Vikash Polytechnic, Bargarh

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Lecture Note on *Mathematics-I*

Diploma 1st Semester

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Trigonometry

~~Angles~~

Meaning of Trigonometry :

It is a resulting form of two Greek words "Trigonon" and "Metron"

Trigonon → Triangle , Metron → Measurement

This means the study of properties of triangles. This includes measurement of angles and lengths.

We know, Ray (→), Line (↔), line Segment (•—•)

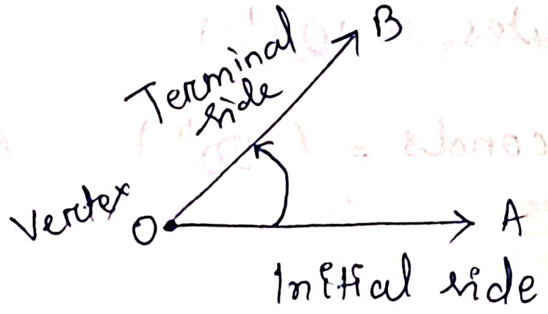
By angle Δ : Acute (<90°) , Right (90°), Obtuse (>90°)

By sides Δ : Equilateral (3 side same), Isoceles (2 side same) Scalene (no equal side).

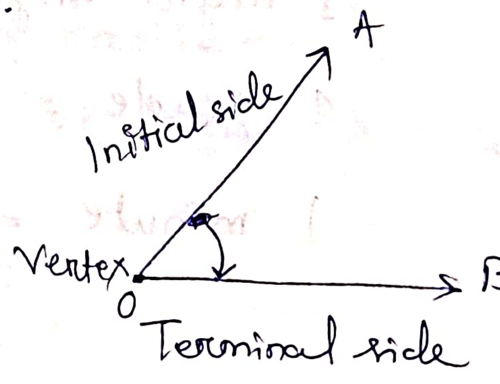
Angle :

It is a measure of rotation of a given ray about its initial point.

Original ray is called initial side of ~~the~~ angle. The final position of ray after rotation is called Terminal side of the angle.



1. Positive angle (anticlockwise)



2. Negative angle (clockwise direction)

The point of rotation is called Vertex.

Positive Angle:

If the direction of rotation is anticlockwise then the angle is said to be positive angle.

Negative Angle:

If the direction of rotation is clockwise then the angle is said to be Negative angle.

System of Measurement of Angles

There are two-system of Measuring angles.

① English system or Sexagesimal:

1 right angle = 90° = 90 degree

1° = 60 minutes = $(60')$

$1'$ = 60 seconds = $(60'')$

② French System or Centesimal:

1 right angle = 100 grades (100^G)

1 grade = 100 minutes $(100')$

1 minute = 100 seconds = $(100'')$

Units of Measurement of Angles :

There are 3- units of measurement for angles

1) Degree
2) Grade
3) Radian

1) **Degree** : device - protractor

A semi circular disk with degree from $0°$ to $180°$.

2) **Grade** :

In trigonometry the gradian also known as the "gon".

3) **Radian** :

The angle subtended at the centre of unit circle by a unit arc length on the circumference is called one radian. It is denoted by $1R$.

Relation between 3- Systems of Measurement of Angles

Let D be the number of degrees.
R be the number of radians
G be the number of Grades in an angle 'a'.

then

$$\frac{D}{90°} = \frac{G}{100} = \frac{2R}{\pi}$$

where approximate value of π is $\boxed{\frac{22}{7} \text{ or } 3.14}$

Now, by the relation

$$\frac{D}{90^\circ} = \frac{2R}{\pi}$$

$$1 \text{ Radian} = .\pi^\circ$$

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① π radian = 180°

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$$

② π Radian = 2×100^G

$$1 \text{ Radian} = \left(\frac{200}{\pi}\right)^G$$

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③ $$1^\circ = \left(\frac{\pi}{180}\right)^R$$

④ $$90^\circ = 100^G$$

$$1^\circ = \left(\frac{100}{90}\right)^G = \frac{10}{9}^G$$

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⑤ $$1^G = \frac{9}{10}^\circ$$

⑥ $$100^G = \frac{\pi}{2}^R$$

$$1^G = \left(\frac{\pi}{200}\right)^R$$

Example (i) $30^\circ = 30 \times \frac{\pi}{180} = \left(\frac{\pi}{6}\right)^R$, $30 \times \frac{10}{9} = \left(\frac{300}{9}\right)^G$

$$2^G = \left(\frac{9}{10} \times 2\right)^\circ = \frac{9}{5}^\circ = 1.8^\circ = 1\frac{4}{5}$$

$$= 1^\circ 48'$$

$$2^G = 2 \times \frac{\pi}{200} = \frac{\pi}{100}$$

$$\left(\frac{\pi}{3}\right)^R = \frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$$

$$\left(\frac{\pi}{3}\right)^R = \frac{\pi}{3} \times \frac{200}{\pi} = \left(\frac{200}{3}\right)^G$$

$1' = 60''$

$1^\circ = 60'$

$1' = \left(\frac{1}{60}\right)^\circ$

$\frac{4}{5} \times 60 = 48'$

Q. $40^\circ 20' \rightarrow$ Radian

$$40^\circ = \cancel{40}^{2} \times \frac{\pi}{\cancel{180}_{9}} = \frac{2\pi}{9}$$

$$20' = 20 \times \left(\frac{1}{60}\right)^\circ = \left(\frac{1}{3}\right)^\circ$$

$$\left(\frac{1}{3}\right)^\circ = \frac{1}{3} \times \frac{\pi}{180} =$$

$$\frac{2\pi}{9} + \frac{\pi}{3\times 180} = \frac{1}{3}\left(\frac{2\pi}{3} + \frac{\pi}{180}\right) = \frac{\pi}{3}\left(\frac{2\times 60+1}{180}\right)$$

$$= \left(\frac{120+1}{180}\right)\frac{\pi}{3} = \frac{121\pi}{540}$$

OR

$$40^\circ 20' = 40^\circ\left(\cancel{20} \times \frac{1}{\cancel{60}}\right)^\circ$$

$$= 40^\circ\left(\frac{1}{3}\right)^\circ$$

$$= \left(40\frac{1}{3}\right)^\circ = 40\frac{1}{3} \times \frac{\pi}{180}$$

$$= \frac{121}{3} \times \frac{\pi}{180} = \frac{121\pi}{540}$$

Q. Convert 6R to Degree Measure

$$6 \text{ Radian} = 6\times\left(\frac{180}{\pi}\right)^\circ = \left(\frac{1080}{\pi}\right)^\circ = \frac{1080}{22}\times 7 = \frac{\cancel{7560}}{\cancel{22}} = \frac{3780}{11}$$

$$343\frac{7}{11} \text{ degree} = \left(343+\frac{7}{11}\right)^\circ \text{ degree} = 343^\circ + \frac{7}{11}\times 60'$$

$$= 343^\circ + \frac{420'}{11} = 343^\circ + \left(38\frac{2}{11}\right)' = 343^\circ + \left(38 + \frac{2}{11}\right)'$$

$$= 343^\circ + 38' + \frac{2}{11}\times 60'' = 343^\circ 38'\left(\frac{120}{11}\right)'' =$$

$$= 343^\circ + 38' + \left(1\frac{10}{11}\right)'' = 343^\circ + 38' (1 + \frac{10}{}$$

Trigonometrical Ratios of Allied Angle :-

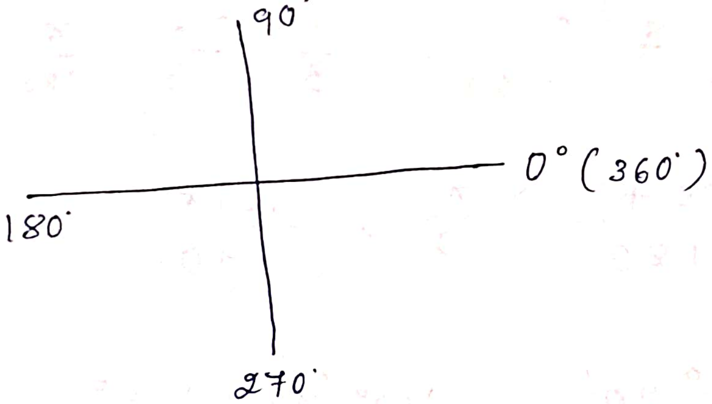
Two angles are said to be allied when their sum or difference is either zero or a multiple of $90°$.

These are usually taken as $(-\theta)$, $(90°-\theta)$, $(90°+\theta)$, $(180-\theta)°$, $(180+\theta)°$, $(270-\theta)°$, $(270+\theta)°$, $(360-\theta)°$, $(360+\theta)$

$270 / 90° \pm \theta$ = fun^n change

$360° / 180° \pm \theta$ = not change

Sign = ASTC



1) $(-\theta)$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

2) $(90°-\theta)°$

$\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

$\tan(90-\theta) = \cancel{\tan\theta}\ \cot\theta$

3) $(90+\theta)°$

$\sin(90°+\theta) = \cos\theta$

$\cos(90°+\theta) = -\sin\theta$

$\tan(90+\theta) = -\cot\theta$

4) $(180-\theta)$

$\sin(180-\theta) = \sin\theta$

$\cos(180-\theta) = -\cos\theta$

$\tan(180-\theta) = -\tan\theta$

5) $\sin(180+\theta) = -\sin\theta$

$\cos(180+\theta) = -\cos\theta$

$\tan(180+\theta) = \tan\theta$

6) $\sin(270-\theta) =$

$\cos(270-\theta) =$

$\tan(270-\theta) =$

7) $\sin(360-\theta) =$
$\cos(360-\theta) =$
$\tan(360-\theta) =$

8) $\sin(360+\theta) = \sin\theta$
$\cos(360+\theta) = \cos\theta$
$\tan(360+\theta) = \tan\theta$

T-ratios Table

$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$(\frac{\pi}{6})\,30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$(\frac{\pi}{4})\,45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$(\frac{\pi}{3})\,60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$(\frac{\pi}{2})\,90°$	1	0	∞
$(\pi)\,180°$	0	-1	0
$(270°)\,\frac{3\pi}{2}$	-1	0	∞

<u>Rule-1</u>: If the angle θ (θ is an acute angle) is increased or decreased by a multiple of $360°$ ($2\pi R$) then t-ratios of an angle remain same.

<u>Rule-2</u>:
t-ratios of an angle $(-\theta), \pi \pm \theta, 2\pi \pm \theta, \ldots$ remain same.

<u>Rule-3</u>: t-ratios of an angle $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta, \ldots$ changes as given below

$\sin \leftrightarrow \cos$
$\tan \leftrightarrow \cot$
$\sec \leftrightarrow \mathrm{cosec}$.

