Vikash Polytechnic, Bargarh

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Diploma 6th Semester

BRANCH-ECE

Submitted By:- Kalyani Behera

CONTENTS

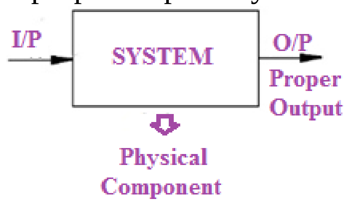
BASIC CONCEPT OF CONTROL SYSTEM ENGINEERING

INTRODUCTION

A Control systems play a vital role in our day to day life. The automatic control systems play an important role in the advancement and improvement of Engineering skills. The control systems is implementations from a Traffic signals, automatic washing machines, automatic electric Iron and also in the working of Satellites, Guided Missiles, etc.

BASIC CONCEPT OF CONTROL SYSTEM

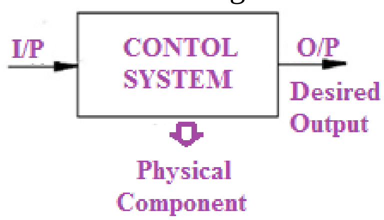
System: A system is a group of physical components arranged in such a way that it gives the proper output to the given input. The proper output may or may not be the desired output.



Example: A fan without blades is not a system. Because no proper output. But a fan without regulator that is system cum air flow (Proper output)

Control: The action to command, direct or regulate a system.
Plant or process**:** The part or component of a system that is required to be controlled.
Input: It is the signal or excitation supplied to a control system.
Output: It is the actual response obtained from the control system.
Controller: The part or component of a system that controls the plant.
Disturbances**:** The signal that has adverse effect on the performance of a control system.

Control System: A control system is a group of physical components arranged in such a way that it gives the desired output by means of control or regulation.



Example: A fan with regulator that is control system(Desired output)
Automation: The control of a process by automatic means
Actuator*:* It is the device that causes the process to provide the output. It is the device that provides the motive power to the process.
Design*:* The process of conceiving or inventing the forms, parts, and details of system to achieve a specified purpose.

Simulation: A model of a system that is used to investigate the behavior of a system by utilizing actual input signals**.**

Optimization: The adjustment of the parameters to achieve the most favorable design.

Feedback Signal*:* A measure of the output of the system used for feedback to control the system.

Negative feedback*:* The output signal is feedback so that it subtracts from the input signal.

Block diagrams: Unidirectional, operational blocks that represent the transfer functions of the elements of the system.

***Signal Flow Graph* (SFG):** A diagram that consists of nodes connected by several directed branches and that is a graphical representation of a set of linear relations.

Specifications: Statements that explicitly state what the device or product is to be and to do. It is also defined as a set of prescribed performance criteria.

Open-loop control system: A system that utilizes a device to control the process without using feedback. Thus the output has no effect upon the signal to the process.

Closed-loop feedback control system**:** A system that uses a measurement of the output and compares it with the desired output.

Regulator*:* The control system where the desired values of the controlled outputs are more or less fixed and the main problem is to reject disturbance effects.

Servo system: The control system where the outputs are mechanical quantities like acceleration, velocity or position.

Stability: It is a notion that describes whether the system will be able to follow the input command. In a non-rigorous sense, a system is said to be unstable if its output is out of control or increases without bound.

Multivariable Control System*:* A system with more than one input variable or more than one output variable.

Trade-off: The result of making a judgment about how much compromise must be made between conflicting criteria.

CHAPTER - 1
FUNDAMENTAL OF CONTROL SYSTEM

1.1 CLASSIFICATION OF CONTROL SYSTEM:

1.1.1 *Natural control system and Man-made control system:*

Natural control system: It is a control systemthat is created by nature.
i.e. solar system, digestive system of any animal, etc.

Man-made control system: It is a control system that is created by humans, i.e. automobile, power plants etc.

1.1.2 *Automatic control system and Combinational control system:*

Automatic control system: It is a control system that is made by using basic theories from mathematics and engineering.This system mainly has sensors, actuators and responders.

Combinational control system: It is a control system that is a combination of natural and man made control systems, **i.e**. driving a car etc.

1.1.3 *Time-variant control system and Time-invariant control system:*

Time-variant control system: It is a control system where any one or more parameters of the control system vary with time i.e. driving a vehicle.

Time-invariant control system: It is a control system where none of its parameters vary with time i.e. control system made up of inductors, capacitors and resistors only.

1.1.4 *Linear control system and Non-linear control system:*

Linear control system: It is a control system that satisfies properties of homogeneity and Superposition.

- Homogeneous property: $f(x+y) = f(x)+f(y)$
- Superposition properties: $f(ax) = a.f(x)$

Non-linear control system: It is a control system that does not satisfy properties of homogeneity and Superposition.**i.e**. $f(x) = x^3$

1.1.5 *Continuous-Time control system and Discrete-Time control system:*

Continuous-Time control system: It is a control system where performances of all of its parameters are function of time, i.e. armature type speed control of motor.

Discrete -Time control system: It is a control system where performances of all of its parameters are function of discrete time i.e. microprocessor type speed control of motor.

1.1.6 *Deterministic control system and Stochastic control system:*

Deterministic control system: It is a control system where its output is predictable or repetitive for certain input signal or disturbance signal.

***Stochastic control system*:**It is a control system where its output is unpredictableor non-repetitive for certain input signal or disturbance signal.

1.1.7 *Lumped-parameter control system and Distributed-parameter control system:*

Lumped-parameter control system: It is a control system where its mathematical model is represented by ordinary differential equations.

Distributed-parameter control system:It is a control system where its mathematical model is represented by an electrical network that is a combination of resistors, inductors and capacitors.

1.1.8 *Single-input-single-output (SISO) and Multi-input-multi-output (MIMO) control system:*

SISO control system: It is a control system that has only one input and one output.

MIMO control system: It is a control system that has only more than one input and more than one output.

1.2 OPEN-LOOP CONTROL SYSTEM AND CLOSED LOOP CONTROL SYSTEM:

The control systems are classified in to two ways based on controlling action.
(1) Open-loop control system (2) Closed-loop control system

1.2.1 ***Open-loop control system:*** It is a control system where its control action only depends on input signal and does not depend on its output response. A system with manual operation is open-loop control system.Ex.-Manual Iron Box

Open-loop control system:

It is a control system where its control action only depends on input signal and does not depend on its output response as shown in Fig.1.2.1

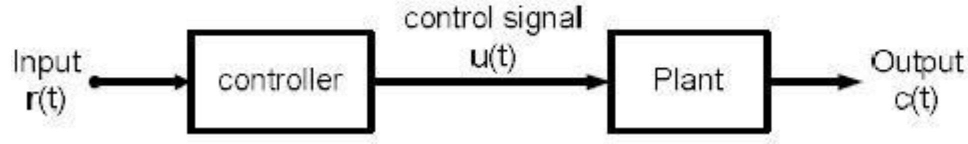


Fig.1.2.1 An open-loop system

Examples 1.2.1: Traffic signal, Automatic Washing machine, Bread toaster, etc.

Advantages	Disadvantages
Simple design and easy to construct Economical Easy for maintenance Highly stable operation	Not accurate and reliable when input or system parameters are variable in nature Recalibration of the parameters are required time to time

1.2.2 ***Closed-loop control system:*** It is a control system where its control action depends on both of its input signal and output response. A system with automatic is closed-loop control system.Ex.- Automatic Iron Box.

Closed-loop control system:

It is a control system where its control action depends on both of its input signal and output response as shown in Fig.1.2.2

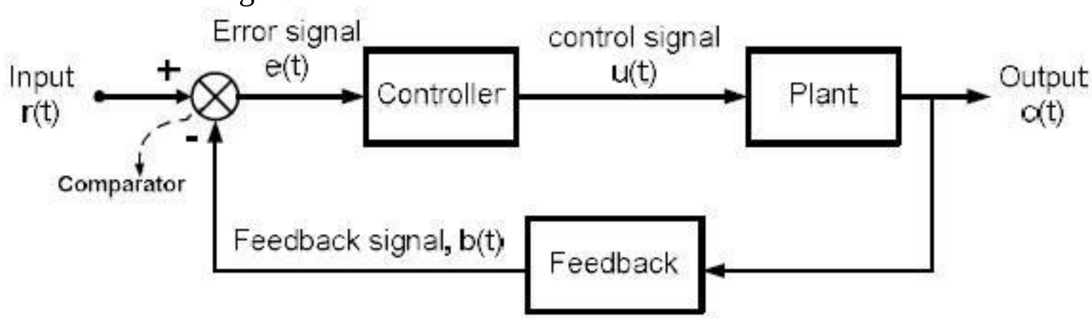


Fig.1.2.2A closed-loop system

Examples 1.2.2: Automatic electric iron, Missile launcher, Voltage Stabilizer, etc.

Advantages:

- More accurate operation than that of open-loop control system
- Can operate efficiently when input or system parameters are variable in nature
- Less nonlinearity effect of these systems on output response
- High bandwidth of operation
- There is facility of automation
- Time to time recalibration of the parameters are not required

Dis-advantages:

- Complex design and difficult to construct
- Expensive than that of open-loop control system
- Complicate for maintenance
- Less stable operation than that of open-loop control system

1.2.3 COMPARISON BETWEEN OPEN-LOOP AND CLOSED-LOOP CONTROL SYSTEMS

It is a control system where its control action depends on both of its input signal and output response.

Open Loop Control System	Closed Loop Control System
1. These are not reliable 2. If calibration is good, they perform accurately. 3. It is easier to build. 4. It is more stable. 5. If non- linearity's are present; the system operation is not good. 6. Feed back is absent. Example: (i) Traffic Control System. (ii) Control of furnace for coal heating. (iii) An Electric Washing Machine.	1. These are reliable. 2. It has got the ability toperform accurately because of the feed back. 3. It is difficult to build. 4. Less Stable Comparatively. 5. Even under the presence of non-linearity's the system operates better than open loop system. 6. Feed back is present. Example: (i) Pressure Control System. (ii) Speed Control System. (iii) Robot Control System. (iv) Temperature Control System

1.3 EFFECTS OF FEED BACK:

1.3.1 Feed Back:

Normally, the feed back signal has opposite polarity to the input signal. This is called negative feed back. The advantage is the resultant signal obtained from the comparator being difference of the two signals is of smaller magnitude.

It can be handled easily by the control system. The resulting signal is called Actuating Signal [E(S)]. This signal has zero value when the desired output is obtained. In that condition, control system will not operate.

1.3.2 Effects of Feed Back:

Let the system has open loop gain [G(S)], feed back loop gain [H(S)], Output signal [C(S)] & Input signal [R(S)].Fig.1.3.2

Then the feed back signal [B(S)] is

B(S) = H(S).C(S)

$G(S) = \frac{C(S)}{R(S)}$ & E(S) = R(S) - B(S)

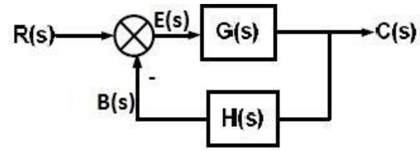


Fig.1.3.2

Hence,

C(S) = G(S).E(S)

= G(S) [R(S) - B(S)]

= G(S) [R(S) - H(S).C(S)]

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

This eq.(1), we can write the effects of feed back as follows:

(a) Overall Gain*:*Eq. (1) shows that the gain of the open loop system is reduced by a factor [1+G(S).H(S)] in a feed back system.

Here the feed back signal is negative. If the feed back gain has positive value, the overall gain will be reduced. If the feed back gain has negative value, the overall gain may increase.

(b) Stability: If a system is able to follow the input command signal, the system is said to be Stable.A system is said to be Unstable, if its output is out of control. In eq.(1), if GH = - 1 the output of the system is infinite for any finite input. The stable system may become unstable for certain value of a feed back gain. Therefore if the feed back is not properly used, the system can be harmful.

(c) Sensitivity: This depends on the system parameters. For a good control system, it is desirable that the system should be insensitive to its parameter changes.

Sensitivity, $S_G = \frac{1}{1+GH}$

This function of the system can be reduced by increasing the value of GH.

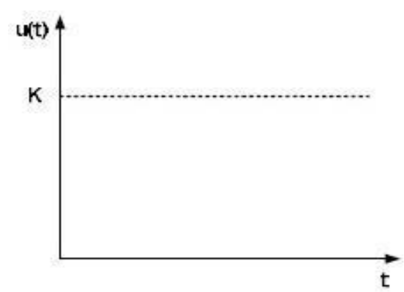
This can be done by selecting proper feed back.

(d) *Noise*: Examples are brush & commutation noise in electrical machines, Vibrations in moving system etc.The effect of feed back on these noise signals will be greatly influenced by the point at which these signals are introduced in the system. It is possible to reduce the effect of noise by proper design of feed back system.

1.4 STANDARD TEST SIGNALS:

1.4.1 Step Signal*:*A step signal u(t) is mathematically defined as follows.

$$u(t) = \begin{cases} 0 & ; t < 0 \\ K & ; t \geq 0 \end{cases}$$



Laplace transform of step signal is

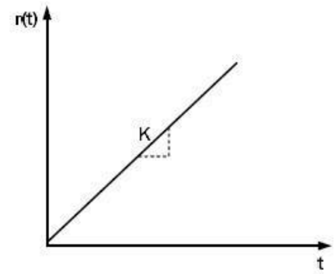
$$U(s) = \frac{K}{s}$$

Step signal

1.4.2 Ramp Signal: A step signal r(t) is mathematically defined as follows.

$$r(t) = \left.\begin{matrix} 0 & ;t<0 \\ Kt & ;t \geq 0 \end{matrix}\right\}$$

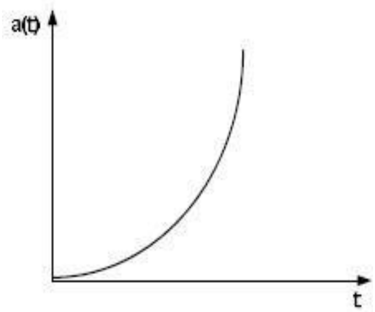
Laplace transform ramp signal is $R(s) = \frac{K}{s^2}$



Ramp Signal

1.4.3 Parabolic Signal: A step signal a(t) is mathematically defined as follows.

$$a(t) = \left.\begin{matrix} 0 & ;t<0 \\ \frac{Kt^2}{2} & ;t \geq 0 \end{matrix}\right\}$$



Parabolic Signal

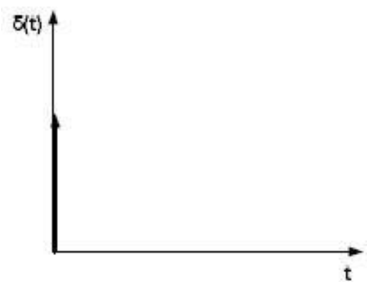
Laplace transform of parabolic signal is

$$A(s) = \frac{K}{s^3}$$

1.4.4 Impulse Signal: An impulse signal δ(t) is mathematically defined as follows.

$$\delta(t) = \left.\begin{matrix} undefined & ;t=0 \\ 0 & ;t \neq 0 \end{matrix}\right\}$$

Laplace transform of impulse signal is δ(s) = 1



Impulse Signal

1.5 SERVOMECHANISM:

It is the feedback unit used in a control system. In this system, the control variable is a mechanical signal such as position, velocity or acceleration. Here, the output signal is directly fed to the comparator as the feedback signal, b(t) of the closed-loop control system. This type of system is used where both the command and output signals are mechanical in nature. A position control system as shown in Fig.1.5(a) is a simple example of this type mechanism. The block diagram of the servomechanism of an automatic steering system is shown in Fig.1.5 (b).

