = \sin(x+2\pi) = \sin(x+4\pi) \ldots \text{ so on.}$$

Continuous function:

Dir

Fourier Series:

The fourier series for the function $f(x)$ in the interval $\alpha \le x < \alpha + 2\pi$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{——— (1)}$$

where

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin(nx)\,dx$$

Even function :

A function $y = f(x)$ is said to be even if

$$f(-x) = f(x) \;,\; \forall x$$

Ex $y = x^2$, $y = \cos x$.

Odd function :

A function $y = f(x)$ is said to be odd if

$$f(-x) = -f(x) \;,\; \forall x.$$

Ex $y = x^n$, $y = \sin x$.

Fourier Series in $0 \le x \le 2\pi$

Put $\alpha = 0$ in eqn 1 then interval becomes $0 < x < 2\pi$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin(nx)\,dx$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$ | $\sin(2n+1)\frac{\pi}{2} = (-1)^n$, $\cos(2n+1)\frac{\pi}{2} = 0$

Fourier Series in $-\pi < x < \pi$

Put $\alpha = -\pi$ Eqn ① becomes in Interval $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Q. Find the fourier Series for $f(x) = e^{ax}$ in the interval $0 < x < 2\pi$.

Sol. i.e to find $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} e^{ax}\, dx$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a}\right]_0^{2\pi} = \frac{1}{\pi a}\left(e^{2\pi a} - 1\right)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{ax} \cos nx\, dx \qquad \because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a^2+n^2} (a\cos nx + n \sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi a}}{a^2+n^2} (a \underbrace{\cos 2n\pi}_{1} + n \underbrace{\sin 2\pi n}_{0}) - \frac{1}{a^2+n^2}(a)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi a}}{a^2+n^2}(a+0) - \frac{a}{a^2+n^2}\right]$$

$$= \frac{a}{\pi(a^2+n^2)}\left(e^{2\pi a}-1\right)$$

Now,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi a}}{a^2+n^2}(a\sin 2n\pi - n\cos 2n\pi) - \frac{1}{a^2+n^2}(a\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{2\pi a}(a\times 0 - n) - (0-n)\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{2\pi a}(-n) + n\right] = \frac{(-n)}{\pi(a^2+n^2)}\left(e^{2\pi a}-1\right)$$

$$= \frac{n}{\pi(a^2+n^2)}\left(1-e^{2\pi a}\right)$$

Now,

$$f(x) = \frac{1}{2\pi a}\left(e^{2\pi a}-1\right) + \sum_{n=1}^{\infty}\left[\frac{a}{\pi(a^2+n^2)}\left(e^{2\pi a}-1\right)\cos nx + \frac{n}{\pi(a^2+n^2)}\left(1-e^{2\pi a}\right)\sin nx\right]$$

Functions Having points of discontinuity

Q. Find the fourier series to represent the function

(i) if

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$ (point of dis-continuity)

Solⁿ

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left\{(-\pi)[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left[(-\pi)(0+\pi) + \frac{\pi^2}{2}\right] = \frac{1}{2\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right)$$

$$= \frac{-\pi^2}{2\pi \times 2} = \frac{-\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= -\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx \qquad \int uv = u\int v - \int u'\int v$$

$$= -\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-1}{n}\left(0-(-0)\right)+\frac{1}{\pi n}\left[0+\frac{(-1)^n}{n^2}-\left(0+\frac{1}{n^2}\right)\right]$$

$$= 0 + \frac{1}{n^2\pi}\left((-1)^n - \cancel{1}\right)$$

$$= \frac{1}{\pi n^2}\left\{(-1)^n - \cancel{1}\right\} = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= (-1)\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \underset{I}{x}\,\underset{II}{\sin nx}\,dx$$

$$= -\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\frac{(-\cos nx)}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{n}\left(\cos 0 - \cos n\pi\right) + \frac{1}{\pi}\left[\frac{\pi}{n}\left(-\cos n\pi + 0\cos 0\right) + 0\right]$$

$$= \frac{1}{n}\left\{1 - \frac{(-1)^n}{\cancel{n}}\right\} + \frac{1}{\pi}\times\frac{\pi}{n}\left[0\times 1 - \frac{(-1)^n}{\cancel{n}}\right]$$

$$= \frac{1}{n}\left[1-(-1)^n \cancel{} - \frac{(-1)^n}{\cancel{n}}\right]$$

$$= \frac{1}{n}\left[1-2(-1)^n\right]$$

Now, $f(x) = \frac{-\pi}{4} + \sum \left[\frac{1}{\pi n^2}\left[(-1)^n - 1\right]\cos nx + \frac{1}{n}\left\{1 - 2(-1)^n\right\}\sin nx \right]$