Rule-4 : for sign fixing
follow "ASTC"

A = All ratios are positive
S = only sin & cosec are positive
T = only tan & cot are positive
C = only cos & sec are positive

$\sin\theta$	All
$\tan\theta$	$\cos\theta$

Ex (i) $\sin 480^\circ = \sin(450^\circ + 30^\circ)$
$= \cos 30^\circ = \frac{\sqrt{3}}{2}$

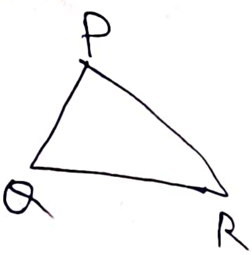
(ii) $\tan(\pi + \theta) = \tan\theta$

(iii) $\sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec}\theta$

(iv) $\operatorname{cosec}^2\left(\frac{7\pi}{6}\right) = \left[\operatorname{cosec}\frac{7\pi}{6}\right]^2 = \left[\operatorname{cosec}\left(\pi + \frac{\pi}{6}\right)\right]^2$
$= \left[-\operatorname{cosec}\frac{\pi}{6}\right]^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

(v) $\cot\left(\frac{5\pi}{6}\right) = \cot\left(\pi - \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$

Q For ΔPQR, Prove that $\sin(Q+R) = \sin P$



$$m\angle P + m\angle Q + m\angle R = 180^\circ = \pi$$

L.H.S $\quad \sin(Q+R) = \sin(\pi - P)$

$$= \sin P$$

$$= R.H.S$$

Sum & Difference formulae :-

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$$

$$\cot(A+B) = \frac{\cot A . \cot B - 1}{\cot A + \cot B}$$

$$\cot(A-B) = \frac{\cot A . \cot B + 1}{\cot A - \cot B}$$

$$\sin(A+B) . \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$\cos(A+B) . \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

Applications of Sum-Difference formula :-

Q1) Evaluate $\sin 75^\circ$ $\quad (75^\circ = 45^\circ + 30^\circ)$

We know, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cdot \cos 30^\circ + \cos 45^\circ \sin 30$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right) = \frac{1}{2\sqrt{2}}(\sqrt{3}+1)$ (Ans.)

$= \frac{\sqrt{2}}{2\sqrt{2}\sqrt{2}}(\sqrt{3}+1) = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

Q2 Evaluate $\tan 15^\circ$,

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$ $\quad \frac{(30+15^\circ)}{(45-30)}$

$\tan(45-30) = \frac{\tan 45 - \tan 30^\circ}{1 + \tan 45^\circ \cdot \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{3-1} = \frac{3+1-2\sqrt{3}}{2} = \frac{4-2\sqrt{3}}{2}$

$= \frac{\cancel{2}(2-\sqrt{3})}{\cancel{2}} = 2-\sqrt{3}$

Q Prove that $\tan 57^\circ = \dfrac{\cos 12^\circ + \sin 12}{\cos 12 - \sin 12^\circ}$

$57 - 12 = 45$, $\tan 45 = 1$

Soln $\tan 57 = \tan(45+12) = \dfrac{\tan 45 + \tan 12}{1 - \tan 45 . \tan 12}$

$$= \left(\frac{\frac{1}{\sqrt{2}} + \tan 12}{1 - \frac{1}{\sqrt{2}}\tan 12}\right) = \frac{\cancel{\sqrt{2}}\tan 12 + \cancel{\sqrt{2}}\,1}{1\,\cancel{\sqrt{2}} - \tan 12}$$

$$= \frac{1 + \tan 12}{1 - \tan 12} = \frac{1 + \frac{\sin 12}{\cos 12}}{1 - \frac{\sin 12}{\cos 12}}$$

$$= \frac{\cos 12^\circ + \sin 12}{\cos 12^\circ - \sin 12^\circ} =$$

Q Evaluate $\cos^2\left(\frac{\pi}{4} + x\right) - \sin^2\left(\frac{\pi}{4} - x\right)$

$\Rightarrow \cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$

Now, $\cos^2\left(\frac{\pi}{4}+x\right) - \sin^2\left(\frac{\pi}{4}-x\right) = \cos\left(\frac{\pi}{4}+x+\frac{\pi}{4}-x\right).\cos\left(\frac{\pi}{4}+x-\frac{\pi}{4}+\right.$

$$= \cos\frac{\pi}{2} \cdot \cos 2x$$

$$= 0 \times \cos 2x$$

$$= 0$$

sum / diff to

Product formulae (after product to sum / diff formula)

(1) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$

(2) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(3) $\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$

(4) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

or $= 2\sin\frac{D+C}{2}\sin\frac{D-C}{2}$

Q1 $\sin 65^\circ + \cos 65^\circ = \sqrt{2}\cos 20^\circ$

LHS:

Sol^n $\sin 65^\circ + \cos(90^\circ - 25^\circ)$

$= \sin 65^\circ + \sin 25^\circ$ $\quad 65^\circ = 90^\circ - 25^\circ$

$= 2\sin\left(\frac{65+25}{2}\right)\cdot\cos\left(\frac{65-25}{2}\right)$

$= 2\sin 45^\circ\cos 20^\circ = \frac{2}{\sqrt{2}}\cos 20^\circ = \sqrt{2}\cos 20^\circ = RHS$

Q2 Prove that $\cot 2\theta + \tan\theta = \operatorname{cosec} 2\theta$

LHS: $\cot 2\theta + \tan\theta = \frac{\cos 2\theta}{\sin 2\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta\cdot\cos 2\theta + \sin\theta\sin 2\theta}{\cos\theta\cdot\sin 2\theta}$

$= \frac{\cos(\theta - 2\theta)}{\cos\theta\sin 2\theta} = \frac{\cos(-\theta)}{\cos\theta\cdot\sin 2\theta}$

$= \frac{\cos\theta}{\cos\theta\cdot\sin 2\theta} = \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta$

Product to sum or difference :-

1) $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

2) $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

3) $2\cos\alpha\cdot\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

4) $-2\sin\alpha\cdot\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

or $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

Short Trick :

$2SC = S+S$	$2CC = C+C$
$2CS = S-S$	$-2SS = C-C$

Q) Prove that $\sin 10^\circ \cdot \sin 30^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ = \frac{1}{16}$

Soln L.H.S : $\sin 10^\circ \cdot \sin 30^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ$

$= \sin 10 \cdot \frac{1}{2} \cdot \sin 50^\circ \cdot \sin 70^\circ$

$= \frac{1}{2}(\sin 10 \cdot \sin 50^\circ \cdot \sin 70^\circ)$

$= \frac{1}{2}\times\frac{1}{2}\{(2\sin 10\cdot\sin 50^\circ)\sin 70^\circ\}$

$= \frac{1}{4}\left[-\{\cos(10+50^\circ) - \cos(10-50)\}\sin 70^\circ\right]$

$= \frac{1}{4}\left[-\{\cos 60^\circ - \cos(-40^\circ)\}\right]\sin 70^\circ$

$= \frac{1}{4}\left\{\left(-\frac{1}{2} + \cos 40^\circ\right)\sin 70^\circ\right\}$

$= \frac{1}{4}\left[-\frac{1}{2}\sin 70^\circ + \sin 70^\circ\cos 40^\circ\right]$

$= \frac{1}{4}\left[-\frac{1}{2}\sin 70^\circ + \frac{1}{2}(2\sin 70^\circ \cos 40^\circ)\right]$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}\left[+\sin(70^\circ - 40) + \sin(70^\circ + 40^\circ)\right]$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}(+\sin 30^\circ + \sin 110^\circ)$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}(+\frac{1}{2} + \sin 110^\circ)$

$= \frac{1}{8}\left[\frac{1}{2} + \sin 110^\circ - \sin 70^\circ\right]$

$= \frac{1}{8}\times\left[\frac{1}{2} + \sin(180 - 70^\circ) - \sin 70^\circ\right]$

$= \frac{1}{16} + \frac{1}{8}(\sin 70^\circ - \sin 70^\circ) = \frac{1}{16} = R.H.S$ (proved)

(1) Prove $2\cos\frac{5\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}$

L.H.S : $2\cos\frac{5\pi}{12}\cos\frac{\pi}{12}$

$= \cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right)$

$= \cos\left(\frac{6\pi}{12}\right) + \cos\left(\frac{4\pi}{12}\right)$

$= \cos\frac{\pi}{2} + \cos\frac{\pi}{3} = 0 + \frac{1}{2} = \frac{1}{2} = R.H.S$

Q $2\sin\frac{5\pi}{12}\cdot\cos\frac{\pi}{12} = ?$

$\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{3}\right) = 1 + \frac{\sqrt{3}}{2}$

(8) t-ratios of $2A, 3A, \frac{A}{2}$

(1) $\sin 2A = 2\sin A\cos A$

$$= \frac{2\tan A}{1+\tan^2 A}$$

$\cos 2A = \cos^2 A - \sin^2 A$

$$= 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}, \quad A \neq (2n+1)\frac{\pi}{4}$$

(2) $\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}, \quad A \neq (2n+1)\frac{\pi}{6}$$

(3) $\sin A = 2\sin\frac{A}{2}\cdot\cos\frac{A}{2}$

$$= \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$$= 2\cos^2\frac{A}{2} - 1$$

$$= 1 - 2\sin^2\frac{A}{2}$$

$$= \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

Q Prove that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

A L.H.S : $\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta \cdot \cos\theta}{1 + 2\cos^2\theta - 1}$ DC

$$= \frac{\cancel{2}\cos\theta \sin\theta}{\cancel{2}\cos^2\theta} = \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta = R.H.S$$

Q Prove that : $\frac{1+\sin 2\theta}{1-\sin 2\theta} = \tan^2\left(\frac{\pi}{4}+\theta\right)$ DC

L.H.S : $\frac{1+\sin 2\theta}{1-\sin 2\theta} = \frac{1 + \frac{2\tan\theta}{1+\tan^2\theta}}{1 - \frac{2\tan\theta}{1+\tan^2\theta}}$

$$= \frac{1+\tan^2\theta + 2\tan\theta}{1+\tan^2\theta - 2\tan\theta} = \frac{(1+\tan\theta)^2}{(1-\tan\theta)^2}$$

$$= \left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2 = \tan^2\left(\frac{\pi}{4}+\theta\right)$$

$$\because \frac{1+\tan\theta}{1-\tan\theta} = \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4} \cdot \tan\theta}$$

$$= \tan\left(\frac{\pi}{4}+\theta\right)$$

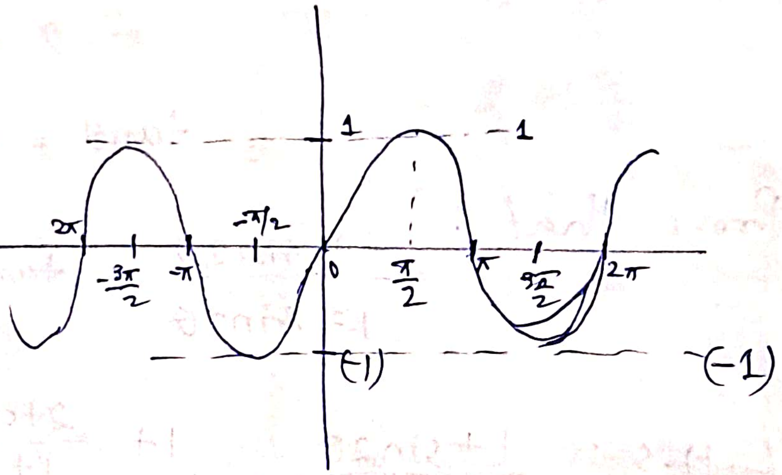
GRAPH OF THE FUNCTIONS :-

Periodic graph - The shape / graph repeats itself exactly after a certain amount of time.