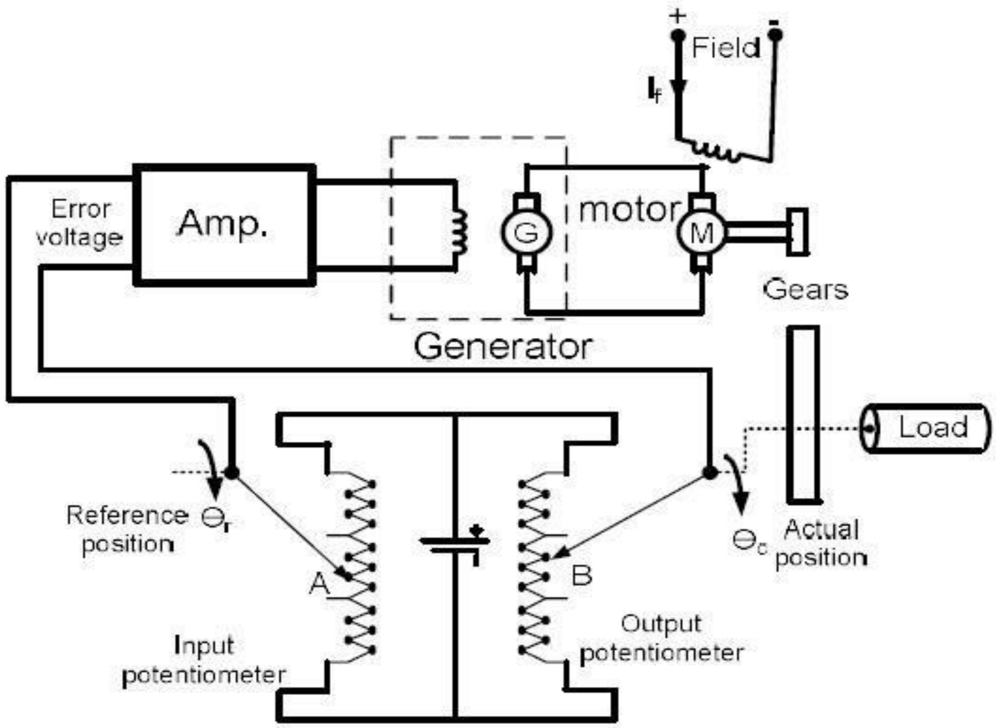


Fig.1.5(a) Schematic diagram of a servomechanism

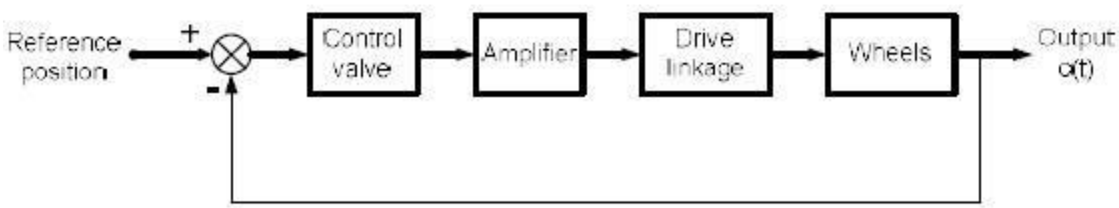


Fig.1.5(b) Block diagram of a servomechanism

Examples:

- Missile launcher
- Machine tool position control
- Power steering for an automobile
- Roll stabilization in ships, etc.

1.6 REGULATORS:

It is also a feedback unit used in a control system like servomechanism. But, the output is kept constant at its desired value. The schematic diagram of a regulating system is shown in Fig.1.6(a) Its corresponding simplified block diagram model is shown in Fig.1.6(b)

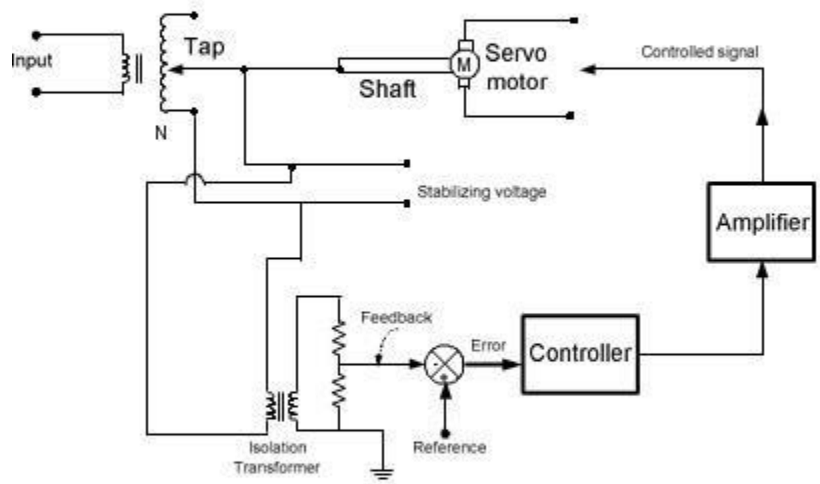


Fig.1.6(a) Schematic diagram of a regulating system

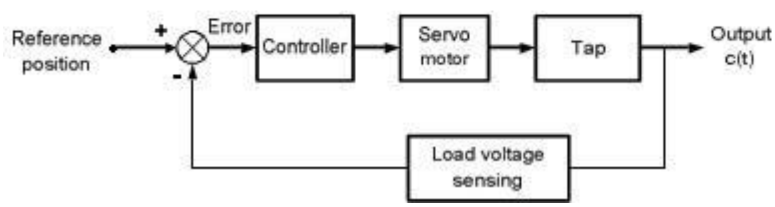


Fig.1.6(b) Block diagram of a regulating system

Examples:

- Temperature regulator
- Speed governor
- Frequency regulators, etc.

CHAPTER - 2
TRANSFER FUNCTION

2.1.1 DEFINITION OF TRANSFER FUNCTION:

It is the ratio of Laplace transform of output signal to Laplace transform of inputsignal assuming all the initial conditions to be zero, i.e.
Let, there is a given system with input r(t) and output c(t) as shown in Fig. (i), then its Laplace domain is shown in Fig. (ii). Here, input and output are R(s) and C(s) respectively.

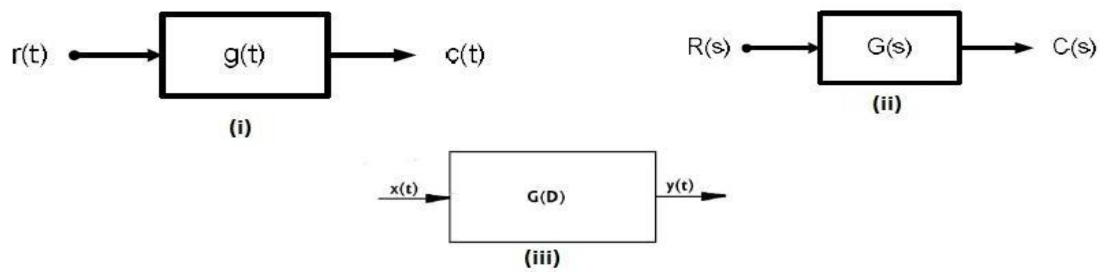


Fig. (i) A system in time domain, (ii) A system in frequency domainand (iii) transfer function with differential operator.

G(s) is the transfer function of the system. It can be mathematically represented as follows.

$$G(s) = \left.\frac{C(s)}{R(s)}\right|_{zero \quad initial \quad condition}$$

An equation describing the physical system has integrals & differentials, the step involved in obtaining the transfer function are;
(1) Write the differential equation of the system.
(2) Replace the terms as,

$$\frac{d}{dt} \text{ by 'S' \& } \int dt \text{ by } 1/S.$$

(3) Eliminate all the variables except the desired variables.

2.1.2 IMPULSE RESPONSE:

In a control system, when there is a single i/p of unit impulse function, then there will be some response of the Linear System.

$$G(S) = \frac{C(s)}{R(s)}$$

$$R(S) = L\{\delta(t)\} = 1$$

The Laplace Transform of the i/p will be R(S) = 1
C(S) = G(S).R(S) = G(S).1
⟹ C(S) = G(S)
i.e., the Laplace Transform of the system o/p will be simply the 'Transfer function' of the system.

Taking L^{-1}

$$\boxed{C(t) = G(t)}$$

Here G(t) will be impulse response of the Linear System. This is called Weighing Function. Hence LT of the impulse response is the Transfer function of the system itself.

2.2 PROPERTIES,ADVANTAGES & DISADVANTAGES OF TRANSFER FUNCTION

2.2.1 Properties:

- Zero initial condition
- It is same as Laplace transform of its impulse response
- Replacing 's' by $\frac{d}{dt}$ in the transfer function, the differential equation can be obtained
- Poles and zeros can be obtained from the transfer function
- Stability can be known
- Can be applicable to linear system only

2.2.2 Advantages of transfer function:

- It is a mathematical model and gain of the system
- Replacing 's' by $\frac{\mathrm{d}}{\mathrm{dt}}$ in the transfer function, the differential equation can be obtained
- Poles and zeros can be obtained from the transfer function
- Stability can be known
- Impulse response can be found

2.2.3 Disadvantages of transfer function:

- Applicable only to linear system
- Not applicable if initial condition cannot be neglected
- It gives no information about the actual structure of a physical system

2.3 POLES & ZEROES OF TRANSFER FUNCTION:

$$G(s)=\frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}=K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-z_j)}$$

Where, Z_1, Z_2… Z_m are called **zeros**.
P_1, P_2….P_n are called **poles**.
Number of poles n will always be greater than the number of zeros m

Example 2.4.1: Obtain the pole-zero map of the following transfer function.

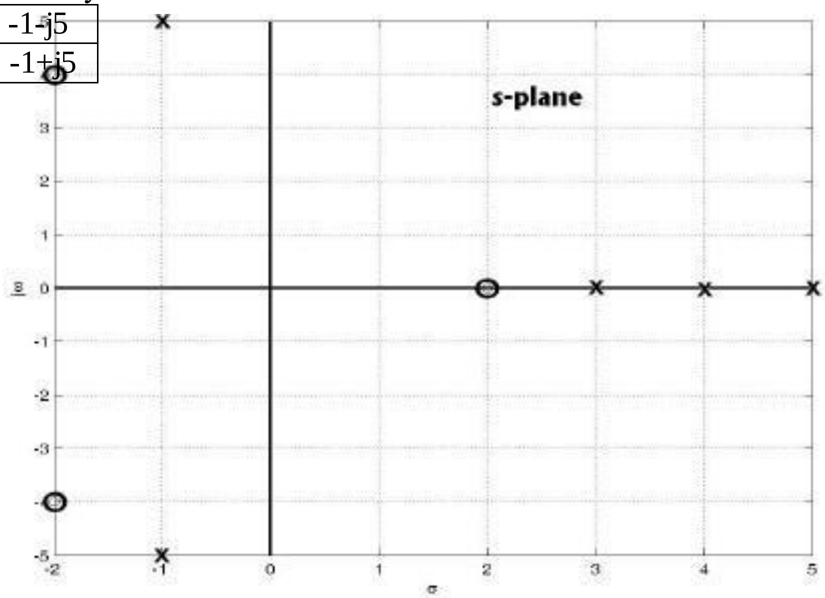
$$G(s)=\frac{(s-2)(s+2+j4)(s+2-j4)}{(s-3)(s-4)(s-5)(s+1+j5)(s+1-j5)}$$

Solution:
The following equation in its partial fractions as follows Laplacetransform is expandedwith

Zeros	Poles
S = 2	S = 3
S = -2-j4	S = 4
S = -2+j4	S = 5

Whereas, Pole is denoted by '×'
and Zero is denoted by'0'

	S = -1-j5
	S = -1+j5



Pole-Zero Map

OR

<u>POLES AND ZEROS OF TRANSFER FUNCTION</u>

Generally a function can be represented to its polynomial form. For example,

$$F(s) = f_0 s^n + f_1 s^{n-1} + f_2 s^{n-2} + f_3 s^{n-3} + \cdots\cdots + f_{n-1} s^1 + f_n$$

Now similarly transfer function of a control system can also be represented as

$$G(s) = \frac{C(s)}{R(s)} = \frac{C_0 S^m + C_1 S^{m-1} + C_2 S^{m-2} + \ldots\ldots\ldots + C_{m-1} S + C_m}{R_0 S^n + R_1 S^{n-1} + R_2 S^{n-2} + \ldots\ldots + R_{n-1} S + R_n}$$

$$G(s) = \frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

Where, K is known as gain factor of the transfer function. Now in the above function if s = z_1 or s = z_2 or s = z_3,...s = z_m, the value of transfer function becomes zero. These z_1,z_2, z_3, ...$z_m$are roots of the numerator polynomial. As for these roots the numerator polynomial, the transfer function becomes zero, these roots are called zeros of the transfer function. Now, if s = p_1 or s = p_2 or s = p_3,....s = p_n the value of transfer function becomes infinite. Thus the roots of denominator are called the poles of the function.

Now let us rewrite the transfer function in its polynomial form.

$$G(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\ldots\ldots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\ldots\ldots(s-p_n)}$$

Now, let us consider s approaches to infinity as the roots are all finite number, they can be ignored compared to the infinite s. Therefore

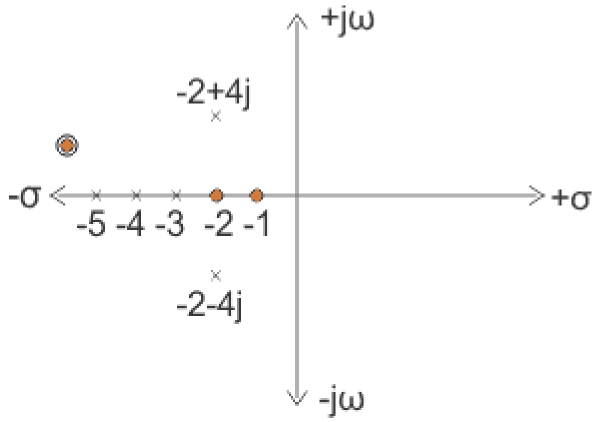
G(s) = K$\frac{s^m}{s^n}$ = Ks^{m-n}

Hence, when s → ∞ and m > n, the function will have also value of infinity, that means the transfer function has poles at infinite s, and the multiplicity or order of such pole is m - n.
Again, when s → ∞ and m < n, the transfer function will have value of zero that means the transfer function has zeros at infinite s, and the multiplicity or order of such zeros is n - m.

EXAMPLE 2.4.2:
The transfer function of the system, G(s) = I(s)/V(s), the ratio of output to input.
1) Let us explain the concept of poles and zeros of transfer function through an example.

$$G(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)(s+2-4j)(s+2+4j)}$$

SOLUTION:
The zeros of the function are, -1, -2 and the poles of the functions are -3, -4, -5, -2 + 4j, -2 - 4j. Here m = 2 and n = 5, as m < n and n - m = 3, the function will have 3 zeros at s → ∞. The poles and zeros are plotted in the figure below



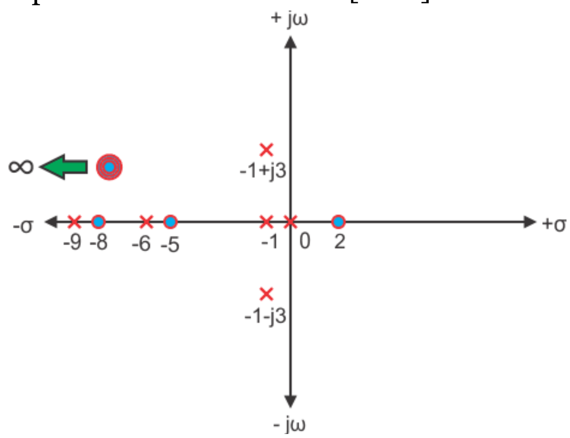
EXAMPLE 2.4.3:
The transfer function of control system is

$$G(s) = \frac{(s-2)(s+5)(s+8)}{s(s+1)(s+6)(s+9)(s+1-j3)(s+1+j3)}$$

SOLUTION:
In the above transfer function, if the value of numerator is zero, then

$$(s-2)(s+5)(s+8) = 0$$
$$\Rightarrow s = 2,\ -5,\ -8$$
$$s(s+1)(s+6)(s+9)(s+1-j3)(s+1+j3) = 0$$
$$\Rightarrow s = 0,\ -1,\ -6,\ -9,\ -1+j3,\ -1-j3$$

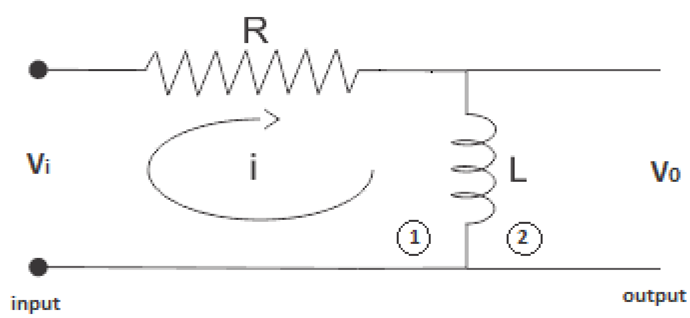
These are the location of poles of the function.