When Interval is in $(0, 2l)$ or $(-l, l)$

If any function $f(x)$ is given with interval $(-l, l)$ then fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Q find the fourier series of $f(x) = \begin{cases} \pi x & : 0 < x < 1 \\ \pi(2-x) & : 1 < x < 2 \end{cases}$

Sol^n Interval is $0 < x < 2l$

here $0 < x < 2\times 1$, $l = 1$

Hence $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}\right]$

$$a_0 = \frac{1}{2\times 1}\int_0^2 f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

Fourier Series if $f(x)$ is Even / odd : in $[-\pi, \pi]$

(1) $f(x)$ is Even then F.S in interval $[-\pi, \pi]$ becomes

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \cancel{\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$$

$= 0$

($\sin nx$ = odd)

(2) $f(x)$ is odd then F.S becomes

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$$

odd × even

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

(odd × odd)

Q. Obtain F.S for $f(x) = x\cos x$, $-\pi < x < \pi$

Sol^n We know that $f(x) = x\cos x$ is odd function

$\therefore$ F.S will be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad , \quad a_0 = b_n = 0$$

Now, $b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x . \sin nx \, dx$ ——— (1)

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} x . \left[\frac{1}{\cancel{2}}\left\{\sin\left(\frac{x+nx}{2}\right) - \sin(x-nx)\right\}\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x . \left[\sin(1+n)x - \sin(1-n)x\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin\{(1+n)x\}\, dx - \frac{1}{\pi}\int_0^{\pi} x \sin\{(1-n)x\}\, dx$$

$$= \frac{1}{\pi}\left[-x\frac{\cos(1+n)x}{(1+n)} - 1\left(-\frac{\sin(1+n)x}{(1+n)^2}\right)\right]_0^{\pi} -$$

$$\frac{1}{\pi}\left[-x\frac{\cos(1-n)x}{(1-n)} - 1\left(-\frac{\sin(1-n)x}{(1-n)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi(1+n)^2}\left[-(1+n)x\cos(1+n)x + \sin(1+n)x\right]_0^{\pi} -$$

$$\frac{1}{\pi(1-n)^2}\left[(n-1)x\cos(1-n)x + \sin(1-n)x\right]_0^{\pi}$$

$$= \frac{1}{(1+n)^2\pi}\left[-\pi(1+n)\cos(1+n)\pi + \cancel{\sin(1+n)\pi}\right] + \frac{1}{}$$

$$= \frac{1}{(1+n)^2\pi}\left[-(1+n)\pi\cos\{(1+n)\pi\} + \cancel{\sin\{(1+n)\pi\}}^{\,0} - 0\right]$$

$$- \frac{1}{(1-n)^2\pi}\left[-(1-n)\pi\cos\{(1-n)\pi\} + \cancel{\sin(1-n)\pi}^{\,0}\right]$$

$$= \cancel{\frac{1}{(1+n)^2\pi}}\Big[$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos\{(\pi+n\pi)\}}{(1+n)} + \cancel{\frac{\sin(\pi+n\pi)}{(1+n)^2}}^{\,0} + \frac{\pi\cos(\pi-n\pi)}{(1-n)} - \cancel{\frac{\sin(\pi-n\pi)}{(1-n)^2}}^{\,0}\right]$$

$$= \begin{cases} \because \cos(\pi+n\pi) = -\cos n\pi = (-1)^n \\ \cos(\pi-n\pi) = -\cos n\pi = (-1)^n \\ \sin(\pi+n\pi) = -\sin n\pi = 0 \\ \sin(\pi-n\pi) = \sin n\pi = 0 \end{cases}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(\pi+n\pi)}{1+n} + \frac{\cos(\pi-n\pi)}{1-n}\right\}\right]$$

$$= -\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} = (-1)^n\left\{\frac{-\cancel{1}+n + \cancel{1}+n}{1-n^2}\right\}$$

$$= \frac{-2n(-1)^n}{(1-n^2)} \qquad \{n \neq 1\}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