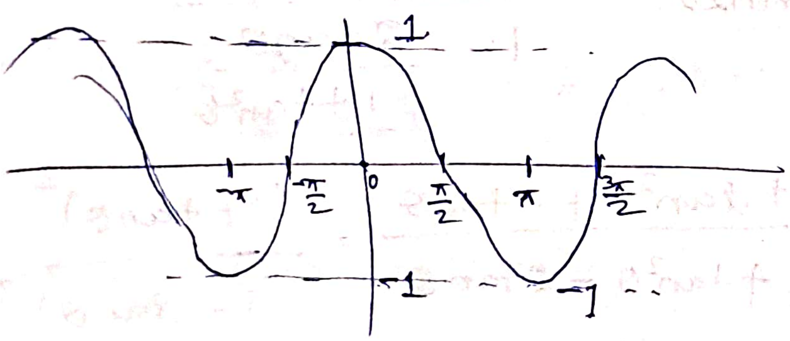
ⓐ Sine function

Domain = R

Range = [-1, 1]



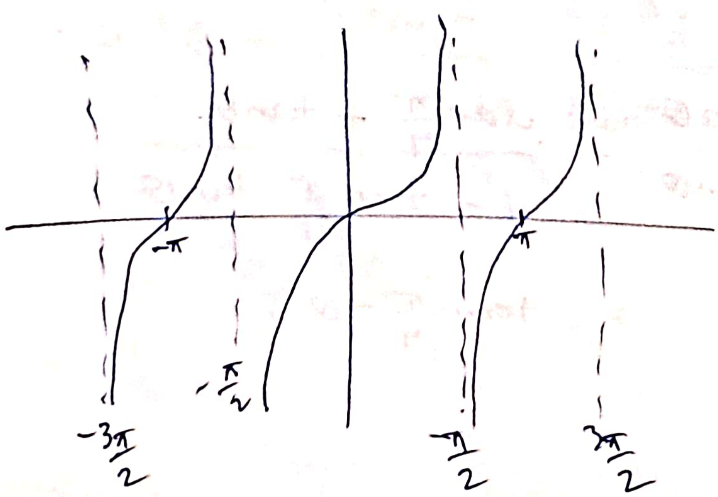
ⓑ Cosine



Domain = R

Range = [-1, 1]

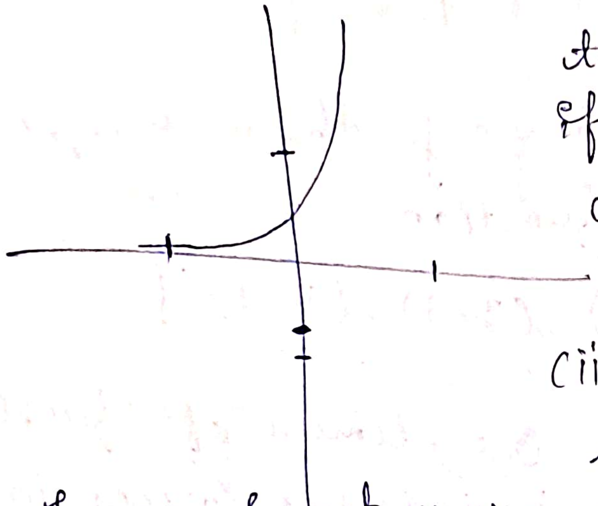
③ Tangent



Domain = $R - [(2n+1)\frac{\pi}{2}]$

Range = R

(4) Exponential function



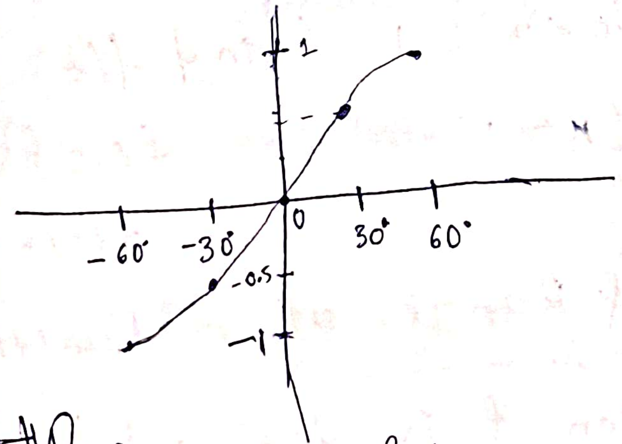
Q. Draw the graph of $y = \sin x$ $(-90° < x < 90°)$

Let $x = -60°$

$y = \sin(-60°) = -\sin 60°$

$= \frac{-1.73}{2} = -0.86$

x	$-60°$	$-30°$	$0°$	$30°$	$60°$
$y = \sin x$	-0.86	-0.5	0	0.5	0.86



HW Draw graph by Table making.

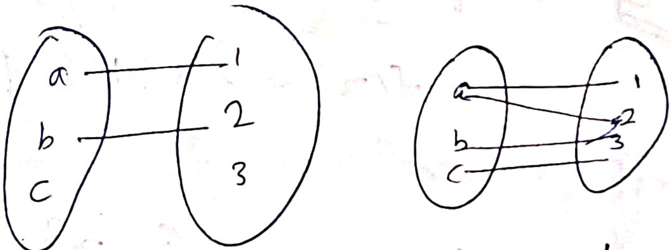
Function

A relation from a set X to Y is said to be a function if

(i) For all elements in X there is an image in Y.

(ii) Every element in set X should have one and only ~~set~~ image in Y.

function　　　　function



not a function　　　　not a functi

Q. Check whether it is a function or not ?

(i) $R = \{(2,1), (3,1), (4,2)\}$

Since 2, 3, 4 are having their unique image this relation is a function.

(ii) $R = \{(2,2), (2,4), (3,3), (4,4)\}$

Since $\left.\begin{array}{l} f(2) = 2 \\ f(2) = 4 \end{array}\right\}$ One element of first set having two images in second set

(iii) $R = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)\}$

It is a function.

Q. A function is defined by $f: N \longrightarrow N$

$f(x) = 2x+1$, find $f(1)$, $f(2)$, $f(3)$

$f(1) = 3$, $f(2) = 5$, $f(3) = 7$

Q. If $f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \leq x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ find $f(\frac{1}{2})$, $\frac{1}{2}$

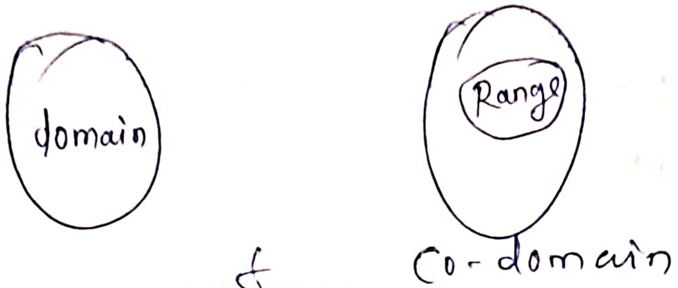
$f(-2) = 4$

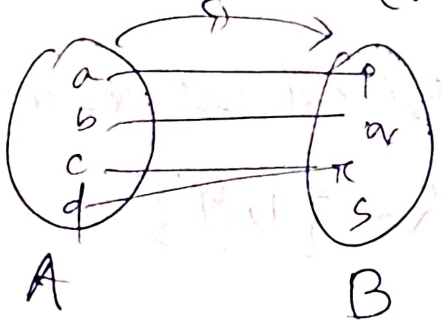
$f(2) = \frac{1}{2}$

Domain, Co domain & Range of a function

① ~~Domain~~: If a function f is defined from a set A to set B then for $f: A \rightarrow B$, the first set A is called as ~~the~~ domain of the function f and the set B is called co-domain of the function f.

The set of all ~~fun~~ image of the ~~function~~ elements of A is called the range of function f.





Domain : $\{a, b, c, d\} = A$

co-domain : $\{p, q, r, s\} = B$

Range $= \{p, q, r\}$

Methods for finding Domain :

(*) ~~for~~ $\sqrt{f(x)}$ and $\frac{f(x)}{g(x)}$, $g(x) \neq 0$

(1) for expression like $\sqrt{f(x)}$, & D_1 = Domain of $f(x)$

$$\text{Domain of } \left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$$

(2) for expression $\frac{f(x)}{g(x)}$, $g(x) \neq 0$, D_1 = Domain of $f(x)$

D_2 = " of $g(x)$

$$\text{Domain of } \frac{f(x)}{g(x)} = D_1 \cap D_2 - \{g(x) = 0\}$$

(3) $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$

Q. Find domain of the function $f(x) = \dfrac{x^2+3x+5}{x^2-5x+4}$

Soln find $g(x) = 0$

$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$
$$x = 4, x = 1$$

The function f is not defined at $x = 4, x = 1$.

Hence domain $f(x) = R - \{1, 4\}$

Q2 $f(x) = \sqrt{x-1}$

$f(x) \geq 0$

$\Rightarrow \sqrt{x-1} \geq 0 \Rightarrow x - 1 \geq 0$

$\Rightarrow x \geq 1$

i.e domain of $f(x)$ is $[1, \infty)$

Range $[0, \infty)$

Special functions:

1) Constant function:

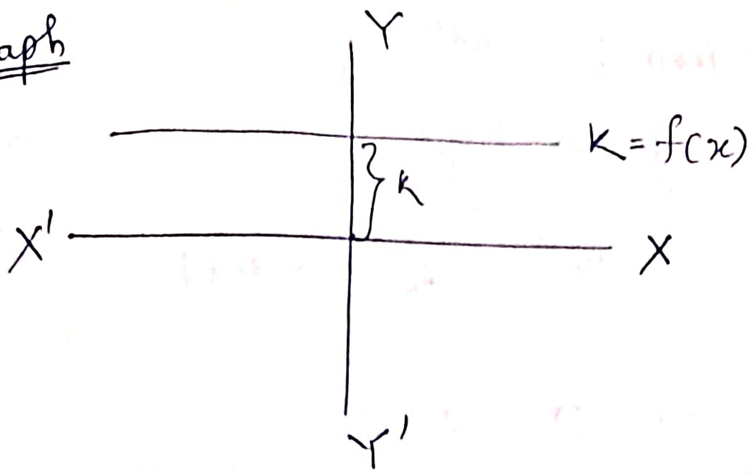
A function is said to be a constant function if

$$f(x) = k, \forall x \in \mathbb{R}$$

domain $= \mathbb{R}$

Range $= \{k\}$

graph



Ex//

$f(x) = 2$

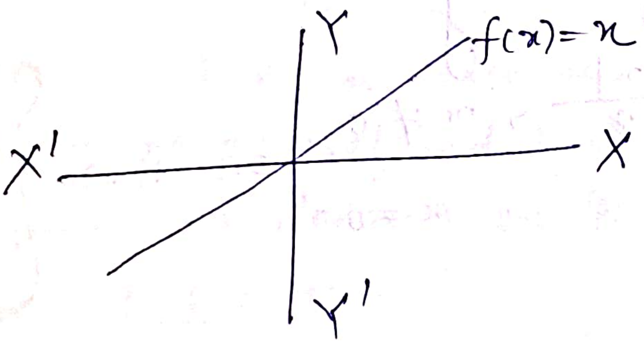
$f(x) = -1$

2) Identity function:

A function f(x) is ~~identifi~~ defined as an identity function if

$$f(x) = x, \quad \forall \text{ (for all) } x \in \mathbb{R}.$$

domain = R = Range



$f(1) = 1$

$f(2) = 2$

$f(-3) = -3$

3) Modulus function:

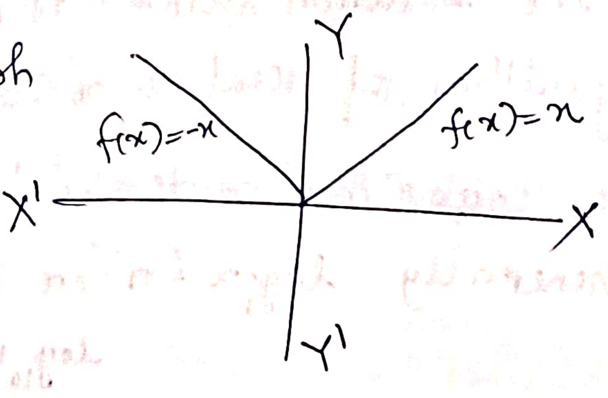
A function f(x) is said to be a Modulus function if

$$f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

domain = R

Range = R^+

graph



4) **Greatest Integer function :**

This function is given by

$f(x) = [x] = n$, where $n \leq x < n+1$

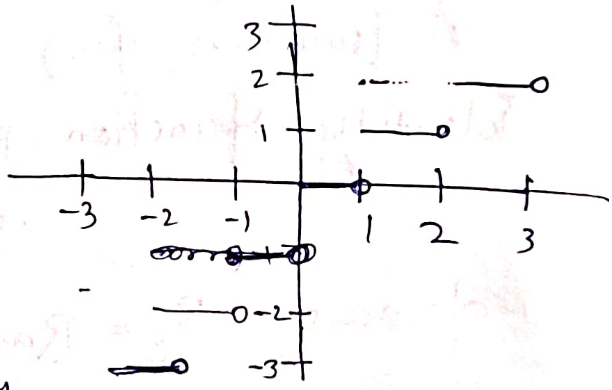
Ex

$[2.1] = 2$, as $2 \leq 2.1 < 3$

$[-2.1] = -3$, as $-3 \leq (-2.1) < -2$

domain = $\mathbb{R}$

Range = Integers ($\mathbb{Z}$)



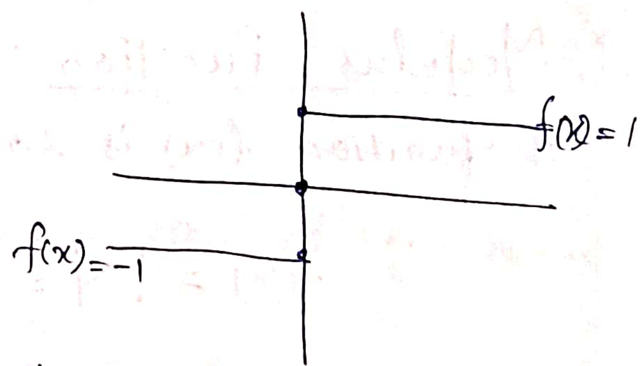
5) **Signum function :**

This function is given by

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Domain = $\mathbb{R}$

Range = $\{-1, 0, 1\}$



6) **Logarithmic function :**

The notation ($\ln x$) is used to denote the natural logarithm of real no x. and Log is $\log_e x$ (read as logarithm x to the base e)

Generally $\log_b x = n$ or $b^n = x$ Ex $\log_3 27 = ?$ (3)

$\log_{10} 100 = 2$ $\log_2 8 = ?$ (3)

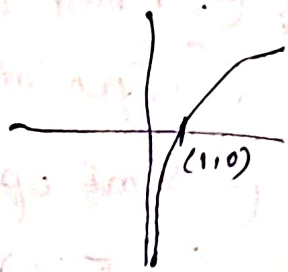
$\log_5(125) = 3$

Natural log = base "e" = 2.71828, Common log base = 10

Properties of logarithm:

Let m and n be arbitrary positive numbers s.t $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ then

1) $\log_a a = 1$, $\log_a 1 = 0$, $\log_b 0 =$ undefined



2) $\log_a b \cdot \log_b a = 1 \rightarrow \log_a b = \frac{1}{\log_b a}$

3) $\log_a (mn) = \log_a m + \log_a n$

4) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

5) $\log_a m^n = n \log_a m$ 6) $a^{\log_a m} = m$

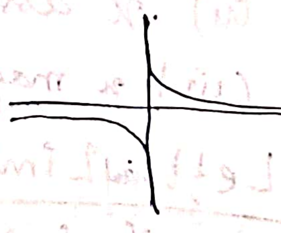
Exponential function:

Let $a \neq 1$ be a positive real number, then $f: R \rightarrow (0, \infty)$ defined by $f(x) = a^x$ called exponential function. Its domain is R and range is $(0, \infty)$.

Reciprocal function:

~~The function that associates~~ for each non-zero value of x, $f(x) = \frac{1}{x}$ is called Reciprocal function.

Domain = Range = $R - \{0\}$



Even function:

A function f is even if for all $x \in D$

$$f(-x) = f(x).$$

Odd function:

A function f is odd if for all $x \in D$

$$f(-x) = -f(x).$$

Interval

The set of the numbers between any two real number is called interval.

(a) closed intervals : $[a,b] = \{x \mid a \le x \le b\}$

(b) Open interval : (a,b) or $]a,b[= \{x \mid a < x < b\}$

(c) Semi open or semi close interval

$[a,b)$ or $[a,b[= \{x \mid a \le x < b\}$

$(a,b]$ or $]a,b] = \{x \mid a < x \le b\}$

LIMIT OF A FUNCTION

Indeterminate form, $0 \times \infty$, 0^0, 1^∞, $\infty - \infty$, $\frac{\infty}{\infty}$, ∞^0, $\frac{0}{0}$

If $y = f(x)$ has a indeterminate value ~~than~~ at $x = a$ then we consider points nearest to 'a' (from left or right) which gives a definite unique number 'l' is called limit of $f(x)$ at $x = a$. We write it as

$$\lim_{x \to a} f(x) = l$$

$(x \to a)$ or x tends to a meaning:-

(i) $x \neq a$

(ii) x assumes values nearer to 'a'

(iii) x may approach from left or right.

Left hand Limit

If $f(x)$ gives finite value for x tends to 'a' from left is called LHL. It is given by

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Right hand limit:

If $f(x)$ gives finite value for x tends to 'a' from right is called RHL. It is given by

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Existence of limit:

The limit 'l' of a function at $x = a$ exists. if

(i) $\lim_{x \to a} f(x)$ exists

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Ex If $f(x) = \begin{cases} x^2+2, & x \geq 1 \\ 2x+1, & x < 1 \end{cases}$ then $\lim_{x \to 1} f(x) = ?$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} [2(1-h)+1] = 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} [(1+h)^2+2] = 3$$

Since LHL = RHL. So $\lim_{x \to 1} f(x) = 3$.

Q ~~Some standard limits~~

$$\lim_{x \to (-1)} \frac{x^2-1}{x^2+3x+2}$$

$$\Rightarrow \lim_{x \to (-1)} \frac{(x-1)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}} = \frac{-1-1}{-1+2} = -2$$

Standard limits

(i) $\lim_{x \to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$

(ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a, \ a > 0$

(iv) $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

$n =$ +ive Int

Q1 $\lim_{x\to 2} \frac{x^n - 2^n}{x-2} = 80$, find $n = ?$

$\Rightarrow n 2^{n-1} = 80$

$\Rightarrow n 2^{n-1} = 80$

$\Rightarrow$ ~~$n2^{n}=160$~~

$$\begin{array}{r|l} 2 & 80 \\ 2 & 40 \\ 2 & 20 \\ 2 & 10 \\ & 5 \end{array}$$

$80 = 5 \times 2^4$

Q2 $\lim_{x\to\sqrt{2}} \frac{x^4 - 4}{x - \sqrt{2}}$

$= \lim_{x\to\sqrt{2}} \frac{x^4 - (\sqrt{2})^4}{x-\sqrt{2}}$

$= 4(\sqrt{2})^{4-1} = 4(\sqrt{2})^3 = 8\sqrt{2}$

Q3 $\lim_{x\to 16} \frac{x^{3/4} - 8}{x - 16}$

$\Rightarrow \lim_{x\to 16} \frac{x^{3/4} - (16)^{3/4}}{x-16}$

$= \frac{3}{4}(16)^{\frac{3}{4}-1} = \frac{3}{4}(16)^{-\frac{1}{4}} = \frac{3}{4}\left(\frac{1}{16}\right)^{\frac{1}{4}}$

$= \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$

Q4 $\lim_{x\to 0} \frac{\sin 3x}{5x}$

$= \lim_{x\to 0} \left(\frac{\sin 3x}{5x} \times \frac{3}{3}\right)$

$= \frac{3}{5} \lim_{3x\to 0} \frac{\sin 3x}{3x} \quad \left[\begin{array}{l}\text{as } x\to 0 \\ 3x \to 0\end{array}\right)$

$= \frac{3}{5} \times 1 = \frac{3}{5}$

Q5 $\lim_{x\to 0} \frac{5x\cos x - 2\sin x}{7x + 5\sin x}$

$= \lim_{x\to 0} \frac{\left(\frac{5x\cos x - 2\sin x}{x}\right)}{\left(\frac{7x + 5\sin x}{x}\right)}$

$= \lim_{x\to 0} \frac{\left(\frac{5x\cos x}{x} - \frac{2\sin x}{x}\right)}{\left(\frac{7x}{x} + \frac{5\sin x}{x}\right)}$

$= \lim_{x\to 0} \frac{\left(5\cos x - 2\frac{\sin x}{x}\right)}{\left(7 + 5\frac{\sin x}{x}\right)}$

$= \frac{5\lim_{x\to 0}\cos x - 2\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0} 7 + 5\lim_{x\to 0}\frac{\sin x}{x}}$

$= \frac{5\times 1 - 2\times 1}{7 + 5\times 1} = \frac{3}{12} = \frac{1}{4}$

Q6 $\lim_{x\to 0}\left(\frac{2^{x+3} - 8}{x}\right)$

$= \lim_{x\to 0} \frac{2^x \cdot 2^3 - 2^3}{x} = \lim_{x\to 0} \frac{2^3(2^x - 1)}{x}$

$= 2^3 \lim_{x\to 0} \frac{2^x - 1}{x}$

$= 2^3 \log_e 2$

$= 8\log_e 2 = 8\log_e 2^8$

Functions & Limit (Unit-2)

Definition

A function is one type of relation between two sets.

" A function from a set A to set B is a rule that assigns to each elements of set A to one and only one element of the set B."

Mathematically it is given by $f : A \longrightarrow B$ where

$$y = f(x), \quad x \in A, \quad y \in B$$

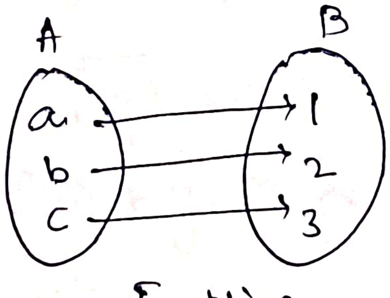
we say 'y' is the image of 'x' under f.