As the number of zeros should be equal to number of poles, the remaining three zeros are located at s →∞.

2.5 CONCEPT OF TRANSFER FUNCTION

The transfer function is generally expressed in Laplace Transform and it is nothing but the relation between input and output of a system. Let us consider a system consists of a series connected resistance (R) and inductance (L) across a voltage source (V_i).



In this circuit, the current **'i'** is the response due to applied voltage(V_i) as cause. Hence the voltage and current of the circuit can be considered as input and output of the system respectively. From the circuit, we get,

Apply KVL in mesh (1)

$V_i = R_i + L\frac{di}{dt}$

Apply KVL in mesh (2)

$V_0 = L\frac{di}{dt}$

Now applying Laplace Transform, we get,

$V_i(s) = RI(s) + sLI(s)$

$V_0(s) = sLI(s)$

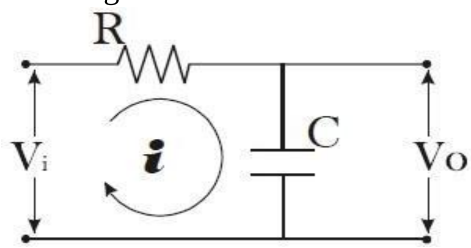
Calculation of Transfer function

$\frac{V_0(s)}{V_i(s)} = \frac{sLI(s)}{(R+sL)I(s)}$

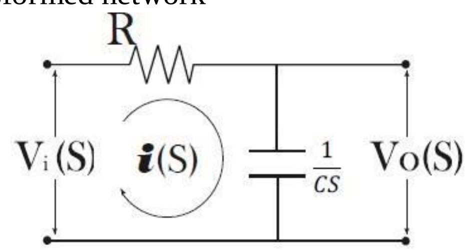
$\frac{V_0(s)}{V_i(s)} = \frac{sL}{R+sL}$

2.5 SIMPLE PROBLEMS OF TRANSFER FUNCTION OF NETWORK

Problem 2.5.1: Determine the TF of the given circuit.



Solution: The Laplace Transformed network



Applying KVL in i/p loop, we get

$V_i(t) = iR + \frac{1}{C}\int i\,dt$

$V_i(S) = I(S)\,R + \frac{1}{C} \cdot \frac{I(S)}{S}$ .... (1)

Applying KVL in o/p loop, we get

$$Vo = \frac{1}{C}\int i\,dt$$

$Vo(S) = \frac{1}{CS} I(S)$ ....(2)

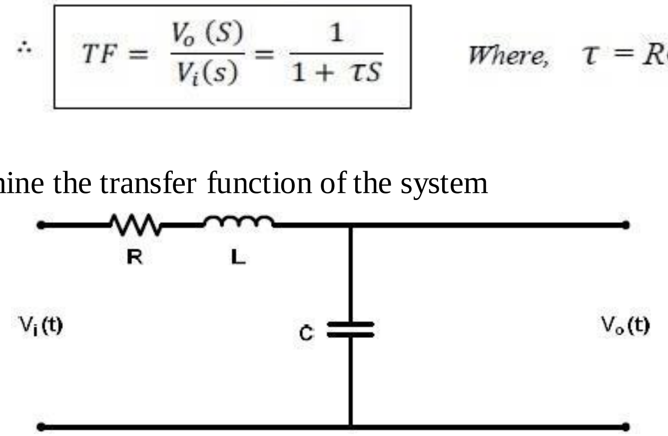
$$i.e.,\ V_o(S) = \frac{1}{CS} I(S) \quad \& \quad V_i(S) = I(S)\left[R + \frac{1}{CS}\right]$$

$$\Rightarrow CS.V_o(S) = I(S)$$

$$CS.V_o(S) = \frac{V_i(s)}{[R + \frac{1}{CS}]}$$

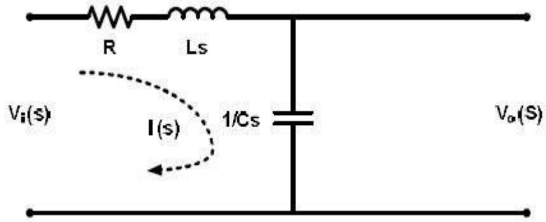
$$\therefore \boxed{TF = \frac{V_o(S)}{V_i(s)} = \frac{1}{1+\tau S}} \quad Where,\ \tau = RC$$

Problem 2.5.2: Determine the transfer function of the system



A system in time domain

Solution:



A system in frequency domain

Applying KVL to loop-1

$$V_i(s) = \left(R + Ls + \frac{1}{Cs}\right) I(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

Applying KVL to loop-2

$$V_o(s) = \left(\frac{1}{Cs}\right) I(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

$$I(s) = V_o(s) / \left(\frac{1}{Cs}\right) = CsV_o(s)$$

Now, using equation (2) in equation (1)

$$V_i(s) = \left(R + Ls + \frac{1}{Cs}\right) CsV_o(s)$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{1}{\left(R + Ls + \frac{1}{Cs}\right) Cs} = \frac{1}{LCs^2 + RCs + 1}$$

Then transfer function of the given system is $G(s) = \frac{1}{LCs^2 + RCs + 1}$

CHAPTER - 3

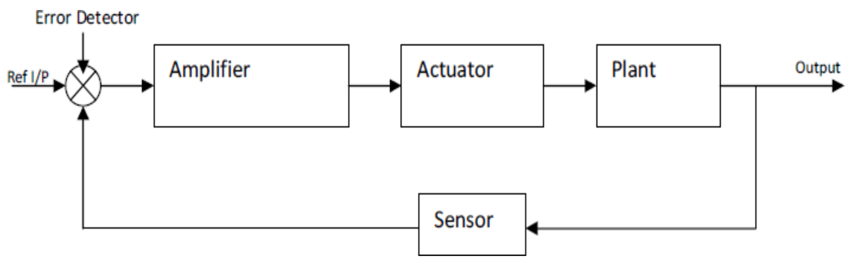
CONTROL SYSTEM COMPONENTS & MATHEMATICAL MODELLING OF PHYSICAL SYSTEM

A physical system is a collection of physical objects connected together to serve an objective. An idealized physical system is called a Physical model. Once a physical model is obtained, the next step is to obtain Mathematical model. When a mathematical model is solved for various i/p conditions, the result represents the dynamic behavior of the system.

3.1 COMPONENTS OF CONTROL SYSTEM:

The components of automatic control systems are

- Error detector
- Amplifier & Controller
- Actuator
- Plant
- Sensor of feedback system



3.2 POTENTIOMETER, SYNCHRONOUS, DIODE MODULATOR & DEMODULATOR:

3.2.1 POTENTIOMETER:

- It is the device converts a linear or angular displacement in to a voltage.
- It is an example of error detector.
- It is avariable resistance whose value varies according to angular position or linear displacement of the wiper contact.
- The resistance element of the potentiometer is constructed by winding resistance wire on a conducting material on a plastic base. The wiper is attached to the i/p shaft of the potentiometer.
- On application of the displacement to i/p shaft, the shaft moves & hence the wiper contact slides over the resistance material. The excitation of the potentiometer is done by either DC or AC voltage source and the o/p is measured at the wiper contact w.r.t reference.

Consider the fig.3.2.1 (a)given below

Let E_i= input voltage

E_0 = output voltage

x_i = displacement from zero position

x_t = total length of translational potentiometer

R = total resistance of potentiometer

Linear relationship ofthe displacement potentiometer

Under ideal condition the o/p voltage E_0 is given by

$\mathbf{E_0 = \frac{x_i}{x_t} E_i}$

Fig. (a) shows a linear relationship shown in fig. (b)

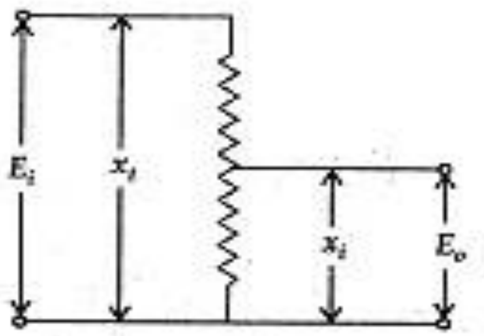


Fig. 3.2.1 (a)

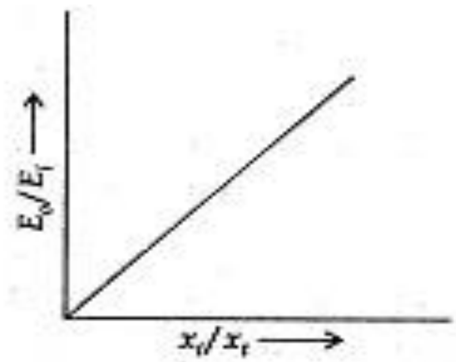


fig. 3.2.1 (b)

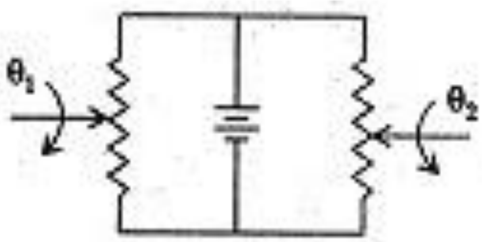
Similarly, for rotational motion, the o/p voltage E_0 is given by $E_0 = \frac{\theta i}{\theta t} E_i$

Where θ_i = input angular displacement

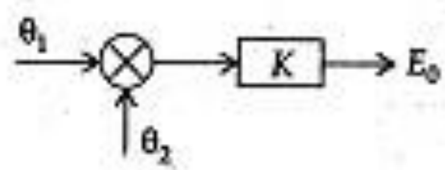
θ_t = total travel of wiper

The fig. (i) shows an arrangement of error sensing transducer.

The two potentiometer are connected in parallel and o/p voltage taken across the variable terminals of the two potentiometer.



3.2.1 Fig. (i)



3.2.1Fig. (ii)

The o/p voltage E_0 is given by

$E_0 = K(\theta_1 - \theta_2)$

The block diagram is shown in fig. (ii)

Where K = Constant and sensitivity of the potentiometer

Unit = volt/degree

E_i= applied input voltage

θ_1 and θ_2 = angular displacement of the wiper

CHARACTERISTICS OF POTENTIOMETER:

- The linear variation of resistance is the ideal characteristics of the potentiometer.
- The impedance of the device is used to measure the o/p voltage of the potentiometer must be high, so that the loading error can be avoided.

3.2.2 SYNCHRONOUS:

It is a self-synchronizing device widely used in servomechanisms as a position indicator.

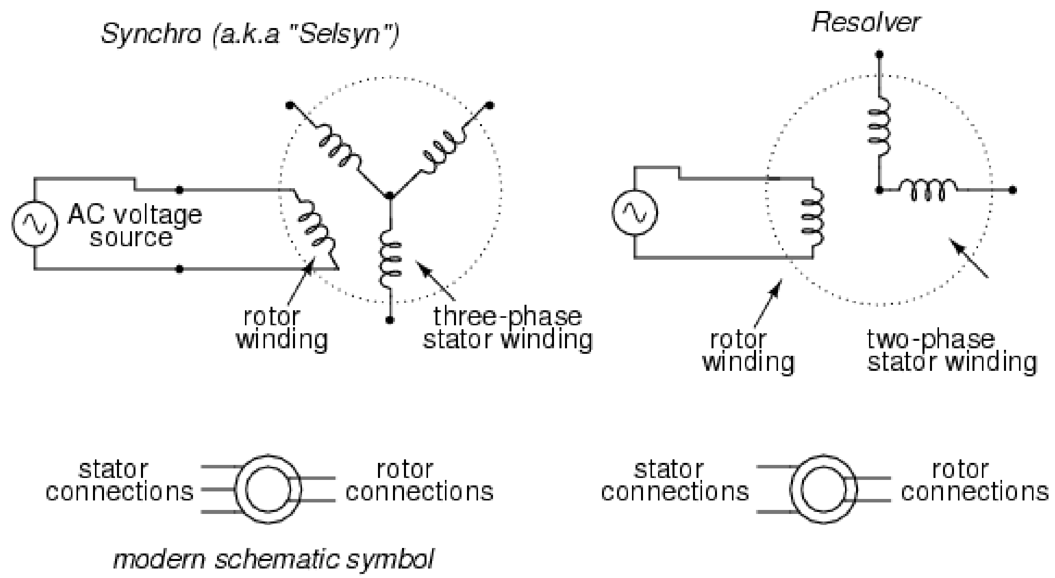
Important synchro systems are,

- Synchro system with transmitter and control transformer
- Synchro system with synchro transmitter and motor
- Synchro system with transmitter, differential and motor

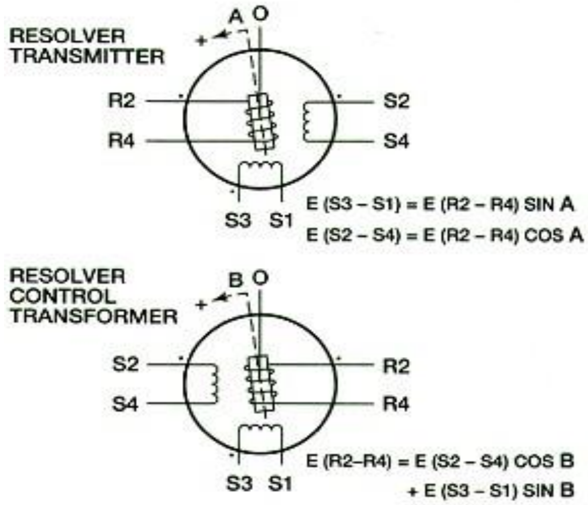
GENERAL CONSTRUCTIONAL FEATURES OF SYNCHRO:

- The construction of synchro transmitter, motor and transformer are almost same.
- Stator laminated silicon steel, slotted to house distributed 3-ϕ,Y-connectedwindings with axes 1200 apart.
- Stator not directly connected to supply
- Rotor is 2-pole (dumb-bell shaped for synchro transmitter and cylindrical shape forcontrol transformer) with single winding connected to AC source. The magneticfield in excited rotor induces voltages in stator coils. The magnitude of voltageinduced in any stator coil depends on the angular position of coil's axis with respectto rotor axis.
- Synchro control transformer has cylindrical shape rotor so that air gap flux isuniformly distributed around the rotor.

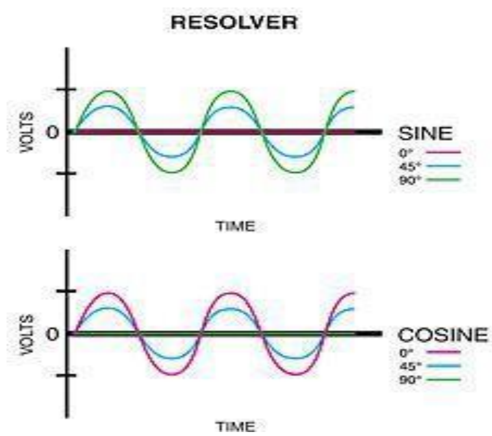
CONSTRUCTIONAL FEATURES:



(a)Constructional features



(b)Electrical Circuit



(c)Schematic Symbol
(Fig. Synchro Transmitter)

SYNCHRO TRANSMITTER:

It is not a three phase machine. It is a single phase machine. Here, input is angular position of its rotor shaft.Output is a set of three stator coil-to-coil voltages. Common connection between the stator coils is not accessible.