to find b_1 from ① $\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin x \, dx = \frac{2}{\pi}\left[\int_0^{\pi} x\left(\frac{1}{2}\left\{\sin(x+x) - \sin(x-x)\right\}\right) dx\right.$$

$$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\pi} x \cdot \sin 2x \, dx = \frac{1}{\pi}\left[x \cdot \left(\frac{-\cos 2x}{2}\right) - 1 \cdot \left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\cancel{4}2\pi}\left[\left\{2\pi(-1) - 0\right\} - \left\{2 \times 0 (-1) - 0\right\}\right]$$

$$= \frac{(-1)2\pi}{4\cancel{2}\pi} = -\frac{1}{2}$$

Now, $f(x) = \left(-\frac{1}{2}\right)\sin x + \sum_{n=2}^{\infty} \frac{(-2)(-1)^n n}{(1-n^2)} \sin nx$

$$= -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{(n^2-1)} \sin nx$$

Numerical Methods

Computers & Mathematics $\xrightarrow{\text{Imp tools}}$ N. Method

tabulated data of an exp, } N. Method
algebraic equation solution } (accurate more answer)

ROOTS OF EQUATION

$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ are polynomials. or $f(x)$ is algebraic function.

Transcendental function:

A non-algebraic function is called a transcendental function.

Ex $f(x) = \ln x^3 - 0.7$

$g(x) = e^{-0.5x} - 5x$

$h(x) = \sin^2 x - x^2 - 2$ etc.

Root of $f(x)$:

A number α is said to be root or zero or solution of an equation $f(x) = 0$ if

$$f(\alpha) = 0.$$

Method of finding Solution:

1. Direct Methods (give exact value of roots in finite no. of steps)
2. Iterative methods (based on successive approx

① BISECTION METHOD:

1) find two points, say a and b such that $a < b$ and $f(a) \times f(b) < 0$
2) find the midpoint of a and b, say 't'
3) t is the root of the given function, else follow the next step.
4) Divide the interval $[a, b]$. Let $t = \frac{a+b}{2}$

If $f(t)f(a) < 0$, there exist a root betn (t, a)

If $f(t)f(b) < 0$, there exist a root betn (t, b)

Q. Perform 5 iterations/approximatn of the bisection method to obtain the smallest root of the eqn

$$f(x) = x^3 - 5x + 1$$

Soln $f(0) = 0^3 - 0 + 1 = 1$ (+ve)

$f(1) = 1^3 - 5\times1 + 1 = 1 - 5 + 1 = -3$ (-ve)

As $f(0)$ and $f(1)$ are of opposite sign roots lies between $(0, 1)$

① 1st Iteration / approximation

$$x_1 = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

$f(x_1) = f(0.5) = -1.375$ (-ve)

$\therefore$ The root lies between $(0, 0.5)$.

(2) 2nd Iteration:

$$x_2 = \frac{0+0.5}{2} = 0.25$$

$f(0.25) = -0.234375$ (-ve)

$\therefore$ The root lies between $(0, 0.25)$

(3) 3rd Iteration

$$x_3 = \frac{0+0.25}{2} = 0.125$$

$f(0.125) = 0.3769$ (+ve)

$\therefore$ The roots lies between $(0.125, 0.25)$

(4) 4th Iteration:

$$x_4 = \frac{0.125+0.25}{2} = 0.1875$$

$f(0.1875) = 0.06909$ (+ve)

$\therefore$ Root lies between $(0.1875, 0.25)$

(5) 5th Iteration:

$$x_5 = \frac{0.1875+0.25}{2} = 0.21875$$

~~f(0.218~~ $f(0.21875) = -0.0832$ (-ve)

~~Hence the approximation root is 0.21875.~~

$\therefore$ Root lies between $(0.1875, 0.21875)$

$$x = \frac{0.1875+0.21875}{2} = 0.203125$$

Q. Perform 5-Iterations of the bisection method to of

$$f(x) = \cos x - xe^x$$

Sol $f(x) = \cos x - xe^x$

$f(0) = 1$ (+ve

$f(1) = -2.1780$ (-ve)

$\therefore$ The root lies between in the internal $(0,1)$

1st Iteration

$$x_1 = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

$f(0.5) = \cos(0.5) - (0.5)e^{0.5} = 0.0532$ (+ve

$\therefore$ The root lies between $(0.5, 1)$

2nd Iteration :

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$f(0.75) = -0.8560$ (-ve)

$\therefore$ The root lies between $(0.5, 0.75)$

3rd Iteration :

$$x_3 = \frac{0.5+0.75}{2} = 0.625$$

$f(0.625) = -0.3566$ (-ve)

$\therefore$ The root lies between $(0.5, 0.625)$

4th Iteration :

$$x_4 = \frac{0.5+0.625}{2} = 0.5625$$

$f(0.5625) = -0.14129$ (-ve)

$\therefore$ The root lies between $(0.5, 0.5625)$.