'x' is the pre-image of 'y'.

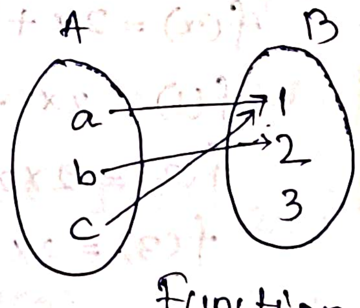
OR two things always be kept in mind

(i) Each element of the set A has its image in B.

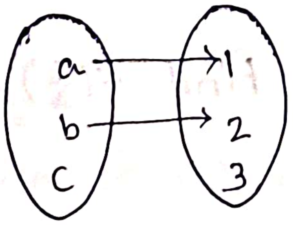
(ii) Every element in A set should have one and only one image. (unique image)



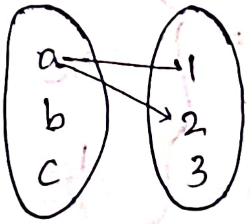
Function



Function



not a function



not a function

Example :- 1

find whether R is a function or not. (Give reasons)

(i) $R = \{(2,1), (3,1), (4,2)\}$

Since 2, 3, 4 are the elements of R having their unique images, this R is a function.

(ii) $R = \{(2,2), (2,4), (3,3), (4,4)\}$

Since the same first element 2 have two different images 2 and 4, R is not a function.

(iii) $R = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)\}$

Every element has one and only one image, thus relation is a function.

Example - 2

If $f: N \longrightarrow N$ given by $f(x) = 2x+1$ find $f(1), f(2), f(3)$.

Ans.

$f(x) = 2x+1$

$f(1) = 2\times1+1 = 2+1 = 3$

$f(2) = 2\times2+1 = 4+1 = 5$

$f(3) = 3\times2+1 = 6+1 = 7$

EX - 3

If $f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x < 1 \\ \frac{1}{x}, & x > 1. \end{cases}$ find $f(\frac{1}{2}), f(-2), f(2)$.

Ans.

$f(\frac{1}{2}) = \frac{1}{2}$

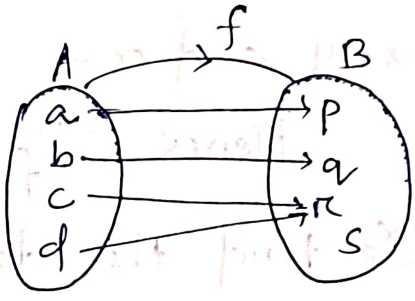
$f(-2) = (-2)^2 = 4$, $f(2) = \frac{1}{2}$

Domain, Codomain, Range of the function

If $f: A \to B$ then the set A is called the <u>domain</u> of the function.

Second set B is called <u>co-domain</u> of the function.

The set of all images of the elements of A is called the range of the function f.



Domain $\{a, b, c, d\} = A$

codomain $\{p, q, r, s\} = B$

Range $\{p, q, r\}$

Methods for finding domain and range of function:

(1) <u>Domain</u>: $\{ D_1 = \text{domain of } f(x),\ D_2 = \text{domain of } g(x) \}$

for $\sqrt{f(x)}$ or $\sqrt[4]{f(x)} \geq 0$, domain is $D_1 \cap \{x \mid f(x) \geq 0\}$

for $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ have $D_1 \cap D_2$, domain

for $\dfrac{f(x)}{g(x)}$ have $D_1 \cap D_2 - \{x \mid g(x) = 0\}$, domain

(2) <u>Range</u>:

If domain = finite no. then R = set of $f(x)$ values

If domain = finite interval then R = find least & greatest value for range.

Ex Find domain of $f(x) = \dfrac{x^2+3x+5}{x^2-5x+4}$.

Let here $f(x) = \dfrac{g(x)}{h(x)}$, $h(x) = x^2-5x+4$

$= (x-4)(x-1)$

f is defined for all real numbers except at $x=4$ and $x=1$.

Hence $D_f = R - \{1,4\}$.

Ex find the domain and the range of $f(x) = \sqrt{x-1}$

$f(x)$ is defined when $x-1 \geq 0$

$x \geq 1$

i.e D_f of f is $x \in [1, \infty)$

Clearly Range will be $[0, \infty)$

Special functions

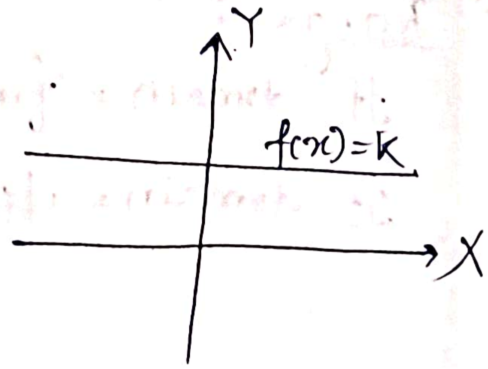
1) Constant function:

A function $f(x)$ is constant function if

$f(x) = k, \ \forall x \in \mathbb{R}$

Domain = $\mathbb{R}$

Range = $\{k\}$



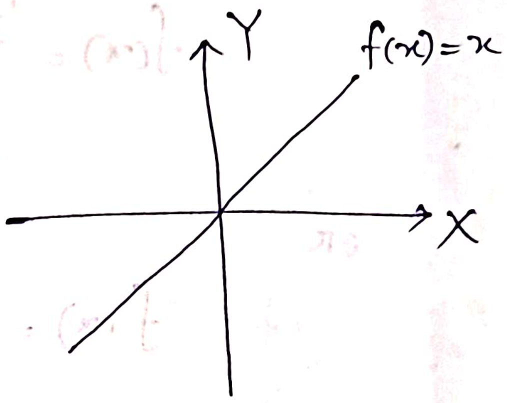
2) Identity function :

A function f(x) is called identity function if

$$f(x) = x, \ \forall \ x \in \mathbb{R}$$

Domain = $\mathbb{R}$

Range = $\mathbb{R}$



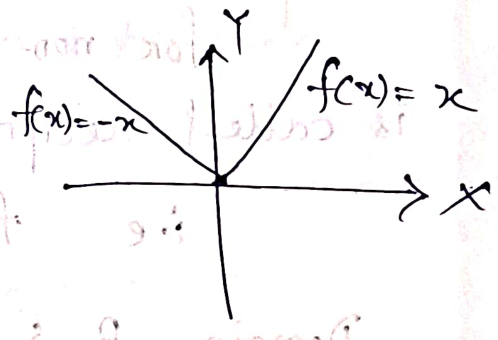
3) Modulus function :

If the function is given by

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$ is called Modulus function.

Domain = $\mathbb{R}$

Range = $\mathbb{R}^+$



4) Greatest Integer function :

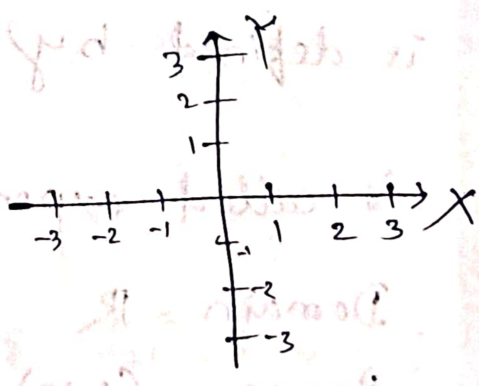
If the function is given by

$$f(x) = [x] = n \quad \text{where } n \leq x < n+1$$

Domain = $\mathbb{R}$

Range = $\mathbb{Z}$

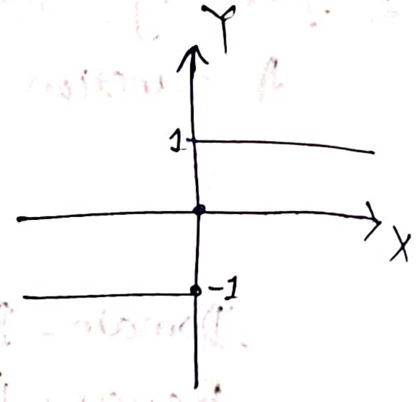
= set of integers



5) Signum function :

If the function is given by

$$f(x) = \begin{cases} \frac{|x|}{x} & ; \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$



or

$$f(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -1 & , \quad x < 0 \end{cases}$$

Domain = $\mathbb{R}$

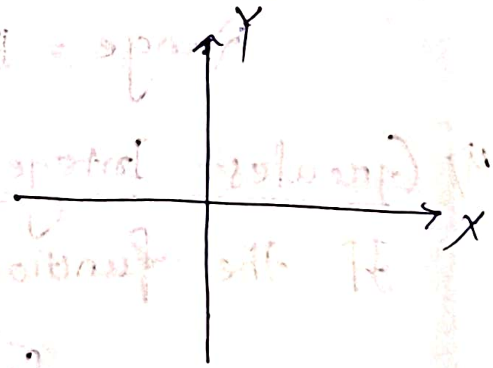
Range = $\{-1, 0, 1\}$

is called the signum function.

6) Reciprocal function :

for non-zero real number x the reciprocal $\frac{1}{x}$ is called reciprocal function.

i.e $\quad f(x) = \frac{1}{x}$



Domain = $\mathbb{R} - \{0\}$ = Range

7) Exponential function :

for $a \neq 1$ be a positive number then $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ is defined by $\quad f(x) = a^x$

is called exponential function.

Domain = $\mathbb{R}$

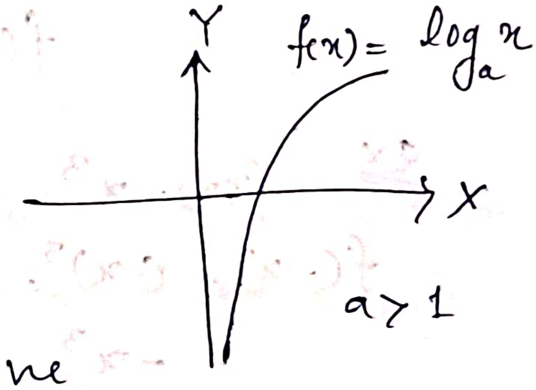
Range = $(0, \infty)$

8) **Logarithmic function:**

A function of the form $f(x) = \log_b x$ where the logarithm of base b is defined as follows $b > 0$ & $b \neq 1$, $\boxed{y = \log_b x \Leftrightarrow b^y = x}$

OR

Generally, $\boxed{\log_b x = n \text{ or } b^n = x}$



Pr.

Domain $= (0, \infty)$

Range $= \mathbb{R}$

Properties:

Let m and n be arbitrary positive numbers such that $a > 0, a \neq 1, b > 0, b \neq 1$ then

1) $\log_a a = 1$, $\log_a 1 = 0$

2) $\log_a b \cdot \log_b a = 1 \Rightarrow \log_a b = \dfrac{1}{\log_b a}$

3) $\log_a (mn) = \log_a m + \log_a n$

4) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

5) $\log_a m^n = n \log_a m$

6) $a^{\log_a m} = m$

Even function:

A function f is even function if for all $x \in D$

$$f(-x) = f(x)$$

Ex $f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

$\Rightarrow f(x) = f(-x)$

$f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x$

$\Rightarrow f(x) = f(-x)$

Odd function :

A function f is called odd function iff for all $x \in D_f$

$$f(-x) = -f(x)$$

Ex $f(x) = x^3$

$f(-x) = (-x)^3 = (-1)^3 x^3$

$= -x^3$

$f(-x) = -f(x)$

$f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x$

$= -f(x)$

$\Rightarrow f(-x) = -f(x)$

Interval

The set of the numbers between any two real numbers is called interval.