SYNCHRO SYSTEM WITH TRANSMITTER AND CONTROL TRANSFORMER:

1. A synchro error detector system may consist of synchro transmitter and synchro control transformer.
2. It compares two angular displacements and the output voltage is approximately linear with angular difference or misalignment between shafts of transmitter & Control transformer.
3. Usedas error detector in feedback systems.

SYNCHRO SYSTEM WITH SYNCHRO TRANSMITTER AND SYNCHRO MOTOR:

The rotors of both the synchro devices are connected to same AC source. Figure (b) shows a circuit configuration, using two synchros, for maintaining synchronism between two shafts. When rotor windings are excited, emfs are induced by transformer action in the stator windings of transmitter and motor. If the two shafts are in similar positions (relative to that of the stator windings), then there are two emfs of equal value are induced in the two stator windings.Also no circulating current exists and hence no torque is produced. If the two shaft positions do not match, the emfs are unequal and result circulating current to flow. The circulating current in conjunction with air gap magnetic field produce torque which tend to align the shafts.

SYNCHRO SYSTEM WITH TRANSMITTER, DIFFERENTIAL AND MOTOR:

The function of this system is to permit the rotation of a shaft to be a function of sum or difference of the rotations of two other shafts. The differential has 3-phase distributed windings on both stator and rotor. The voltages impressed on its stator windings induce corresponding voltages in its rotor windings.

α_r =Displacement of receiver shaft

α_s=Displacement of transmitter shaft

α_d=Displacement of differential shaft

Then, $\alpha_r = \alpha_t - \alpha_d t$

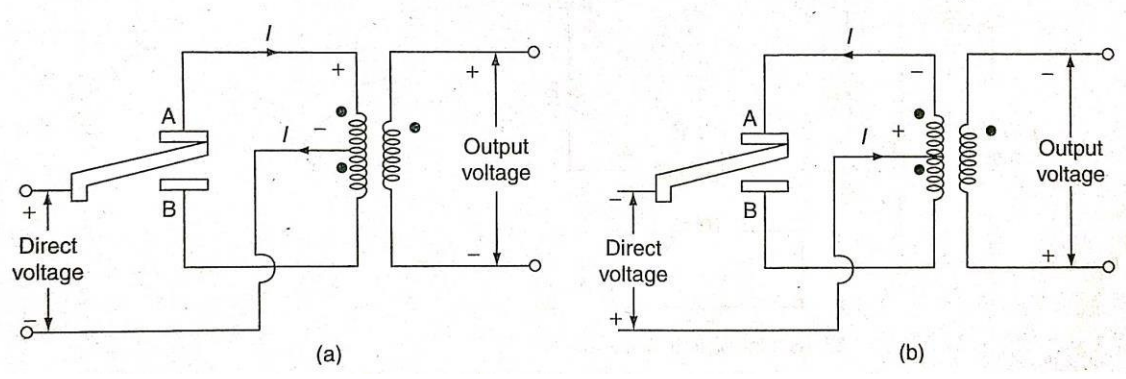
If the phase sequence of stator and rotor windings of differential are reversed then

$$\alpha_r = \alpha_t + \alpha_d t$$

3.2.3 DIODE MODULATOR & DEMODULATOR:

4.6.2 Diode Modulator

Figure 4.17 shows a diode modulation in which two diodes have been used. A sinusoidal supply of frequency ω_c was used here to excite the primary of the centre-tapped transformer. The centre point of the



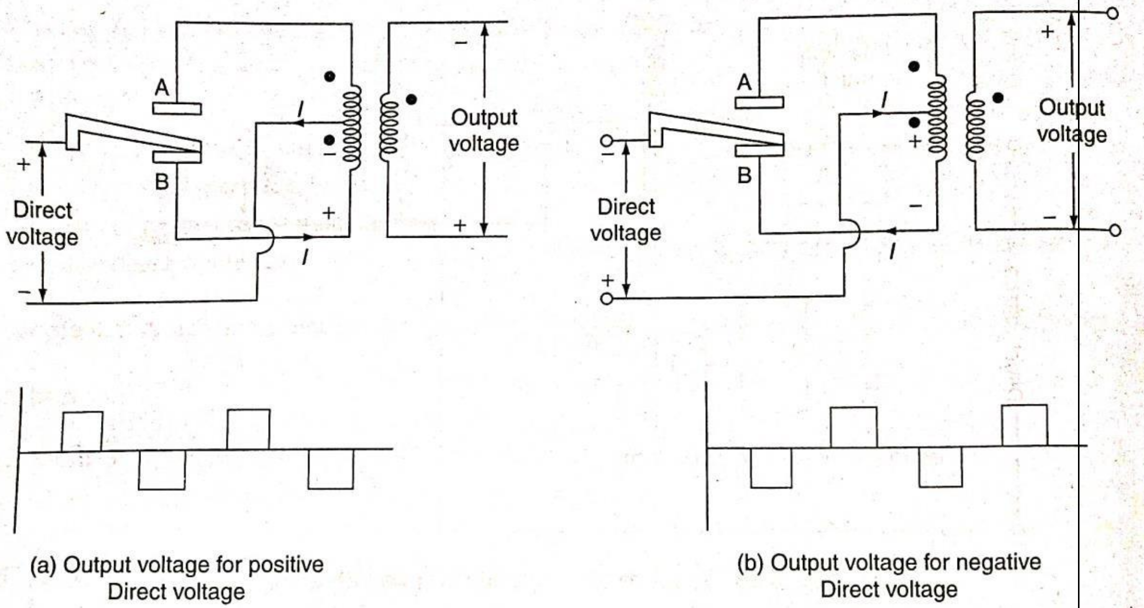
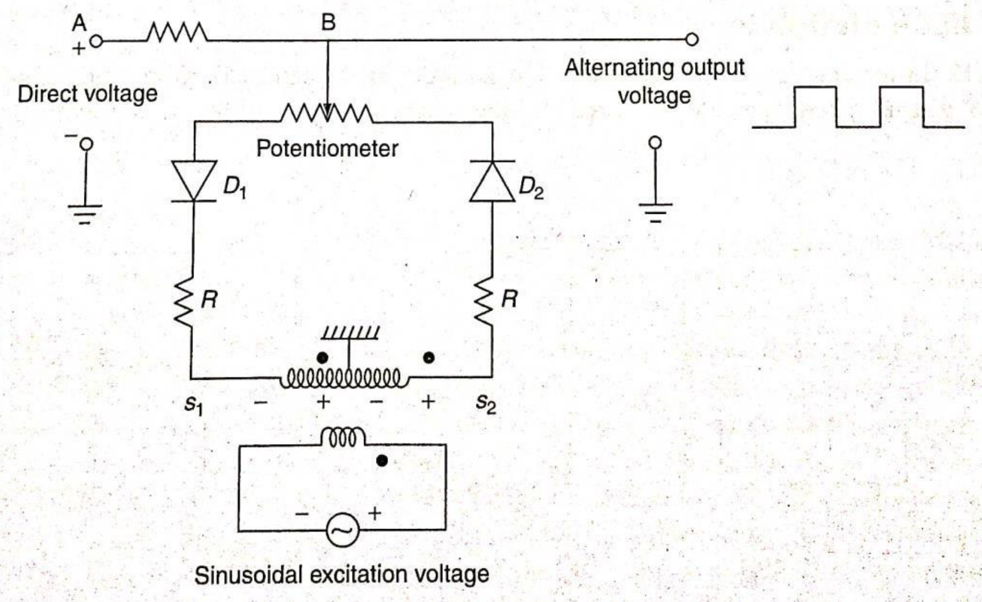


Fig. 4.16 Polarity of output voltages

secondary winding of the transformer is grounded and hence the phase shift of the emfs included in the two halves of secondary will be 180°.

The end s_2 is positive while the end s_1 is negative during the positive half cycle of the supply voltage, which leads the diode D_1 and D_2 to be forward biased. Hence, a current circulates in the secondary of transformer. The voltage between point B and ground becomes zero when the moving contact of



potentiometer is located exactly at the midpoint. This is valid strictly under the assumption that the diodes are identical and the resistances in series with the diode are identical. Therefore, the output voltage becomes zero.

The end s_2 becomes negative while the end s_1 becomes positive during the negative half cycle of the supply voltage. The diodes D_1 and D_2 become reverse biased. The point B is disconnected from the transformer and it is at the same potential of A. Therefore, the output voltage is equal to the DC voltage applied between point A and ground.

The output voltage is shown in Fig. 4.17, which oscillates between two voltage levels-ground potential and direct voltage and hence it becomes a square wave having frequency same as that of excitation voltage and an amplitude proportional to direct voltage applied at the input.

4.6.3 Diode Bridge Modulator

Figure 4.18 shows a bridge modulator, which consists of four diodes. These diodes are connected such that a bridge is formed. The circuit of bridge modulator consists of two centre-tapped transformers T_1 and T_2. The primary of the centre-tapped transformer T_1 is excited by the sinusoidal voltage having frequency ω_c. This voltage is known as reference voltage. The direct voltage, which needs to be modulated, is applied between the centre point of secondary of the transformer (T_1) and ground. On the other hand, the primary of the transformer (T_2) is centre tapped and the output is available from secondary side of T_2.

The polarity of emf induced in the secondary of T_1 during one half cycle is such that it forward biases the diodes D_1 and D_2. The diodes D_3 and D_4 are reverse biased. The point A becomes at the potential, which is equal to that of DC voltage. The DC voltage is available across the upper half of the transformer T_2 and hence the current flows from P_1 to ground in the primary of T_2, which induces an emf in the secondary of T_2.

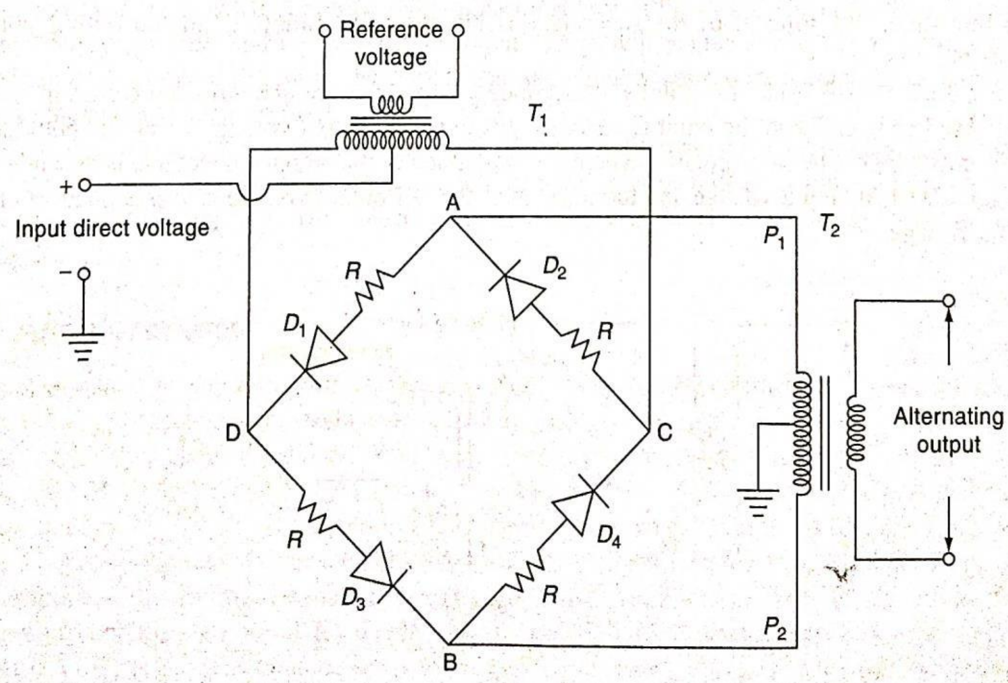


Fig. 4.18 Diode bridge modulator

The polarity of the emf induced in the secondary of T_1 during the next half cycle of reference voltage is such that it forward biases the diodes D_3 and D_4 and reverse biases the diodes D_1 and D_2. This leads to the case that the point B is at the potential same that of direct voltage. The direct voltage is available across the lower half primary winding of T_2. This causes a current to flow from P_2 to ground in the primary of T_2. The direction of flow current in this case is opposite to the previous cycle. This leads to induce an emf in the secondary of T_2 and the polarity of the induced emf is opposite to that of the emf induced during previous cycle.

From above discussion, it is clear that an alternating voltage is developed across the secondary of T_2 due to the circulation of current in the upper and lower halves of primary winding of T_2 in the alternate half cycles of reference voltage. This circulation of current is due to the direct input voltage. The direction of flow of currents is reversed in the alternate half cycles. The magnitude and polarity of alternating output voltage are totally dependent on the magnitude and polarity of DC output voltage, respectively. The frequency of alternating output voltage is same as that of reference voltage.

4.6.4 Synchronous Vibrator as a Demodulator

The reverse process of modulation is known as demodulation. If the modulated alternating voltage is used as input to the transformer, any synchronous vibration will function as demodulator. Figure 4.19 shows a synchronous vibration, which works as a demodulator. The driving coil of this synchronous vibration shown in Fig. 4.19 is excited by an AC signal having frequency same as that of the supply frequency. The secondary winding of the transformer is centre tapped and the DC output is obtained between the vibrator and the centre-tapped transformer, which has been shown in Fig. 4.19.

The vibrator, which is designed to operate in the range of 50–400 Hz, vibrates in synchronism with excitation voltage and the choice of excitation frequency is dependent on the signal frequency. The vibration comes in contact with fixed contact A during the first half cycle of AC input because during this time, the point s_1 of the secondary is positive. The point 'C' of the output voltage becomes positive.

During the next half cycle, the point s_2 of secondary is positive and the vibration comes in contact with the fixed contact B and the point C becomes positive. The output voltage during the both halves becomes positive i.e., the same polarity where the magnitude of the output DC voltage is dependent on the magnitude of AC input voltage and the polarity of the DC output voltage is also dependent on the AC input voltage.

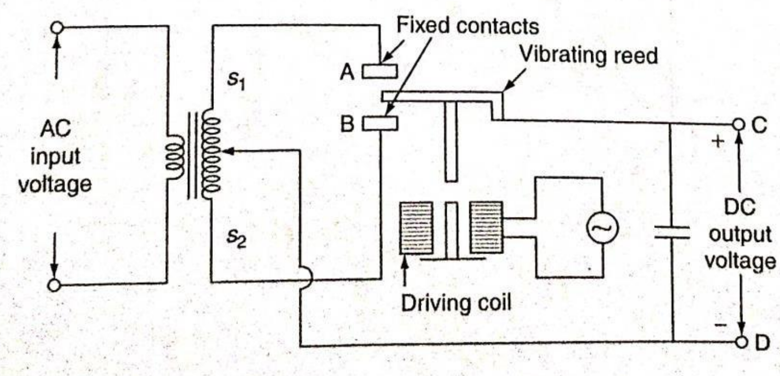


Fig. 4.19 Synchronous vibrator as a demodulator

4.6.5 Diode Demodulator

Figure 4.20 shows a diode modulator in which the excitation voltage is alternating having frequency same as that of input signal. The input signal has been applied to the primary of the transforms (T_1).

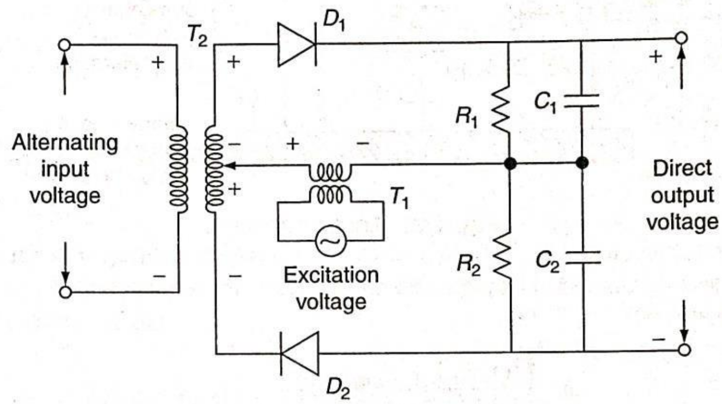


Fig. 4.20 Diode demodulator

The input voltage is assumed to be in phase with the excitation voltage. During positive half cycle, the diodes D_1 and D_2 will conduct due to the nature of the polarities of the induced emfs in the secondary of T_2 as shown in Fig. 4.20. The available voltage across D_1 is more than that of D_2 because there exists an additive secondary voltage in the closed path of D_1 and subtractive voltage in the closed path of D_2. Hence, the diode D_1 conducts more current than D_2 resulting in greater charge storage in C_1 compared to that of C_2. The resulting polarity has been shown in Fig. 4.20.