5th Iteration:

$$x_5 = \frac{0.5 + 0.5625}{2} = 0.53125 \quad (0.5937)$$

$f(0.53125) = -0.0415$ (-ve)

$\therefore$ The root lies between (0.5, 0.53125)

$$x = \frac{0.5 + 0.53125}{2} = 0.515625 \text{ (Ans)}.$$

* Newton-Raphson Method :

1) Find two points a & b such that $a < b$ and $f(a) \cdot f(b) < 0$
2) choose x_0 that $f(x_0)$ nearest to '0'.
3) Root lies between (a, b)
4) find $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ until root are same.

Q Use the Newton-Raphson method to find a root of the equation $f(x) = x^3 - 2x - 5$

Sol^n

$f(x) = x^3 - 2x - 5$	$f(1) = 1 - 2 - 5 = -6$
$f'(x) = 3x^2 - 2$	$f(2) = 8 - 4 - 5 = -1$ (-ve)
	$f(3) = 27 - 6 - 5 = 16$ (+ve)

So, the root lies between (2, 3)

The formula for N-R method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Hence it gives $x_{k+1} = x_k - \frac{(x_k^3 - 2x_k - 5)}{3x_k^2 - 2}$

$$= \frac{3x_k^3 - 2x_k - x_k^3 + 2x_k + 5}{3x_k^2 - 2}$$

$$x_{k+1} = \frac{2x_k^3 + 5}{3x_k^2 - 2}$$

Now, Choose $x_0 = 2$, put $k=0$

$$x_1 = \frac{2x_0^3 + 5}{3x_0^2 - 2} = \frac{2(2)^3 + 5}{3(2)^2 - 2} = \frac{16 + 5}{12-2} = \frac{21}{10}$$

$$= 2.1$$

$$x_2 = \frac{2x_1^3 + 5}{3x_1^2 - 2} = \frac{2(2.1)^3 + 5}{3(2.1)^2 - 2} = \frac{23.522}{11.23} = 2.09$$

$$x_3 = \frac{2x_2^3 + 5}{3x_2^2 - 2} = \frac{2(2.09)^3 + 5}{3(2.09)^2 - 2} = \frac{23.25}{11.10} = 2.09$$

Hence the correct root is 2.09

Finite Differences and Interpolation

If $y = f(x)$ and x_i lies between x_0 and x_n

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	y_2	---	y_n

then the process of finding 'y' corresponding to any x_i between x_0 & x_n is called Interpolation.

The process of finding 'y' outside the given range is called extrapolation.

The study of the interpolation is based on the concept of differences of a function.

Finite differences formula

If $y = f(x)$ is tabulated as

x	x_0	x_0+h	x_0+2h	....	x_0+nh
y	y_0	y_1	y_2	----	y_n

then there are three ~~values~~ a formulas of differences

① forward differences (Δ)

Δ is called forward difference operator.

where $y_1 - y_0 = \Delta y_0$

$y_2 - y_1 = \Delta y_1$

$y_3 - y_2 = \Delta y_2$ ...... $y_{n+1} - y_n = \Delta y_n$

are first forward differences $\Delta y_n = y_{n+1} - y_n$

Second forward differences $\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$

p^{th} ~~[illegible]~~ forward differences $\Delta^p y_n = \Delta^{p-1} y_{n+1} - \Delta^{p-1} y_n$

② **Backward differences (∇)**

∇ is called backward difference operator.

$$y_1 - y_0 = \nabla y_1$$
$$y_2 - y_1 = \nabla y_2 \cdots\cdots$$

first backward differences $\nabla y_n = \nabla y_n - \nabla y_{n-1}$

Second backward differences $\nabla^2 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$

p^{th} backward differences $\nabla^p y_n = \nabla^p y_n - \nabla^p y_{n-1}$

③ **Central differences (δ)**

δ is the central difference operator

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2} \cdots\cdots\ y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$$

first ~~back~~ central difference. $y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$

2nd central differences $\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1 \cdots$ so on.