(a) Close Interval:

$[a,b] = \{x \mid a \leq x \leq b\}$

(b) Open interval :

$(a,b) = \{x \mid a < x < b\}$

(c) Semi open or Semi closed interval

$[a,b) = \{x \mid a \leq x < b\}$

$(a,b] = \{x \mid a < x \leq b\}$

Limit of a function

Some functions sometimes do not give definite value at some point.

Ex $f(x) = \frac{x^2-4}{x-2} \Rightarrow f(2) = \frac{4-4}{2-2} = \frac{0}{0}$

which can not be determined.

Such forms are indeterminate form given below.

$0 \times \infty,\ 0^0,\ 1^\infty,\ \infty - \infty,\ \frac{\infty}{\infty},\ \infty^0,\ \frac{0}{0}$

Then in such cases we approach 'y' at $x=a$ by their neighbourhood points (by left side or right side)

Definition of limit

A point nearest to 'a' (from left or right) which gives a definite unique no. 'l' is called limit of $f(x)$ at $x=a$. We write it as

$$\lim_{x \to a} f(x) = l.$$

Meaning of '$x \to a$' $\Rightarrow$ 'x tends to a'

$\Rightarrow x \neq a$

$\Rightarrow x$ assumes values nearer and nearer to 'a'

$\Rightarrow$ approach may from left or right

Left hand limit:

When x tends to 'a' from left, then this number is called left hand limit (LHL) at $x=a$.

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} (a \pm h)$$

Right hand limit :

When x tends to 'a' from right, then this number is called right hand limit. and given by

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Existance of limit

The limit of $f(x)$ exists only if

(i) $\lim_{x \to a} f(x)$ exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$

where l is called the limit of the function.

Some Standard limits

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$ (ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a, \ a > 0$ (iv) $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

example :

If $f(x) = \begin{cases} x^2 + 2, & x \geq 1 \\ 2x+1, & x < 1 \end{cases}$ then $\lim_{x \to 1} f(x) = ?$

Ans

$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 2(1-h) + 1 = \lim_{h \to 0} (2 - 2h + 1) = 3$

$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \left(2(1+h) + 1 \right) = \lim_{h \to 0} (2 + 2h + 1) = 3$

Since LHL = RHL = 3

$\Rightarrow \lim_{x \to 1} f(x) = 3$

Ex $\lim_{x \to (-1)} \left(\dfrac{x^2 - 1}{x^2 + 3x + 2} \right)$ Evaluate.

Ans $\lim_{x \to (-1)} \dfrac{(x-1)(x+1)}{(x+2)(x+1)} = \lim_{x \to (-1)} \dfrac{x-1}{x+2} = \dfrac{-1-1}{-1+2} = \dfrac{-2}{1} = -2$

Ex $\lim_{x \to \sqrt{2}} \dfrac{x^4 - 4}{x - \sqrt{2}}$ Evaluate.

Sol^n $\Rightarrow \lim_{x \to \sqrt{2}} \dfrac{x^4 - (\sqrt{2})^4}{x - \sqrt{2}} = 4(\sqrt{2})^{4-1} = 4(\sqrt{2} \cdot 2) = 8\sqrt{2}$

Ex If $\lim_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 80$, n = +ve integer find n.

Sol^n

$\lim_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = n \cdot 2^{n-1}$

$\Rightarrow n \cdot 2^{n-1} = 80 = 2^4 \cdot 5$

$\Rightarrow n = 5$

$$\begin{array}{r|l} 2 & 80 \\ \hline 2 & 40 \\ \hline 2 & 20 \\ \hline 2 & 10 \\ \hline & 5 \end{array}$$

$80 = 2^4 \cdot 5$

Ex find $\lim_{x \to 0} \dfrac{\sin 3x}{5x}$

Sol^n $\lim_{x \to 0} \left(\dfrac{\sin 3x}{5x} \times \dfrac{3}{3} \right) = \lim_{x \to 0} \left(\dfrac{\sin 3x}{3x} \times \dfrac{3}{5} \right) = \dfrac{3}{5} \lim_{x \to 0} \dfrac{\sin 3x}{3x}$

$= \dfrac{3}{5} \times 1 = \dfrac{3}{5}$

$\frac{2x + \sin 2x}{5x - \sin x} = 1$

Ex Evaluate $\lim_{x \to 0} \frac{5x\cos x - 2\sin x}{7x + 5\sin x}$ (Ans. $\frac{1}{4}$)

Ans $\lim_{x \to 0} \frac{\left(\frac{5x\cos x - 2\sin x}{x}\right)}{\left(\frac{7x + 5\sin x}{x}\right)} = \lim_{x \to 0} \frac{\left(\frac{5\cancel{x}\cos x}{\cancel{x}} - \frac{2\sin x}{x}\right)}{\left(\frac{7\cancel{x}}{\cancel{x}} + \frac{5\sin x}{x}\right)}$

$\Rightarrow \lim_{x \to 0} \frac{\left(5\cos x - 2\frac{\sin x}{x}\right)}{\left(7 + 5\frac{\sin x}{x}\right)}$

$= \frac{\left(5\lim_{x \to 0} \cos x - 2\lim_{x \to 0} \frac{\sin x}{x}\right)}{\left(7 + 5\lim_{x \to 0} \frac{\sin x}{x}\right)} = \frac{5 \times \cos 0 - 2 \times 1}{7 + 5 \times 1}$

$= \frac{5 \times 1 - 2}{7 + 5} = \frac{5 - 2}{7 + 5} = \frac{3}{12}$

Ex Evaluate $\lim_{x \to 0} \left(\frac{2^{x+3} - 8}{x}\right)$

$= \lim_{x \to 0} \frac{2^x \cdot 2^3 - 8}{x} = \lim_{x \to 0} \frac{2^x \cdot 8 - 8}{x} = \lim_{x \to 0} \frac{8(2^x - 1)}{x}$

$= 8\lim_{x \to 0} \frac{2^x - 1}{x} = 8 \times \log_e 2 = \log_e 2^8$

Ex $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = ?$

$= \lim_{x \to 0} \left(\frac{e^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x}\right) = \lim_{\sin x \to 0} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \to 0} \frac{\sin x}{x}$

$(\because x \to 0$ then $\sin x \to 0)$

$= \log_e e \times 1 = 1 \times 1 = 1$

$\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$

ix Evaluate $\lim_{x\to 0} (1+3x)^{\frac{2}{x}}$ (as $x \to 0, 3x \to 0$)

$= \lim_{3x\to 0} (1+3x)^{\left(\frac{2}{x}\times\frac{3}{3}\right)} = \lim_{3x\to 0} \left[\left\{(1+3x)^{\frac{1}{3x}}\right\}^{6}\right]$

$= \left[\lim_{3x\to 0} (1+3x)^{\frac{1}{3x}}\right]^{6}$

$= [e]^{6} = e^{6}$

Ex $\lim_{x\to\infty} \left(1+\frac{2}{3x}\right)^{5x}$

$= \lim_{x\to\infty} \left(1+\frac{2}{3x}\right)^{\frac{3x}{2}\times\frac{2}{3x}\times 5x}$ as $x \to \infty \Rightarrow \frac{1}{x} \to 0$

$= \lim_{\frac{2}{3x}\to 0} \left(1+\frac{2}{3x}\right)^{\frac{3x}{2}\times\frac{10}{3}} = \left[\lim_{\frac{2}{3x}\to 0}\left(1+\frac{2}{3x}\right)^{\frac{3x}{2}}\right]^{\frac{10}{3}}$

$= e^{\frac{10}{3}}$

Ex Evaluate $\lim_{x\to e} \frac{1-\log_e x}{e-x}$

Sol Put $\log_e x = y$

$e^{\log_e x} = e^{y} \Rightarrow x = e^{y}$

here $x \to e$ then $y \to 1$

$$= \lim_{x \to e} \frac{1 - \log_e x}{e - x}$$

$$= \lim_{y \to 1} \frac{1 - y}{e - e^y} = \lim_{y \to 1} \frac{1 - y}{e(1 - e^{y-1})}$$

$$= \lim_{y \to 1} \frac{1}{e\left(\frac{1 - e^{y-1}}{1 - y}\right)}$$

$$= \lim_{y \to 1} \frac{1}{e} \frac{1}{\frac{-(e^{y-1} - 1)}{-(y-1)}}$$

$$= \frac{1}{e} \frac{1}{\lim_{y-1 \to 0}\left(\frac{e^{y-1} - 1}{y - 1}\right)}$$

$y \to 1$.
$y - 1 \to 0$

$$= \frac{1}{e} \times \frac{1}{1} = \frac{1}{e}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1$$

Ex $\lim_{x \to 0} \frac{x \log(x+1)}{1 - \cos x}$

$$= \lim_{x \to 0} \frac{\left(\frac{x \log(x+1)}{x^2}\right)}{\left(\frac{1 - \cos x}{x^2}\right)} = \lim_{x \to 0} \frac{\left(\frac{\log(x+1)}{x}\right)}{\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}\log(1+x)}{\frac{\cancel{2}\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2 \cdot \cancel{2}^2}} = \lim_{x \to 0} \frac{2\log(1+x)^{\frac{1}{x}}}{\left\{\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right\}^2}$$

$$= \frac{2 \lim_{x \to 0} \log(1+x)^{\frac{1}{x}}}{\lim_{\frac{x}{2} \to 0} \frac{\sin\frac{x}{2}}{\left(\frac{x}{2}\right)}} = 2 \times \log_e e = 2$$

DERIVATIVE

Derivative of a function at a point :

Let $f(x)$ is a real valued function, then the derivative of $f(x)$ at $x=a$ is defined by

$$~~f'(x)~~ \quad f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

put $x=a$, then

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Ex find the derivative of $f(x) = 3x$ at $x=2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

at $x=2 \Rightarrow f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$$= \lim_{h\to 0}\frac{3(2+h)-3\times 2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{6}+3h-\cancel{6}}{h} = \lim_{h\to 0}\frac{3\cancel{h}}{\cancel{h}} = 3$$

Ex $f(x) = 5x+2$ at $x=1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$f'(1) = \lim_{h\to 0}\frac{\{5(1+h)+2\}-(5\times 1+2)}{h}$$

$$= \lim_{h\to 0}\frac{(5+5h+2)-7}{h} = 5$$

Differentiation by definition (x^n, $\sin x$, $\cos x$, $\tan x$, e^x, $\log_a x$)

① $f(x) = x^n$ or $y = f(x) = x^n$ Find derivative of $f(x)$.

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x+h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x} \qquad \text{by } \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\frac{dy}{dx} = nx^{n-1}$$

② $y = f(x) = \sin x$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= 2 \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right) \cdot \sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \left\{ \cos\left(x + \frac{h}{2}\right) \cdot \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)} \right\}$$

$$= \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)}$$

$$= \cos x \times 1 = \cos x$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x.\cos(x+h)} = \lim_{h\to 0} \frac{\sin h}{h} . \lim_{h\to 0} \frac{1}{\cos x(\cos(x+h))}$$

$$= 1 \times \frac{1}{\cos x.\cos x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Ex $y = f(x) = e^x$. find the derivative by definition.