During the negative half cycle, the diodes D_1 and D_2 will be reverse biased because the polarity of the secondary emfs is exactly opposite to the polarity shown in Fig. 4.20. The capacitors C_1 and C_2 in this will function as filter and hence the direct output voltage will be maintained constant. The excitation voltage in the secondary of T_1 must be greater than the voltage induced in one half of secondary of T_1 to make the operation successful.

The diode D_2 conducts more current than the diode D_1 if the alternating voltage applied at the input is in phase opposition with the excitation voltage. In this case, the polarity of direct output voltage is opposite to the polarity shown in Fig. 4.20. The capacitors maintain the same direct voltage levels during the reactive half cycles.

3.3 DC MOTOR

Comparison of Armature control & Field control Excitation:

Armature Control	Field Control
• Large current is required as the source has to meet the full power requirement of the motor. • Time constant is less. • Damping is provide by back e.m.f. • Efficiency is very high. • It is easy to provide constant field current. • Speed of response of the motor to change in current is fast.	• Requirement of current is small. • Time constant will be more. • Damping is provided by motor & load. • Efficiency is low. • It is difficult to provide constant field current. • Speed of response to change in current is low.

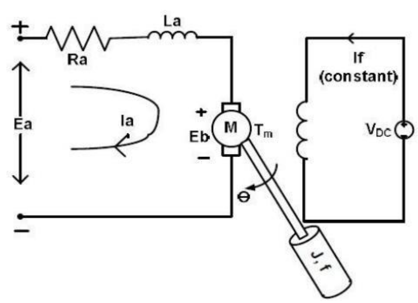
FUTURE DEVELOPMENTS OF DC MOTOR:

(i) Development of rare earth magnet results in DC motor high torque to volume ratio.
(ii) Advances in brush commutator technology make trouble free maintenance.
(iii) Development of brushless DC motors

MERITS OF DC MOTOR:	DEMERITS OF DC MOTOR:
(i) Linear characteristics, (ii) Used for large power applications, (iii) Easier control	(i) Lower torque to volume and (ii) Lower torque to inertia ratio

MATHEMATICAL MODEL OF ARMATURE CONTROLLED DC MOTOR:

The armature control type speed control system of a DC motor is shown in Fig. The following components are used in this system.



R_a=resistance of armature
L_a=inductance of armature winding
i_a=armature current
I_f=field current
E_a=applied armature voltage
E_b=back emf
T_m=torque developed by motor
Θ=angular displacement of motor shaft
J=equivalent moment of inertia and load referred to motor shaft
f=equivalent viscous friction coefficient of motor and load referred to motor shaft
A Schematic diagram of armature control type speed control system of a DC motor
The air-gap flux Øis proportional of the field current i.e.

$$\phi = K_f I_f \quad \text{.................. (1)}$$

The torque Tm developed by the motor is proportional to the product of armature current and air gap flux i.e.

$$T_m = k_1 K_f I_f i_a \quad \text{..................(2)}$$

In armature-controlled D.C. motor,the field current is kept constant,so that equation can be written as follows.

$$T_m = K_t i_a$$ ……………….(3)

The motor back emf being proportional to speed is given as follows.

$$E_b = K_b\left(\frac{d\theta}{dt}\right)$$ …………………………..(4)

The differential equation of the armature circuit is

$$L_a\left(\frac{di_a}{dt}\right) + R_a i_a + E_b = E_a$$ ……………………..(5)

The torque equation is

$$J\left(\frac{d^2\theta}{dt^2}\right) + f\left(\frac{d\theta}{dt}\right) = T_m = K_t I_a$$ …………………….(6)

Taking the Laplace transforms of equations (4), (5) and (6), assuming zero initial conditions, we get

$$E_b(s) = sK_b\theta(s)$$

$$(sL_a + R_a)I_a(s) = E_a(s) - E_b(s)$$

$$(s^2J + sf)\theta(s) = T_m(s) = K_t I_a$$ ……………….(7,8,9)

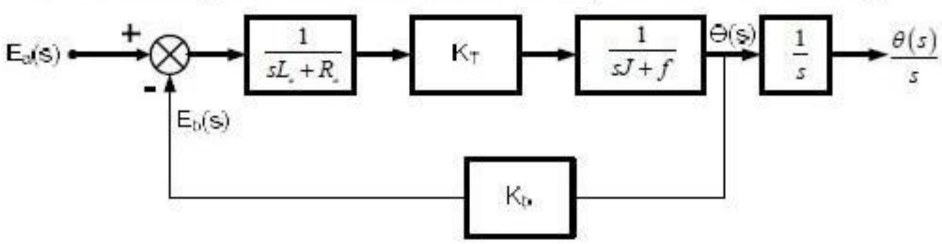
From eq. (7) to (9) the transfer function of the system is obtained as,

$$G(s) = \frac{\theta(s)}{E_a(s)} = \frac{K_t}{s\left[(R_a + sL_a)(sJ + f) + K_t K_b\right]}$$

can be rewritten as

$$G(s) = \frac{\theta(s)}{E_a(s)} = \left[\frac{\dfrac{K_t}{(R_a + sL_a)(sJ + f)}}{1 + \dfrac{K_t K_b}{(R_a + sL_a)(sJ + f)}}\right]\frac{1}{s}$$

The block diagram that is constructed from equations is shown in Fig.



Block diagram of armature control type speed control system of a DC motor

The armature circuit inductance La is usually negligible. Therefore, equation (11) can besimplifiedas follows.

$$\frac{\theta(s)}{E_a(s)} = s^2\left(\frac{K_t}{R_a}\right)J + s\left(f + \frac{K_tK_b}{R_a}\right)$$

The term $\left(f + \frac{K_tK_b}{R_a}\right)$ indicates that the back emf of the motor effectively increases the viscous friction of the system. Let,

$$f' = f + \frac{K_tK_b}{R_a}$$

Where f' be the effective viscous friction coefficient. The transfer function given by eq 13 may be written in the following form.

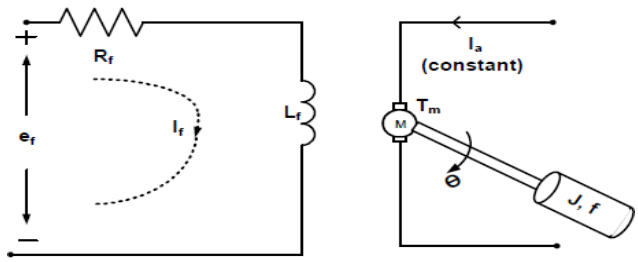
$$\frac{\theta(s)}{E_a(s)} = \frac{K_m}{s(s\tau+1)}$$

Here $K_m = \frac{K_t}{R_a f'}$ = motor gain constant, and $\tau = \frac{J}{f'}$ = motor time constant. Therefore, the motor torque and back emf constant K_t, K_b are interrelated.

MATHEMATICAL MODEL OF FIELD CONTROLLED DC MOTOR:

The field control type speed controlsystem of a DC motor is shown in Fig. below. The following components are used in this system.

R_f=Field winding resistance
L_f=inductance of field winding
I_f=field current
e_f=field control voltage
T_m=torque developed by motor
Ө=angular displacement of motor shaft
J=equivalent moment of inertia and load referred to motor shaft
f=equivalent viscous friction coefficient of motor and load referred to motor shaft



Block diagram of field control type speed control system of a DC motor

In field control motor the armature current is fed from a constant current source.The air-gap flux Φ is proportional of the field current i.e.

$\phi = K_f I_f$ ....(1)

The torque Tm developed by the motor is proportional to the product of armature current and air gap flux i.e.

$T_m = k_1 K_f I_f I_a = K_t I_f$ …………………….(2)

The equation for the field circuit is

$$L_f \frac{dI_f}{dt} + R_f I_f = E_f \qquad \text{……………………….(3)}$$

The torque equation is

$$J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} = T_m = K_t I_f \qquad \text{… ………………(4)}$$

Taking the Laplace transforms of equations (3) and (4) assuming zero initial conditions, we get the following equations.

$$\left(L_f s + R_f\right) I_f(s) = E_f(s)$$

and $$\left(Js^2 + fs\right)\theta(s) = T_m(s) = K_t I_f(s) \qquad \textbf{…………………………………(5,6)}$$

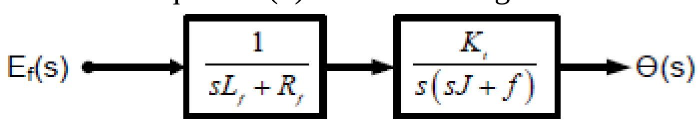
From eq(5) and (6) the transfer function of the system is obtained as,

$$G(s) = \frac{\theta(s)}{E_f(s)} = \frac{K_t}{s\left(R_f + sL_f\right)\left(Js + f\right)} \qquad \textbf{………………………………..(7)}$$

The transfer function given by eq. (7) may be written in the following form.

$$\frac{\theta(s)}{E_a(s)} = \frac{K_t}{s\left(L_f s + R_f\right)\left(Js + f\right)} = \frac{K_m}{s(s\tau + 1)(s\tau' + 1)} \qquad \textbf{……………………….(8)}$$

Here $K_m = \dfrac{K_t}{R_f f}$ = motor gain constant, and $\tau = \dfrac{L_f}{R_f}$ = time constant of field circuit and $\tau' = \dfrac{J}{f}$ = mechanical time constant. For small size motors field control is advantageous. The block diagram that is constructed from equation (8) is shown in Fig. below.

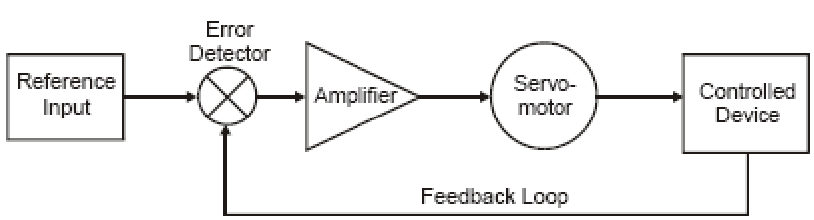


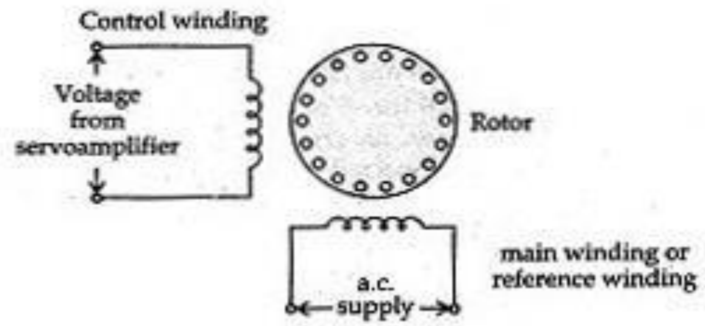
Block diagram of field control type speed control system of a DC motor

3.3 AC SERVOMOTORS

A two phase servomotor (Induction Motor) (A few watts to hundred watts) is commonly usedin feedback control systems. In servo applications, an induction motor is required to produce rapid accelerations from standstill.

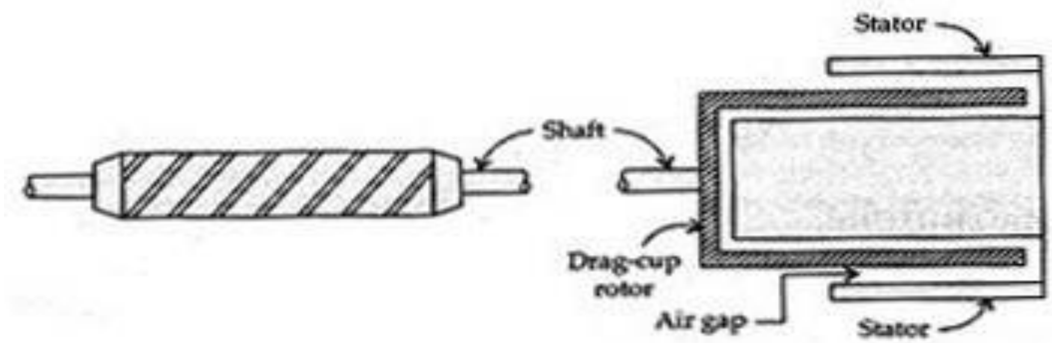
Schematic Diagram





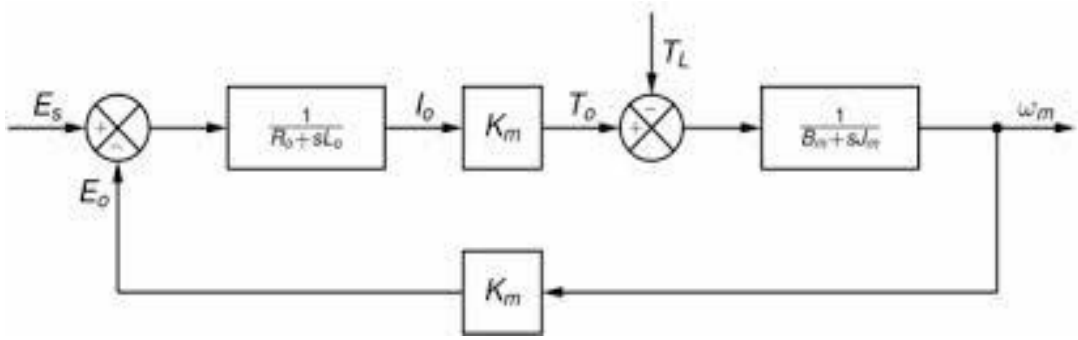
CONSTRUCTIONAL FEATURES:

- Squirrel Cage rotor with Cu or Al conductor
- High Rotor resistance
- Small diameter to length ratio to minimize inertia
- Two stator windings in space quadrature(One called reference winding and the other Controlwinding)
- The two voltages to stator windings must derived from same source



PRINCIPLE OF OPERATION:

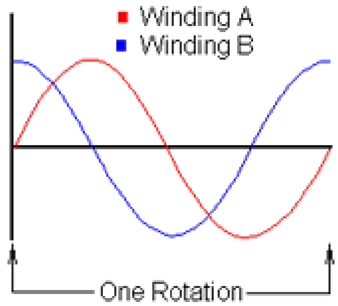
(i) The two applied AC voltage to stators with a phase difference produce a rotating flux.
(ii) As this moving flux sweeps over the rotor conductors, small emf is induced in rotor. Rotor being short circuited, currents will flow and this current interacts with rotating flux to produce a torque in the rotor. This torque causes the rotor to turn so that it chases the rotating magnetic flux.



TORQUE-SPEED CHARACTERISTICS OF AC SERVOMOTOR:

For induction motor in high power applications, rotor resistance is low in order to obtain maximum torque. Positive slope part of the characteristics is not suitable to control applications as this results instability.

In AC servomotor high resistance rotor results in negative torque-speed characteristics. This characteristic is needed for positive damping and good stability. The rotor has a small diameter–to-length ratio to minimize the moment of inertia and to give a good accelerating characteristic. However, more rotor resistance results more loss and less efficiency.