④ **Shift Operator (E)**: This operator is given by

$$E f(x) = f(x+h)$$
$$E^2 f(x) = f(x+2h) \cdots\cdots \boxed{E^n f(x) = f(x+nh) = y_{x+nh}}$$

The inverse operator E^{-1} is defined as $E^{-1} f(x) = f(x-h)$

⑤ **Averaging Operator (μ)**:

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

Relations between the operators :

1> $\Delta = E - 1$

2> $\nabla = 1 - E^{-1}$

3> $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

4> $\delta = E^{1/2} - E^{-1/2}$

5> $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

6> $E = e^{hD}$

Proof:

(1) $\Delta f(x) = f(x+h) - f(x)$

$= E f(x) - f(x)$

$\Delta f(x) = (E-1) f(x)$

$\Rightarrow \Delta = E - 1$

(2) $\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$\nabla f(x) = (1 - E^{-1}) f(x) \Rightarrow$

$\Rightarrow \nabla = (1 - E^{-1})$

3) $E\nabla f(x) = E(f(x) - f(x-h)) = Ef(x) - Ef(x-h)$

$= f(x+h) - f(x) = \Delta f(x) \Rightarrow \boxed{E\nabla = \Delta}$

also, $\nabla E f(x) = \nabla f(x+h) = f(x+h) - f(x) = \Delta f(x)$

$\therefore \boxed{\nabla E = \Delta}$

$\delta E^{1/2} f(x) = \delta f(x + \frac{h}{2}) = f(x + \frac{h}{2} + \frac{h}{2}) - f(x + \frac{h}{2} - \frac{h}{2})$

$= f(x+h) - f(x) = \Delta f(x)$

$\Rightarrow \delta E^{1/2} f(x) = \Delta f(x)$

$\Rightarrow \delta E^{1/2} = \Delta$

1) Interpolation with Equal Intervals

(a) Newton's forward Interpolation ✓
(b) Newton's Backward Interpolation ✓
(c) Central difference Interpolation

(a) Newton's forward Interpolation

For any real number p, we have defined E such that

$E^p f(x) = f(x+ph)$ where $y(x) = f(x_0+ph)$, $x = (x_0+ph)$

or $E^p f(x_0) = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p f(x_0)$

$p = \frac{x-x_0}{h}$, h = diff

$E = 1+\Delta$

$y_0 = f(x_0)$

$\Rightarrow E^p f(x_0) = (1+\Delta)^p f(x_0)$

$\Rightarrow f(x_0+ph) = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right] y_0$

$\Rightarrow f(x) = \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right]$

General

$\Rightarrow f^n$

(b) Newton's Backward Interpolation:

$f(x_n+ph) = E^p f(x_n)$

$= (E^{-1})^{-p} f(x_n)$

$= (1-\nabla)^{-p} f(x_n)$

$E^{-1} = 1-\nabla$

$y_n = f(x_n)$

$p = x - x_0$

$\Rightarrow f_n(x) = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\right] y_n$

$\Rightarrow f_n(x) = \left[f(x_n) + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots\right]$

for (a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

for (b) put $x = -\nabla$, $n = -p$

ii) Interpolation with Unequal Intervals

ⓐ Lagrange's Interpolation ✓

ⓑ Newton's Divided Difference formula

ⓐ Lagrange's Interpolation

If $y = f(x)$ takes the value $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

This is known as Lagrange's Interpolation formula for unequal intervals.

Numerical Integration

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

ⓐ Newton-Cotes formula

ⓑ Trapezoidal Rule

ⓒ Simpson's One-Third {Rule}

ⓐ Newton'- Cotes formula

Let $I = \int_a^b f(x)\,dx$ for

x	x_0	x_1	x_2	$\cdots x_n$
y	y_0	y_1	y_2	$\cdots y_n$

Let us divide (a, b) into n-sub intervals of width 'h' so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ ----, $x_n = x_0 + nh = b$ then

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\cdots\right] \quad (1)$$

This is known as Newton- Cotes quadrature formula. for $n = 1, 2, 3 \ldots$

ⓑ Trapezoidal Rule :- (any interval)

Put $n=1$ in eqn (1), we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

ⓒ Simpson's One-third Rule : (even interval)

Put $n=2$ in eqn (1) we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$