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x.e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e^x \times \log_e e = e^x \times 1 = e^x$$

Ex $y = f(x) = \log_e x$. Find derivative by definition.

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\log_e(x+h) - \log_e x}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{1}{h} \log\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0} \log\left(1+\frac{h}{x}\right)^{\frac{1}{h}\frac{x}{x} \times \frac{1}{x}}$$

$$= \lim_{h\to 0} \left\{\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right\}^{\frac{1}{x}}$$

$$= \lim_{h\to 0} \frac{1}{x} . \lim_{h\to 0} \log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \log\left\{\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right\}^{\frac{1}{x}} = \log_e e^{\frac{1}{x}} = \frac{1}{x}\log_e e = \frac{1}{x}$$

Some Other derivatives formulae

1) $\frac{d}{dx}(x^n) = nx^{n-1}$

2) $\frac{d}{dx}(a^x) = a^x \log a$, $\frac{d}{dx}(e^x) = e^x$ uch that

3) $\frac{d}{dx}(\log x) = \frac{1}{x}$

4) $\frac{d}{dx}(\sin x) = \cos x$

5) $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{dy}{dx}$

6) $\frac{d}{dx}(\tan x) = \sec^2 x$

7) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

8) $\frac{d}{dx}(\sec x) = \sec x . \tan x$ $-\tan^2 x$

9) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x.$

$[3x^2 - 2x + 3]$

is

$\frac{dwz}{dx}$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Algebra of derivative of functions :

Let $f(x)$ & $g(x)$ be two functions such that

(i) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$

(ii) $\frac{d}{dx}(kf) = k\frac{d}{dx}f$

(iii) $\frac{d}{dx}(f.g) = \cancel{f.\frac{dg}{dx}} \; g\frac{df}{dx} + f\frac{dg}{dx}$

(iv) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$

Q (i) $y = \frac{1}{\sqrt{x}} = \frac{dy}{dx} = ?$, $y = \sec^2 x + \tan^2 x$

(ii) $y = \sqrt{x}$, (iii) $y = (x-1)(x^2+3)$ $[3x^2 - 2x + 3]$

(iv) $y = x^2 \cos x$ (v) $y = \frac{2x+1}{x}$,

Chain Rule :

If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In general, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdots \frac{dz}{dx}$

Q $y = \sin x^2$, $2x \cos x^2$

$y = \log(\tan x)$, $2\operatorname{cosec} 2x$

$y = \operatorname{cosec}(2x^2+3)$, $-4x \operatorname{cosec}(2x^2+3) . \cot(2x^2+3)$.

$y = \tan^3 x$, $3\tan^2 x . \sec^2 x$.

$y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

$y = e^x \sin x$

$y = \cos(\sin x^2)$

Differentiation of Logarithmic function & Exponential

Ex $y = x^{\sqrt{x}}$, $y = (\sin x)^x$, $y = x^{\sin x}$, $y = \log_{\cos x} \sin x$

(1) $y = x^{\sqrt{x}}$ (taking both sides log)

$\log y = \log x^{\sqrt{x}}$

$\log y = \sqrt{x} \log x$ (differentiate bothsides)

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \log x + \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\log x + 1\right)$

$\frac{dy}{dx} = \frac{y}{\sqrt{x}} \frac{(\log x + 2)}{2}$

$= \frac{1}{2\sqrt{x}}(\log x + 2) y$

$= \frac{1}{2\sqrt{x}}(\log x + 2)\, x^{\sqrt{x}}$

Complex Number

def^n A no. of the form $x+iy = z$, $x, y \in \mathbb{R}$, $i = \sqrt{-1}$ is called a complex number and 'i' is called iota.

notations: complex number $= z$

Set of complex number $= \mathbb{C}$

i.e $\mathbb{C} = \{x+iy \mid x \in \mathbb{R}, y \in \mathbb{R}, i = \sqrt{-1}\}$

NOTE

$i = \sqrt{-1}$

$i^2 = \sqrt{-1}.\sqrt{-1} = -1$

$i^3 = i^2.i = -i$

$i^4 = i^2.i^2 = (-1)(-1)$

$\Rightarrow i^4 = 1$

Ex $5+3i$, $-1+10i$, $4+0i$, $0+4i$

Real and Imaginary part:

Let $z = x+iy$ is a complex number then

x = real part of $z = Re(z)$

y = imaginary part of $z = Im(z)$

z is purely real if $Im(z)=0$

z is purely imaginary if $Re(z)=0$

Ex $Re(2i-3)$, $Im(3+0i)$

Algebra of Complex Numbers

Let z_1 & z_2 be two complex Numbers $z_1 = x+iy$, $z_2 = a+ib$

(i) $z_1+z_2 = (x+iy)+(a+ib) = (x+a)+(y+b)i$

(ii) $z_1-z_2 = (x+iy)-(a+ib) = (x-a)+(y-b)i$

(iii) $z_1.z_2 = (x+iy).(a+ib) = xa+xbi+ayi+ybi^2$

$= xa+(-1)yb+(xb+ay)i$

$z_1.z_2 = xa-yb+(xb+ay)i$

(iv) $\frac{z_1}{z_2} = \frac{x+iy}{a+ib}$

$= \frac{x+iy}{(a+ib)} \times \frac{(a-ib)}{(a-ib)}$ (Rationalization)

$= \frac{(x+iy).(a-ib)}{a^2-i^2b^2} = \frac{xa - xbi + ayi - i^2yb}{a^2-(-1)b^2}$

$= \frac{xa-(-1)yb + (ay-xb)i}{a^2+b^2}$

$= \frac{xa+yb+(ay-xb)i}{a^2+b^2}$

Q. form the queⁿ to x+iy. take let z = question

(i) $(3-2i) + i(5+2i)$

(ii) $(3-2i)(i+4)$

(iii) $\frac{5-2i}{2+3i}$

(iv) $(2-i)(2+i)(1+i)$

Solⁿ (i) Let $z = (3-2i) + (5+2i)i$

$= 3-2i+5i+2i^2$

$= 3-2+3i = 1+3i$

(ii) $z = (3-2i)(i+4)$

$= 3i+12-2i^2-8i$

$= 12-2(-1)+3i-8i$

$= 14-5i$

(iii) $z = \frac{5-2i}{2+3i}$

$= \frac{(5-2i)(2-3i)}{(2+3i)(2-3i)} = \frac{10-15i-4i+6i^2}{4-9i^2} = \frac{4-19i}{13}$

(iv) $z = (2-i)(2+i)(1+i)$

$= (4-i^2)(1+i)$

$= (4+1)(1+i)$

$= 5(1+i)$

$z = 5+5i$

Equality of Two Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if their real and imaginary parts are separately equal.

i.e $\quad z_1 = z_2$

$\Leftrightarrow \quad x_1 = x_2$ and $y_1 = y_2$

NOTE: $(9+6i) > (3+2i)$ makes no sense.

Q If $(x+iy)(2-3i) = 4+i$ then check the values for

(i) $x = \frac{5}{13}, y = \frac{8}{13}$ $\qquad$ (ii) $x = \frac{5}{13}, y = \frac{14}{13}$

(i) $\left(\frac{5}{13} + \frac{8i}{13}\right)(2-3i) = \frac{(5+8i)(2-3i)}{13}$

$$= \frac{10 - 15i + 16i - 24i^2}{13}$$

$$= \frac{34+i}{13} \neq 4+i$$

(ii) $\left(\frac{5}{13} + \frac{14i}{13}\right)(2-3i) = \frac{(5+14i)(2-3i)}{13}$

$$= \frac{10 - 15i + 28i - 42i^2}{13}$$

$$= \frac{52 + 13i}{13} = 4+i = R.H.S$$

Hence $x = \frac{5}{13}$ & $y = \frac{14}{13}$ satisfies given condition.

Conjugate of a Complex Number

If there ~~exist~~ exist a complex number $z = a+ib$, $a, b \in \mathbb{R}$ then its conjugate is defined as

$$\bar{z} = a - ib \quad \text{for } z = a+ib.$$

also, $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$

Geometrically, the conjugate of z is the reflection on point image of z in the real axis.

Properties

If z_1, z_2 and z are C.N then

1) $\overline{(\bar{z})} = z$

2) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

3) $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

5) $|z|$ = purely real $= \sqrt{x^2+y^2}$ for $z = x+iy$

6) $(\bar{z})^n = \overline{(z^n)}$

Examples

1) $z = 2+3i$, $\bar{z} = 2-3i$

2) $z = -3-5i$, $\bar{z} = -3+5i$

3) $z = 0+3i$
$\bar{z} = -3i$

4) $z = -5+0i$
$\bar{z} = -5$

5) $z = \frac{2+5i}{4-3i} = \frac{(2+5i)(4+3i)}{16-9i^2} = \frac{8+6i+20i+15i^2}{16+9}$

$= \frac{8-15+26i}{25}$

$= \frac{-7+26i}{25}$

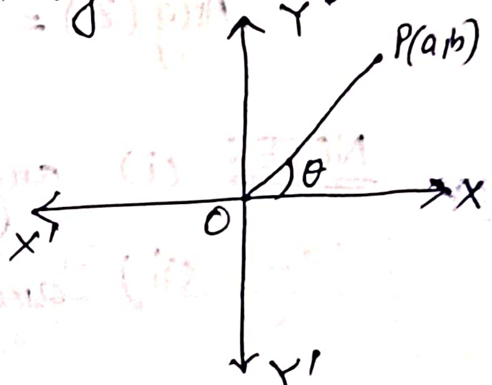
$\bar{z} = \frac{-7-26i}{25}$

$\bar{z} = 4-3i$ $\left| \therefore |z| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5 \right.$

Argument / Amplitude of a Complex Number

Let $z = a+ib$ is represented by a point $P(a,b)$ then the angle made by the OP with real axis (initial axis) is called argument. It is given by

$$\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$



Principal Value of arg(z)

If $\arg(z) = \theta$ lies between $-\pi$ and π then θ is called principal value of z.

i.e for $-\pi < \theta < \pi$, $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$ (acute angle)

$$\text{Principal value}(\alpha) \; \{\mathrm{Arg}(z)\} = \begin{cases} \alpha, & \text{1st quadrant} \\ \pi-\alpha, & \text{2nd quadrant} \\ -\pi+\alpha, & \text{3rd quadrant} \\ -\alpha, & \text{4th quadrant} \end{cases}$$

Ex $z = 1+i$ here $a=1$, $b=1$

$$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{1}{1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$$

(1,1) lies in first quadrant.

$$\mathrm{Arg}(z) = \alpha = \frac{\pi}{4}$$

(ii) $z = -1 - \sqrt{3}i$ here $a = -1$, $b = -\sqrt{3}$

$$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

Since $(-1, -\sqrt{3})$ lies ~~b~~ in 3^{rd} quadrant.