TRANSFER FUNCTION:

The torque developed is a function of shaft angular position (Θ) and control voltage Ec

$$G(s)=\frac{\theta(s)}{E_c(s)}=\frac{K_m}{Js^2+Ds}=\frac{K_m}{s(Js+D)}=\frac{K}{s(T_m s+1)}$$

Where, $K=\frac{K_m}{D}$ = motor gain constant, $T_m=\frac{J}{D}$ = motor time constant

Merits of AC Servomotors	Demerits of AC Servomotors
✦ Lower cost ✦ Less weight and inertia ✦ Higher efficiency ✦ Fewer maintenance requirements (since no commutator or brush)	✦ Nonlinear characteristics ✦ Used for low power applications (e.g. Instrument Servo) ✦ Difficult for speed control and positioning

COMPARISON BETWEEN A.C. AND D.C. SERVOMOTOR:

A.C. Servomotor	D.C. Servomotor
✓ Low power output ✓ Efficiency is less ✓ No brushes and slip rings hence maintenance free ✓ No radio frequency noise ✓ Smooth operation	✓ High power output ✓ High efficiency ✓ Frequent maintenance required ✓ Brushes produce radio frequency noise ✓ Noisy operation

APPLICATION OF SERVOMOTORS:

- Radars
- Electromechanical actuators
- Computers
- Machine tools
- Tracking and guided system
- Process controllers
- Robots

3.4 MODELLING OF AN ELECTRICAL SYSTEMS (R,L,C, ANALOGOUS SYSTEMS)

3.4.1 ANALOGOUS SYSTEM:

It is very useful in practice. Since one type of system may be easier to handle experimentally than another. A given electrical system consisting of resistance, inductance & capacitances may be analogous to the mechanical system consisting of suitable combination of Dash pot, Mass & Spring. The advantages of electrical systems are,

1. Many circuit theorems, impedance concepts can be applicable.
2. An Electrical engineer familiar with electrical systems can easily analyze the system under study & can predict the behavior of the system.
3. The electrical analog system is easy to handle experimentally

3.4.2 COMPONENTS OF AN ELECTRICAL SYSTEM:

There are three basic elements in an electrical system,
i.e. (a) resistor (R), (b) inductor(L) and (c) capacitor (C). Electrical systems are of two types, i.e. (i) voltage source electrical system and (ii) current source electrical system.

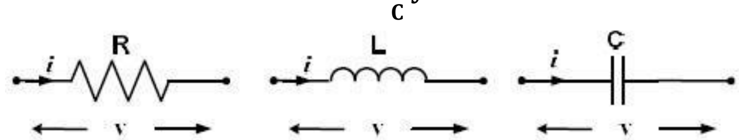
VOLTAGE SOURCE ELECTRICAL SYSTEM:

If 'i' is the current through a resistor and v is the voltage drop in it, then **v = Ri** .
If 'i' is the current through an inductor and v is the voltage developed in it,

$$\text{Then } \mathbf{v = L\frac{di}{dt}}$$

If 'i' is the current through a capacitor and v is the voltage developed in it,

$$\text{Then } \mathbf{v = \frac{1}{C}\int idt}$$



Current and voltage shown in resistor, inductor and capacitor

CURRENT SOURCE ELECTRICAL SYSTEM:

If 'i' is the current through a resistor and v is the voltage drop in it, then $\mathbf{i = \frac{v}{R}}$

If 'i' is the current through an inductor and v is the voltage developed in it, then $\mathbf{i = \frac{1}{L}\int vdt.}$

If 'i' is the current through a capacitor and v is the voltage developed in it, then $\mathbf{i = C\frac{dv}{dt}}$

3.4.3 FORCE-VOLTAGE ANALOGY:

The characteristics of an electrical system is identical to that of a mechanical system, then the electrical system is said to be analogous to the mechanical system.

Mathematically, $f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt}kx$

Consider a series RLC circuit as shown in fig. below

$v(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$....................(1)

But, $i = \frac{dq}{dt}$,

$\therefore v(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$...............................(2)

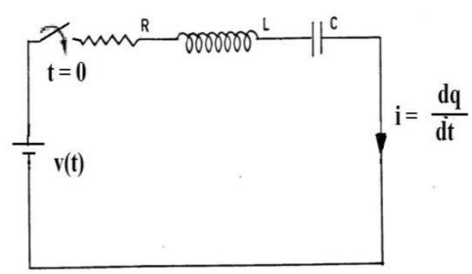


Fig. 3.3.3

Equation(1) with Equation (2) we can written as

Note: The mathematical similarity with respect to the following analogous parameters.

$f(t) \leftrightarrow v(t) x \leftrightarrow q$

k ↔ $\frac{1}{C}$; B ↔ R

M ↔ L; $\frac{dx}{dt}$ ↔ i ∴ $\frac{dx}{dt}$ = velocity

The Electrical systems are

v = Voltage, i = Current, L = Inductance, R = Resistance, q = Charge,

$\frac{1}{C}$ = Reciprocal of Capacitance.

The mass is analogous to inductance, viscous friction is analogous to resistance, spring constant is analogous to reciprocal of capacitance, velocity is analogous to current & displacement is analogous to charge. The force is analogous to voltage is known as force-voltage analogy.

3.4.4 FORCE-CURRENT ANALOGY:

Let the parallel RLC circuit as shown fig. below by using KCL we can write the eq. for total current in the circuit as follow;

i(t) = $C\frac{dv}{dt} + \frac{v}{R} + \frac{1}{L}\int v dt$

= $C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{\phi}{L}$ ……………………..(1)

∴ $v = \frac{d\phi}{dt}$, rate of change of flux.

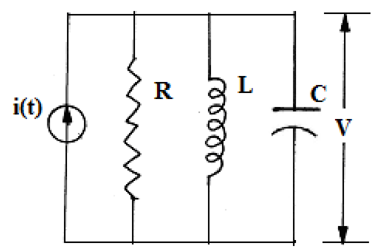


Fig. 3.3.4

We using differential eq. by f(t) = M$\frac{d^2x}{dt^2}$ + B$\frac{dx}{dt}$kx

Put comparing the differential eq. with eq. (1) we can written

f ↔ i(t)

M ↔ C

x ↔ φ

B ↔ $\frac{1}{R}$

v

k↔ $\frac{1}{L}$

dt

The above analogy is called Force-Current analogy. The Electrical Systems are

v = Voltage, i = Current, φ = Flux, C = capacitance,

$\frac{1}{R}$ = Conductance, $\frac{1}{L}$ = Reciprocal of inductance.

Problem 3.4.1:

Find system T.F. between function between the inductance current to the source current in the following RL circuit as shown in Fig. 3.3.1

Solution:

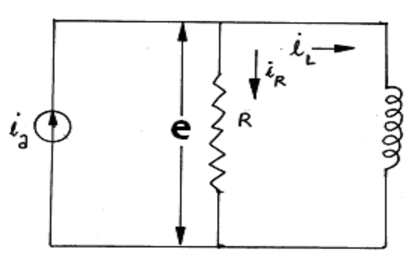


Fig. 3.3.1

Voltage across the Resistance, $e(t) = i_R R \Rightarrow i_R = \frac{e(t)}{R}$

Voltage across the Inductance, $e(t) = L\frac{di_L}{dt} \Rightarrow i_L = \frac{1}{L}\int e(t)dt$

Total current, $i_a = i_R + i_L = \frac{e(t)}{R} + \frac{1}{L}\int e(t)dt$

Laplace transform of the current source,

$I_a(s) = E(s)\left(\frac{1}{R} + \frac{1}{Ls}\right)$ and $I_L(s) = \frac{E}{Ls}$

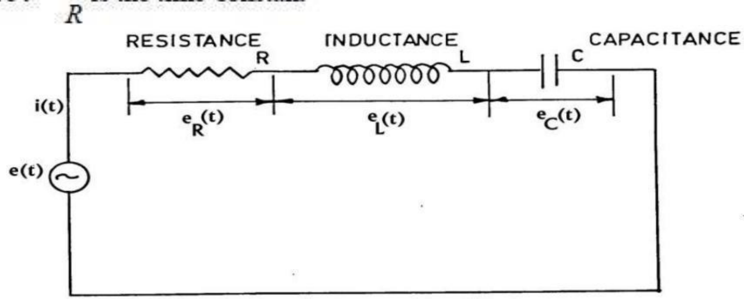
Transfer function between the inductance current to the source current,

$$\frac{I_L(s)}{I_a(s)} = \frac{1}{\frac{L}{R}s+1} = \frac{1}{\tau s+1}$$

where $\tau = \frac{L}{R}$ is the time-constant

Problem 3.4.2:

Find system transfer function betweenfunction between the capacitance voltageto the source voltage in the followingRLC circuit as shown in Fig



Solution:

Voltage across the Resistance, $e_R(t) = iR$

Voltage across the Inductance, $e_L(t) = L\frac{di}{dt}$

Voltage across thecapacitance, $e_C(t) = \frac{1}{C}\int idt$

Total voltage, $e(t) = iR + L\frac{di}{dt} + \frac{1}{C}\int idt$

Laplace transform of the voltage source, $E(s) = I(s)\left(R + Ls + \frac{1}{Cs}\right)$

Transfer function between capacitance voltage and source voltage

$$\frac{E_C(s)}{E(s)} = \frac{1}{Cs\left(R + Ls + \frac{1}{Cs}\right)} = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

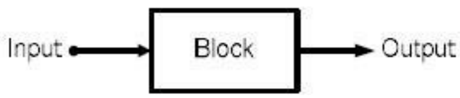
where $\omega_n = \frac{1}{\sqrt{LC}}$ and $\zeta = \frac{R}{2\sqrt{\frac{L}{C}}}$

CHAPTER - 4

BLOCK DIAGRAM& SIGNAL FLOW GRAPHS(SFG)

4.1 DEFINITION OF BASIC ELEMENTS OF BLOCK DIAGRAM :

Block diagram: It is the pictorial representation of each system & it makes easier to understand the system. The short hand pictorial representation the cause-and-response/effect relationshipbetween input and output of a physical system is known as block diagram.



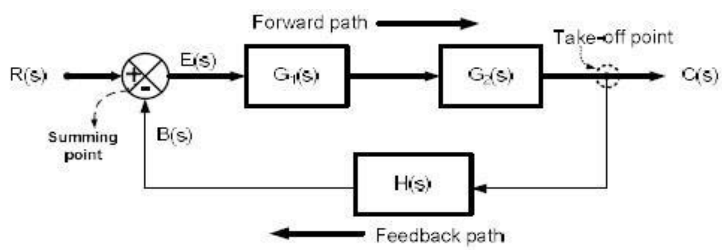
(i) A block representation of a system (ii) A block representation with gain of a system

Output: The value of input multiplied by the gain of the system.

Gain, $G(s)=\frac{C(S)}{R(S)}$

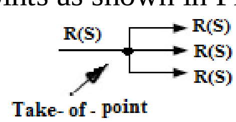
$\therefore$Output $C(s) = G(s)R(s)$

Summing point: It is apoint at which addition or subtraction of two or more signals.In Fig. below, inputs R(s) and B(s) have been given to a summing point and its output signal is E(s). Here, $E(s) = R(s) – B(s)$

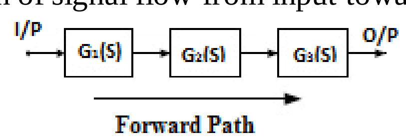


A block diagram representation of a systemshowing its different components

***Take-off- point*:** It is the component of a block diagram model at which a signal can be taken directly and supplied to one or more points as shown in Fig. below



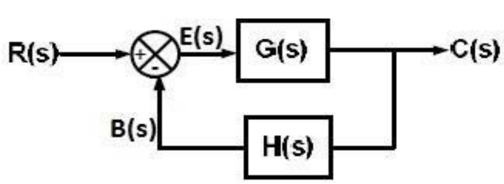
Forward path: It is the direction of signal flow from input towards output.



Feedback path: It is the direction of signal flow from output towards input.

4.2. CANONICAL FORM OF CLOSED LOOP SYSTEM:

It consists of block forward path feedback path summing point & take-off-point.

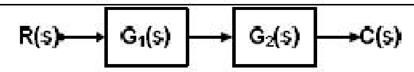
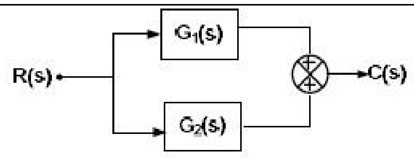
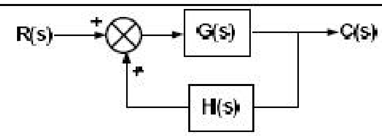
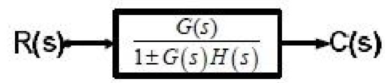
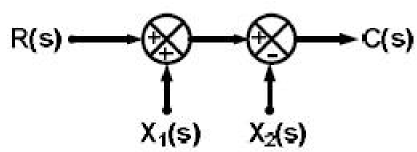
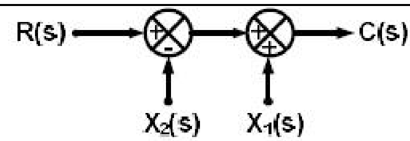
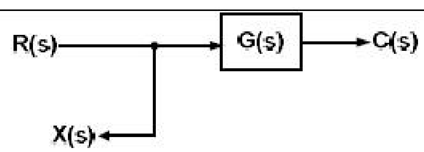
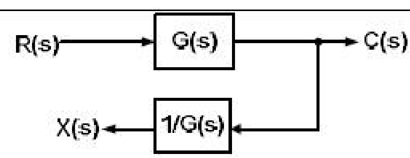
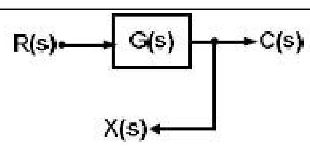
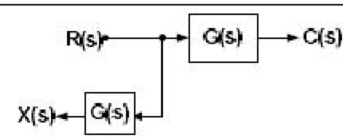
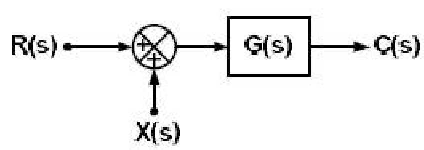
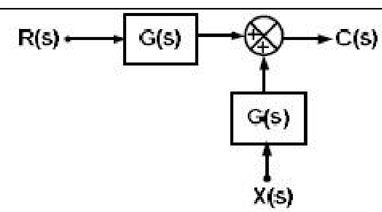
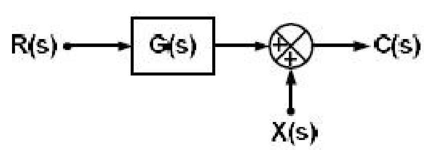
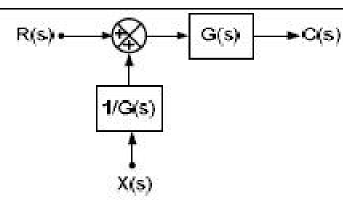


$\frac{C(S)}{R(S)}$ = Closed loop Transfer function =control ratio

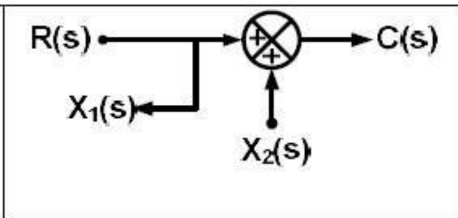
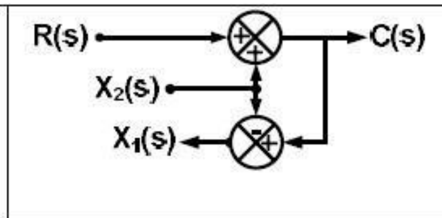
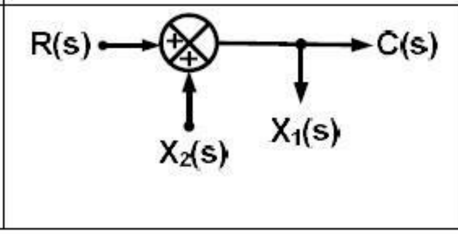
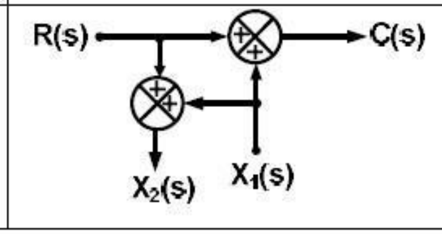
$\frac{E(S)}{R(S)}$ = Error ratio