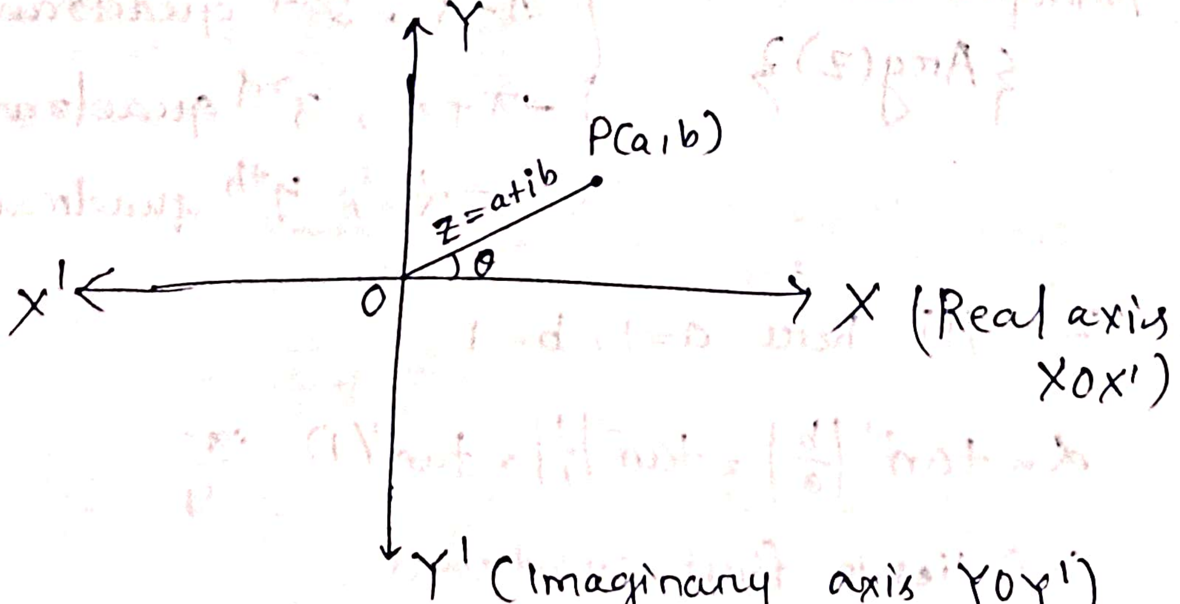
$$\operatorname{Arg}(z) = -\pi + \alpha = -\pi + \frac{\pi}{3} = \frac{2\pi}{3}$$

NOTE: (i) $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

① Cartesian Representation :-

Let $z = a + ib = (a, b)$ is given by a point $P(a, b)$ in a plane (real and imaginary axis) is called argand plane or argand diagram or complex plane.



distance $|OP|$ = modulus of $z = |z| = \sqrt{a^2 + b^2}$

$\operatorname{amp}(z) = \tan^{-1}\left(\frac{b}{a}\right)$

2) Polar Representation :

If we put $|z| = r$, $a = r\cos\theta$, $b = r\sin\theta$

z will converted into polar form $z = r\cos\theta + ir\sin\theta$

$(\cos\theta + i\sin\theta)$ also written as $cis\theta$.

examples

1) write polar form of $\frac{1-i}{1+i}$

Let $z = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1+i^2-2i}{1^2-i^2} = \frac{1-1-2i}{1-(-1)}$

$= \frac{-2i}{2} = -i$

$\therefore z = -i$ here $a = 0$, $b = -1$

$r = |z| = \sqrt{a^2+b^2} = \sqrt{0^2+(-1)^2} = \sqrt{1} = 1$

Now, $a = r\cos\theta = 1.\cos\theta$ and $b = r\sin\theta = 1.\sin\theta$

again $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

So, polar form of z is $(z = -i)$

$z = \cancel{\cos\theta}\ \cos\theta + i\sin\theta$

$z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$.

(ii) Convert $z = i$ into polar form.

here $a = 0$, $b = 1$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$r = |z| = \sqrt{0^2+1^2} = 1$

$\therefore z = r(\cos\theta + i\sin\theta) = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

De-Moivre's Theorem

1) If n is any rational number then (i.e $n=\frac{p}{q}$)

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin\theta(n\theta)$$

2) If $z = r(\cos\theta + i\sin\theta)$, n = positive integer

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

$$k = 0, 1, 2, \ldots, (n-1)$$

Theorem is not valid when n = irrational

Partial Fraction:

1. $\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$ $\longrightarrow \dfrac{A}{x-a} + \dfrac{B}{x-b}$

2. $\dfrac{px+q}{(x-a)^2} \longrightarrow \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$

3. $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)} \longrightarrow \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$

4. $\dfrac{px^2+qx+r}{(x-a)^2(x-b)} \longrightarrow \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$

5. $\dfrac{px^2+qx+r}{(x-a)(ax^2+bx+c)} \longrightarrow \dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$

where x^2+bx+c can not factorized further.

Permutation and Combination

1) Factorial Notation

It is denoted by $n!$ (read as n factorial)

given by $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

$\boxed{0! = 1}$

$1! = 1$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

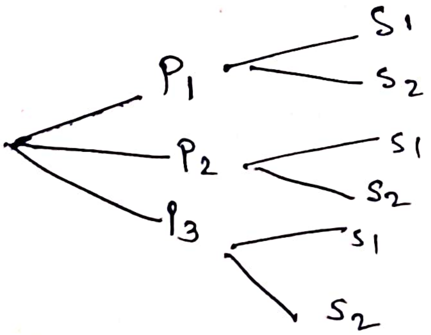
$5! = 5 \times 4 \times 3 \times 2 \times 1$

and so on.

2) Principle of Multiplication :

If an event can occure in 'm' different ways and another event in 'n' different ways then total number of occurance of the events in order is '$m \times n$'. (product)

Example: Mohan has 3 pants & 2 shirts. How many pairs of pant & shirts will be ?



$P_1 S_1$ 1

$P_1 S_2$ 2

$P_2 S_1$ 3

$P_2 S_2$ 4

$P_3 S_1$ 5

$P_3 S_2$ 6 (total pairs)

i.e $2 \times 3 = 6$. options.

3) Principle of Addition :

~~Example~~ **Permutations** (Arrangement purpose)

The number of permutations of n-different objects taken 'r' at a time (not repeat) is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

Ex Arrange 3-letter words (with or without meaning) from NUMBER.

total letters (N, U, M, B, E, R) = 6

to form 3-letters

So, $^nP_r = {}^6P_3 = \frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}} = 120$ ways.

Combinations (order does not matter) (Selection)

The number of combinations of n-different objects taken 'r' at a time given by

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

Properties

1) $^nC_n = 1$, $^nC_0 = 1$, $^nC_1 = n$

2) $^nC_a = {}^nC_b \Rightarrow a = b$ or $n = a + b$

Binomial Expression

An algebraic expression consisting of two terms +ve or −ve sign between them is binomial expression.

e.g. $(a+b)$, $(2x-3y)$, $\left(\frac{p}{x^2}-\frac{q}{x^4}\right)$, $\left(\frac{1}{x}+\frac{y}{y^3}\right)$ etc.

Binomial Theorem (for +ve integral)

for n = +ve integer and $x, y \in$

$$(x+y)^n = {}^nC_0\, x^{n-0} y^0 + {}^nC_1\, x^{n-1} y^1 + \dots + {}^nC_r\, x^{n-r} y^r + \dots + {}^nC_n\, x^{n-n} y^n$$

i.e $$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

for $$(x-y)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r} y^r$$

Other formulas

1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + \infty \dots \infty$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots + \dots \infty$

3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \infty$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \infty$

* General formula for any Index (r^{th} term)

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}\, x^r$$

Partial Fraction Problems

* proper rational function $= \frac{p(x)}{q(x)}$ (degree $p(x) <$ deg $q(x)$)
(i.e like $\frac{2}{3}, \frac{3}{5}, \frac{x+1}{x^2+2}$ etc) otherwise improper.

* Conversion improper to proper (by long division process)

i.e $\quad \frac{p(x)}{q(x)} \longrightarrow T(x) + \frac{p_1(x)}{q(x)}$

Improper $\longrightarrow$ $T(x)$ + proper

ex $\quad \frac{x^2+1}{x^2-5x+6}$ division $\rightarrow$ $x^2-5x+6 \overline{\big) x^2+1 } \big(1$
$\quad x^2-5x+6$
$\quad -\ (+)\ (-)$
$\quad +5x-5$

$$\Rightarrow \frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6}$$

NOTE: We use Partial fraction in proper rational function

Ex Resolve into partial fraction.

(i) $\frac{1}{(x+1)(x+2)}$

$$\Rightarrow \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad \text{——— (1)}$$

$$= \frac{(x+2)A + (x+1)B}{(x+1)(x+2)}$$

$$\Rightarrow \frac{1}{\cancel{(x+1)(x+2)}} = \frac{Ax+2A+Bx+B}{\cancel{(x+1)(x+2)}}$$

$$\Rightarrow 0x+1 = (A+B)x + (2A+B)$$

Equating the coefficients of x of both sides.

$$A+B=0$$

$$\begin{array}{r} 2A+B=1 \\ (-)\quad(-)\quad(-) \\ \hline -A+0=-1 \end{array} \Rightarrow \boxed{A=1} \quad \boxed{B=-1}$$

put values of A & B in equation ① we get

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

(ii) $\dfrac{x^2+1}{x^2-5x+6}$

degrees of x of numerator & denominator are same
~~divide~~ division method applied (previous page)

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6} = 1 + P_1(x) \quad \text{——} ①$$

$P_1(x)$ is proper

Let $\dfrac{5x-5}{x^2-5x+6} = \dfrac{5x-5}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$ —— ②

Now, $\dfrac{5x-5}{(x-2)(x-3)} = \dfrac{(x-3)A + (x-2)B}{(x-2)(x-3)}$

$\Rightarrow \quad 5x-5 = Ax - 3A + Bx - 2B$

$\Rightarrow \quad 5x-5 = (A+B)x + (-3A-2B)$

Equating coefficients of corresponding terms

$$A+B=5 \xrightarrow{\times 3} 3A+3B=15$$
$$-3A-2B=-5 \qquad -3A-2B=-5$$
$$B=10,\ A=-5$$

putting values of A & B in equation (1)

$$\frac{x^2+1}{x^2-5x+6} = 1+P_1(x)$$
$$= 1+\frac{A}{x-2}+\frac{B}{x-3}$$
$$\Rightarrow \frac{x^2+1}{x^2-5x+6} = 1+\frac{-5}{x-2}+\frac{10}{x-3}$$

<u>Ans</u>

iii) $\dfrac{3x-2}{(x+1)^2(x+3)}$

$$\Rightarrow \frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{(x+1)}+\frac{B}{(x+3)}+\frac{C}{(x+1)^2}$$

$$\Rightarrow \frac{3x-2}{(x+1)^2\cancel{(x+3)}} = \frac{A(x+1)(x+3)+B(x+1)^2+C\cancel{(x+1)}(x+3)}{(x+1)^2\cancel{(x+3)}}$$

$$\Rightarrow 3x-2 = A(x^2+3x+x+3)+B(x^2+1+2x)+C(x+3)$$
$$= A(x^2+4x+3)+B(x^2+2x+1)+C(x+3)$$
$$\Rightarrow 3x-2 = (A+B)x^2+(4A+2B)x+(3A+B+C)$$

Equating coefficients of corresponding terms.

$$A+B=0$$
$$4A+2B=3$$
$$3A+B+C=-2$$