$\frac{B(S)}{R(S)}$ = Primary feedback ratio

4.3 RULES FOR REDUCTION OF BLOCK DIAGRAM MODEL:

Sl. No.	Rule No.	Configuration	Equivalent	Name
1	Rule 1			Cascade
2	Rule 2			Parallel
3	Rule 3			Loop
4	Rule 4			Associative Law
5	Rule 5			Move take-off point after a block
6	Rule 6			Move take-off point before a block
7	Rule 7			Move summing-point point after a block
8	Rule 8			Move summing-point point before a block

4.3 RULES FOR REDUCTION OF BLOCK DIAGRAM MODEL:

9	Rule 9			Move take-off point after a summing-point
10	Rule 10			Move take-off point before a summing-point

4.4 PROCEDURE FOR REDUCTION OF BLOCK DIAGRAM MODEL:

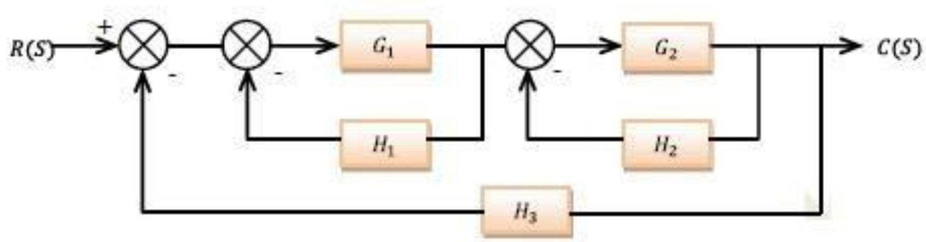
Step 1: Reduce the cascade blocks.
Step 2: Reduce the parallel blocks.
Step 3: Reduce the internal feedback loops.
Step 4: It is ad visibleShift take-off points towards right and summing points towards left.
Step 5: Repeat step 1 to step 4 until the simple form is obtained.
Step 6: Find transfer function of whole system as $\frac{C(S)}{R(S)}$

PROCEDURE FOR FINDING OUTPUT OF BLOCK DIAGRAM MODEL WITH MULTIPLE INPUTS:

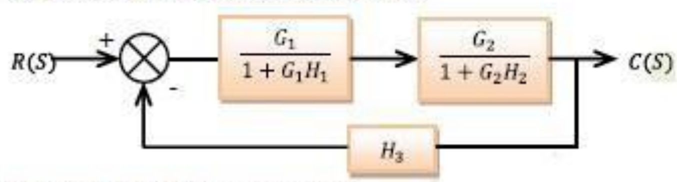
Step 1: Consider one input taking rest of the inputs zero, find output using the procedure described in section.
Step 2: Follow step 1 for each inputs of the given Block Diagram model and find their corresponding outputs.
Step 3: Find the resultant output by adding all individual outputs.

PROBLEMS 4.5.1:

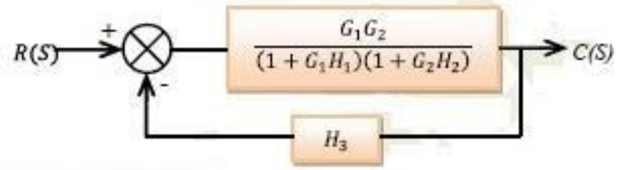
1. Reduce the Block Diagrams shown below:



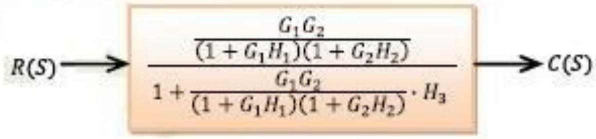
Solution: By eliminating the feed-back paths, we get

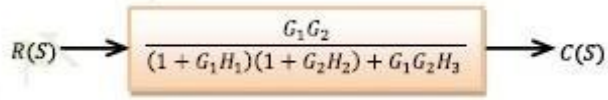


Combining the blocks in series, we get



Eliminating the feed back path, we get





$$\Rightarrow \boxed{TF=\frac{C(S)}{R(S)}=\frac{G_1G_2}{(1+G_1H_1)(1+G_2H_2)+G_1G_2H_3}}$$

4.6 BASIC DEFINITIONS IN SFG &PROPERTIES :

4.6.1 BASIC DEFINITIONS IN SFG:

It is a pictorial representation of a system that graphically displays the signal transmission in it.

4.6.2 PROPERTIES OF SFG

- It is applicable to linear time-invariant systems.
- The signal flow is only along the direction of arrows.
- The value of variable at each node is equal to the algebraic sum of all signals entering at that node.
- It is given by Mason's gain formula.
- It is not be the unique property of the system.

4.7 MASON'S GAIN FORMULA

Transfer function of a system:

$$G(s)=\frac{C(s)}{R(s)}=\frac{\sum_{k=1}^{N}P_k\Delta_k}{\Delta}$$

Where,

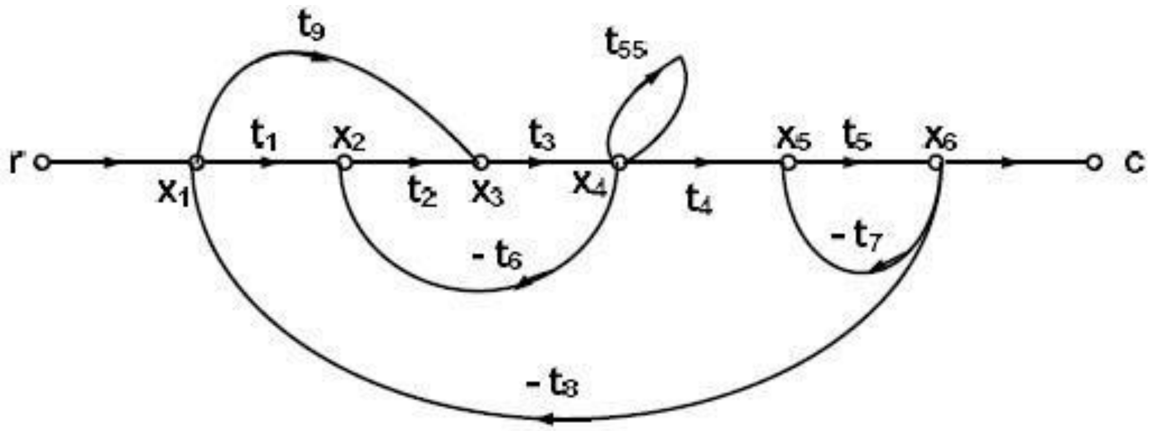
N= total number of forward paths

P$_k$= path gain of k$_{th}$ forward path

Δ= 1 - (Σloop gains of all individual loops) + (Σgain product of loop gains of all possible two non-touching loops) - (Σgain product of loop gains of all possible three non-touching loops) + ……….

Δ$_k$= value of Δ after eliminating all loops that touches k$_{th}$ forward path

4.8 STEPS FOR SOLVING SIGNAL FLOW GRAPH



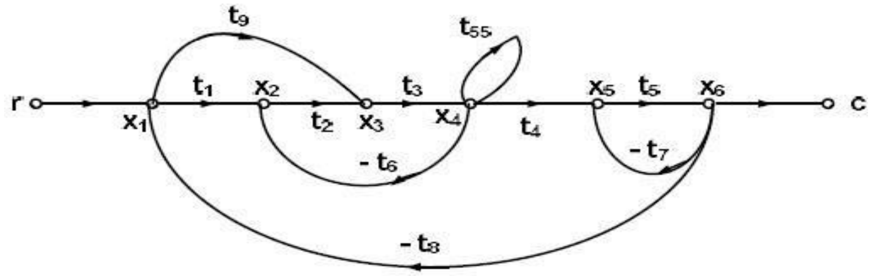
Example of a fig.4.8.1 SFG model

Input or source node: It is a node that has only outgoing branches i.e. node ‘r’ in Fig.4.5.1
Output or sink node: It is a node that has only incoming branches i.e. node ‘c’ in Fig.4.5.1
Chain node**:** It is a node that has both incoming and outgoing branches i.e. nodes ‘X_1’, ‘X_2’,‘X_3’,‘X_4’,‘X_5’and ‘X_6’in Fig.4.5.1
Gain or transmittance**:** It is the relationship between variables denoted by two nodes or value of a branch. In Fig.4.5.1, transmittances are‘t_1’, ‘t_2’,‘t_3’,‘t_4’,‘t_5’and ‘t_6’.
Forward path**:** It is a path from input node to output node without repeating any of the nodes in between them. In Fig., there are two forward paths, i.e. path-1:‘r-X_1-X_2-X_3-X_4-X_5-X_6-C’ and path-2:‘r-X_1-X_3-X_4-X_5-X_6-C’.

Feedback path**:** It is a path from output node or a node near output node to a node near input node without repeating any of the nodes in between them.
Loop: It is a closed path that starts from one node and reaches the same node after trading through other nodes. In Fig.4.5.1, there are four loops, i.e. loop-1:‘X_2-X_3-X_4-X_1’, loop-2:‘X_5-X_6-X_5’,
loop-3:‘X_1-X_2-X_3-X_4-X_5-X_6-X_1’and loop-4:‘X_1-X_3-X_4-X_5-X_6-X_1’.
Self Loop**:** It is a loop that starts from one node and reaches the same node without trading through other nodes i.e. loop in node ‘x_4’ with transmittance ‘t_{55}’ in Fig.4.5.1
Path gain**:** It is the product of gains or transmittances of all branches of a forward path. In Fig., the path gains are $P_1 = t_1t_2t_3t_4t_5$(for path-1) and $P_2 = t_9t_3t_4t_5$(for path-2).
Loop gain**:** It is the product of gains or transmittances of all branches of a loop In Fig.4.5.1 there are four loops, i.e. $L_1 = -t_2t_3t_6$, $L_2 = -t_5t_7$, $L_3 = -t_1t_2t_3t_4t_5t_8$, and $L_4 = -t_9t_3t_4t_5t_8$.
Dummy node**:** If the first node is not an input node and/or the last node is not an output node than a node is connected before the existing first node and a node is connected after the existing last node with unity transmittances. These nodes are called dummy nodes. In Fig.4.5.1 ‘r’ and ‘c’ are the dummy nodes.
Non-touching Loops**:** Two or more loops are non-touching loops if they don’t have any common nodes between them. In Fig.,L_1 and L_2 are non-touching loops.

PROBLEM 4.9.1:

Find the overall transfer function of the system given in Fig. using Mason's gainformula.



Solution:

No. of forward paths: **N =2**

Path gain of forward paths: $P_1 = t_1t_2t_3t_4t_5$ **and** $P_2 = t_6t_3t_4t_5$

Loop gain of individual loops: $L_1 = -t_2t_3t_6$, $L_2 = -t_5t_7$, $L_3 = -t_1t_2t_3t_4t_5t_8$ **and** $L_4 = -t_9t_3t_4t_5t_8$

No. of two non-touching loops = **2** i.e. L_1 **and** L_2

No. of more than two non-touching loops = **0**

$$\Delta = 1-(L_1+L_2+L_3+L_4)+(L_1L_2)-0 = 1-L_1-L_2-L_3-L_4+L_1L_2$$

$$\Delta_1 = 1-0 = 1 \text{ and } \Delta_2 = 1-0 = 1$$

$$G(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\Rightarrow G(s) = \frac{(t_1t_2t_3t_4t_5)(1)+(t_6t_3t_4t_5)(1)}{1+t_2t_3t_6+t_5t_7+t_1t_2t_3t_4t_5t_8+t_9t_3t_4t_5t_8+t_2t_3t_5t_6t_7}$$

$$\Rightarrow G(s) = \frac{t_1t_2t_3t_4t_5+t_6t_3t_4t_5}{1+t_2t_3t_6+t_5t_7+t_1t_2t_3t_4t_5t_8+t_9t_3t_4t_5t_8+t_2t_3t_5t_6t_7}$$

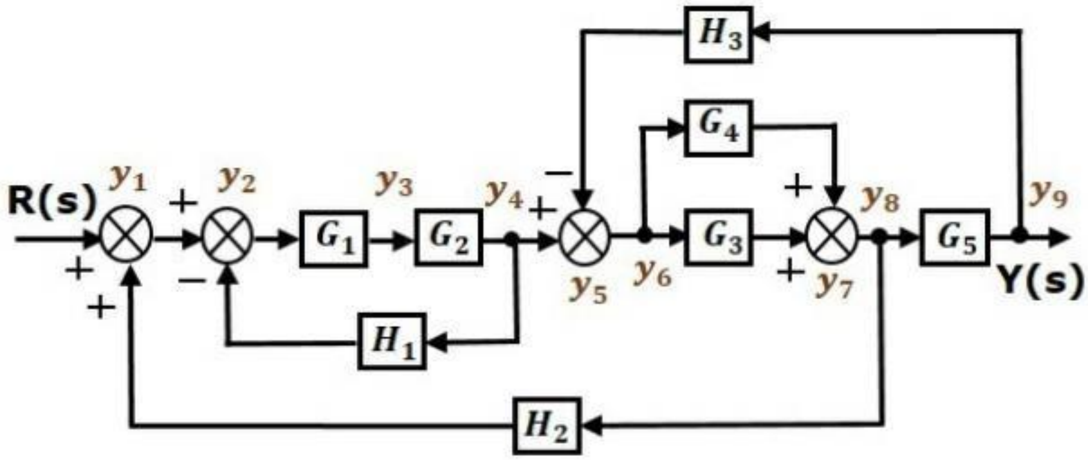
4.6.2 CONVERSION OF BLOCK DIAGRAMS IN TO SFG:

Follow the steps for converting a block diagram in to its equivalent SFG.

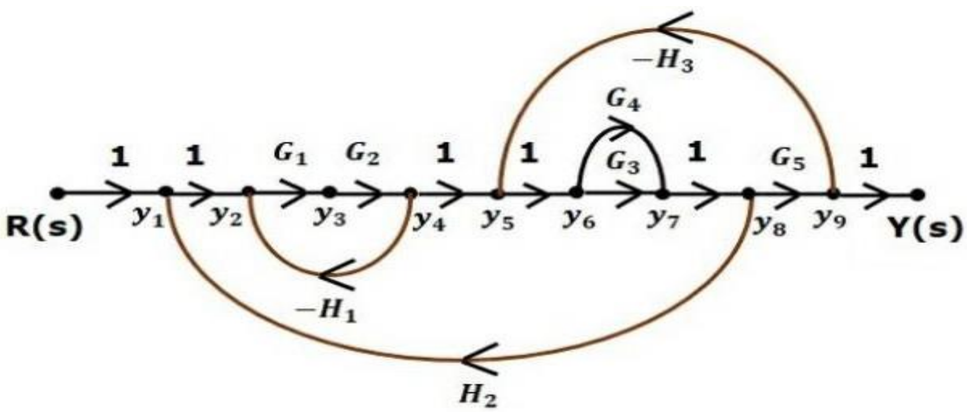
- All the signals, variables summing point and take-off points of block diagram as nodes in SFG.
- The blocks of block diagram as branches in SFG.
- The Transfer function inside the blocks of block diagram as gains of the branches in SFG.
- Connects the node as per the block diagram.

EXAMPLE 4.9.2:

Let us convert the following block diagram in to its equivalent SFG.



Represent the input signal R(s) and output signal C(s) of block diagram as input node R(s) and output node C(s) of SFG. Just for reference, the remaining nodes (y_1 to y_9) are labelled in the block diagram. There are nine nodes other than input and output nodes. That is 4 nodes for 4 summing points, 4 nodes for 4 take-off points and 1 node for the variable between blocks G_1 and G_2. The following fig. given below.



With the help of Mason's gain formula, we can calculate the transfer function of this SFG. It is advantage of SFGs. Here we no need to simplify(reduce) the SFGs for calculating the transfer function.

CHAPTER - 5

TIME DOMAIN ANALYSIS OF CONTROL SYSTEMS

Time domain is the analysis of mathematical functions, physical signals with respect to time. In the time domain, the signal or function's value is known for all real numbers, for the case of continuous time, or at various separate instants in the case of discrete time.

5.1. DEFINITION OF TIME, STABILITY, STEADY-STATE RESPONSE, ACCURACY, TRANSIENT ACCURACY, IN-SENSITIVITY AND ROBUSTNESS

DEFINATION OF TIME

Time is the indefinite continued progress of existence and events that occur in an apparently irreversible succession from the past, through the present, into the future.

DEFINATION OF STABILITY

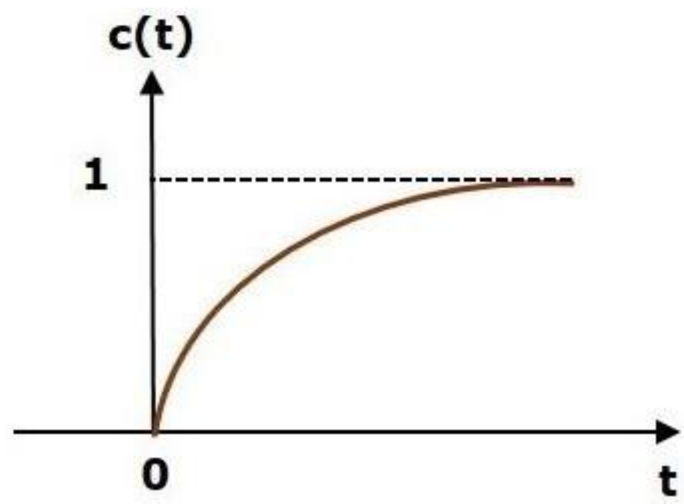
The property of a body that causes it when disturbed from a condition of equilibrium or steady motion to develop forces or moments that restore the original condition.

OR

STABILITY

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of tincluding zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types ofSystems based on StabilityWe can classify the systems based on stability as follows:

- Absolutely stable system
- Marginally stable system
- Absolutely Stable System

DEFINATION OF STEADY STATE RESPONSE

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as steady state response. This means, the transient response will be zero even during the steady state.

OR

5.1.3 STEADY STATE RESPONSE:

Definition: The part of response that remains even after the transients have died the out is said to be steady state response.

From steady state response we get the following information about the system.

(a) The time that the o/p takes to reach the steady state value.
(b) Existence of any error.
(c) Whether the existing error is constant zero or infinite.

DEFINATION OF ACCURACY

In a set of measurements, accuracy is closeness of the measurements to a specific value, while precision is the closeness of the measurements to each other.

DEFINATION OF TRANSIENT ACCURACY

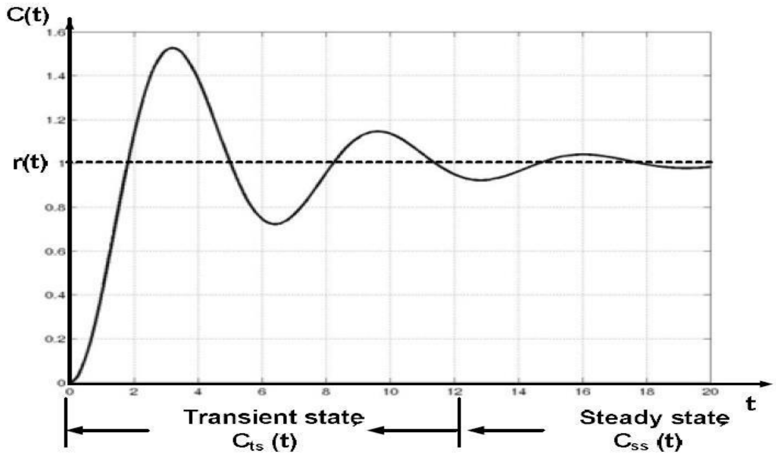
Transient analysis calculates a circuit's response over a period of time defined by the user. The accuracy of the transient analysis is dependent on the size of internal time steps, which together make up the complete simulation time known as the Run to time or Stop time.

ROBUSTNESS

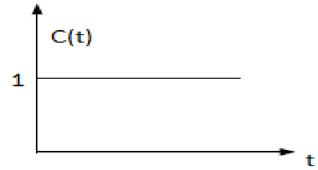
Robustness is the property of being strong and healthy in constitution. When it is transposed into a system, it refers to the ability of tolerating perturbations that might affect the system's functional body.

5.2 SYSTEM TIME RESPONSE:

Time response c(t)is the variation of output with respect to time. The part of time response that goes to zero after large interval of time is called transient response ctr(t). The part of time response that remains after transient response is called steady-state response css(t).



Time response of a system



System Time Response

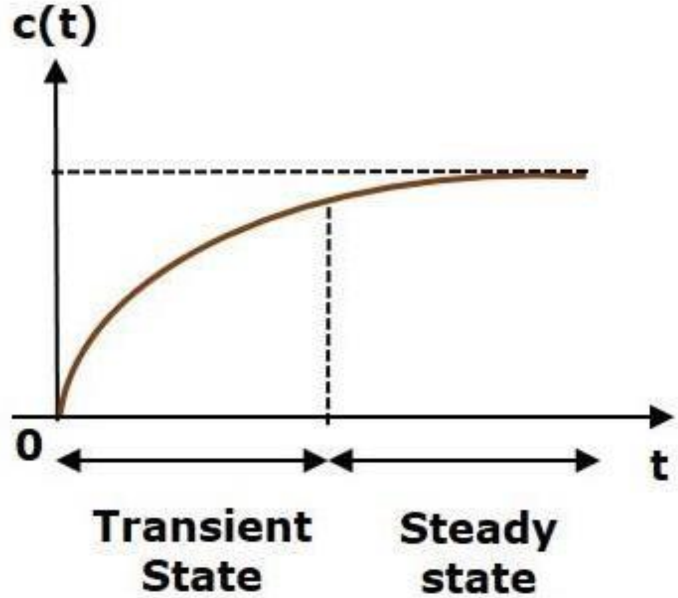
OR

5.2. SYSTEM TIME RESPONSE

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.
Mathematically, we can write the time response c(t) as

$$C(t)=C_{tr}(t)+C_{ss}(t)$$

Where,

- Ctr(t) is the transient response
- Css(t) is the steady state response

TRANSIENT RESPONSE

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

OR

5.1.5 TRANSIENT RESPONSE:

Definition:The part of the time response which goes to zero after a long interval of time is known as transient response. i.e. $\lim_{t\to\infty} c(t) = 0$

From transient response use have the following information.

- The time interval after which the system responses taking the instant of application of excitation as reference.
- The total time it takes to achieve the o/p for the 1^{st} time.
- Whether or not the o/p oscillates about it's final value.
- The time that it takes to settle to the final value.

STEADY STATE RESPONSE

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control system $C(t)=10+5e^{-t}$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

LAPLACE TRANSFORM PAIRS

No.	Function	Time-domain $x(t)=\mathcal{L}^{-1}\{X(s)\}$	Laplace domain $X(s)=\mathcal{L}\{x(t)\}$
1	Delay	$\delta(t-\tau)$	$e^{-\tau s}$
2	Unit impulse	$\delta(t)$	1
3	Unit step	$u(t)$	$\frac{1}{s}$
4	Ramp	t	$\frac{1}{s^2}$
5	Exponential decay	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
6	Exponential approach	$(1-e^{-\alpha t})$	$\frac{\alpha}{s(s+\alpha)}$
7	Sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
8	Cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
9	Hyperbolic sine	$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
10	Hyperbolic cosine	$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$
11	Exponentially decaying sine wave	$e^{-\alpha t}\sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
12	Exponentially decaying cosine wave	$e^{-\alpha t}\cos \omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

5.3 STEADY-STATE ERROR:

For a step excitation the difference between the desired o/p & the final value of the o/p of a system is term as steady state error of the system

5.3. ANALYSIS OF STEADY STATE ERROR

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

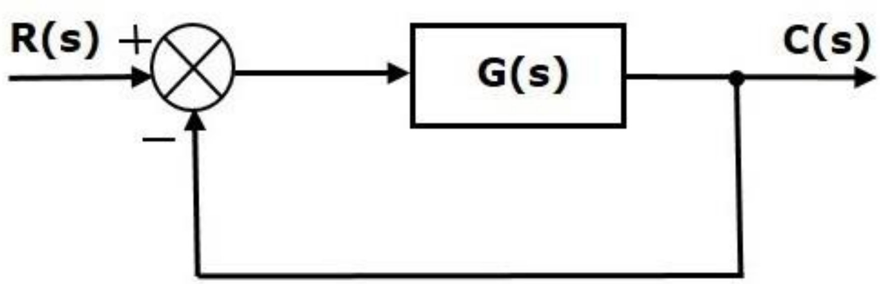
Where,

E(s) is the Laplace transform of the error signal, $e(t)$

Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



Where,

- R(s) is the Laplace transform of the reference Input signal $r(t)$
- C(s) is the Laplace transform of the output signal $c(t)$

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

The following table shows the steady state errors and the error constants for standard input signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s\to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s\to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s\to 0} s^2G(s)$

Where, K_p , K_v and K_a are position error constant, velocity error constant and acceleration error constant respectively.

Example

Let us find the steady state error for an input signal $r(t) = \left(5 + 2t + \frac{t^2}{2}\right) u(t)$ of unity negative feedback control system with $G(s) = \frac{5(s+4)}{s^2(s+1)(s+20)}$

The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s \to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s \to 0} sG(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s \to 0} s^2G(s) = 1$	$e_{ss3} = \frac{1}{k_a} = 1$

We will get the overall steady state error, by adding the above three steady state errors.

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$$

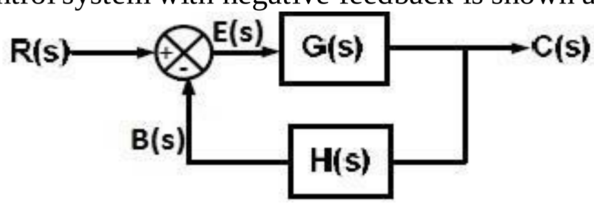
$$\Rightarrow e_{ss} = 0 + 0 + 1 = 1$$

Therefore, we got the steady state error e_{ss} as **1** for this example.

OR

5.3 ANALYSIS OF STEADY-STATE ERROR:

A simple closed-loop control system with negative feedback is shown as follows



$$E(s) = R(s) - B(s)$$
$$B(s) = C(s)H(s)$$
$$C(s) = E(s)G(s)$$
$$E(s) = R(s) - C(s)H(s)$$
$$E(s) = R(s) - E(s)G(s)H(s)$$
$$\Rightarrow [1 + G(s)H(s)]E(s) = R(s)$$
$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Steady-state error,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$
$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

Therefore, steady-state error depends on two factors, i.e.
(a) type and magnitude of R(s)
(b) open-loop transfer function G(s)H(s)

STEADY-STATE ERROR IN TIME DOMAIN:
Steady-state error in time domain (**e_{ss}**)

$$\text{i.e } e_{ss} = \lim_{t\to 0} e(t)$$
$$\therefore e_{ss} = \lim_{s\to 0} sE(S)$$
$$\Rightarrow e_{ss} = \lim_{s\to 0} \frac{sR(S)}{1+G(S)H(S)}$$

5.4. TYPES OF INPUT & STEADY STATE ERROR (STEP ,RAMP, PARABOLIC)

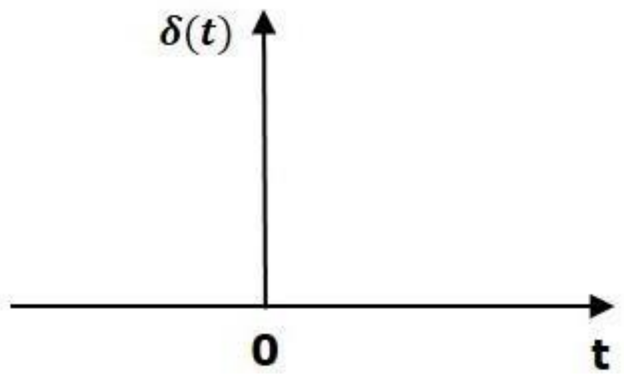
The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

$\delta(t)=0$ for $t\neq 0$

and $\int \delta(t)\, dt = 1$

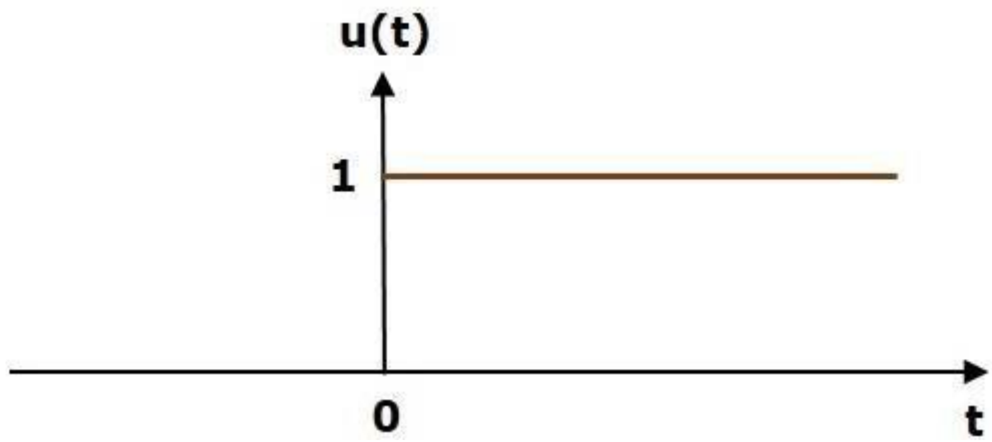


Unit Step Signal

A unit step signal, u(t) is defined as

u(t)=1; t ≥ 0

=0; t < 0



So, the unit step signal exists for all positive values of ‘t’ including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of ‘t’.

Unit Ramp Signal

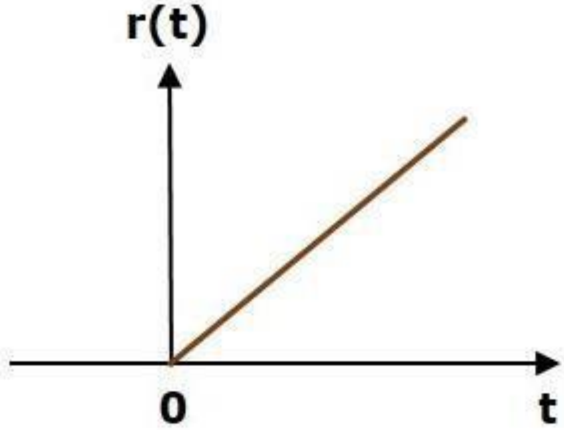
A unit ramp signal, r(t) is defined as

r(t)=t ; t ≥ 0

=0; t < 0

We can write unit ramp signal, r(t) in terms of unit step signal, u(t) as

r(t)=t . u(t)



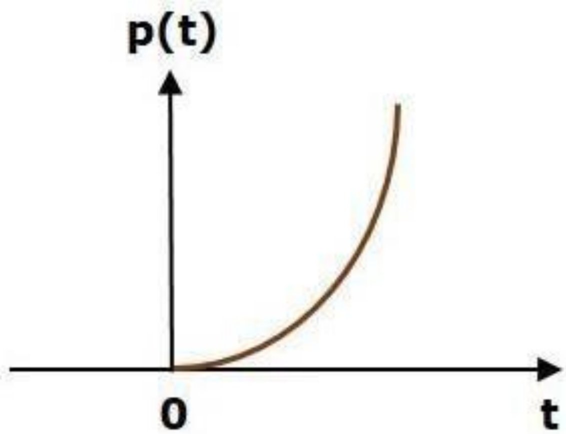
So, the unit ramp signal exists for all positive values of ‘t’ including zero. And its value increases linearly with respect to ‘t’ during this interval. The value of unit ramp signal is zero for all negative values of ‘t’.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,

$p(t)= t^2/2$; $t \geq 0$

$=0$; $t < 0$



So, the unit parabolic signal exists for all the positive values of **‘t’** including zero. And its value increases non-linearly with respect to ‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of ‘t’.

5.4 TYPES OF INPUT AND STEADY-STATE ERROR:

Step Input:

$$R(s)=\frac{A}{s}$$

$$e_{ss}=\lim_{s\to 0}\frac{s\left(\frac{A}{s}\right)}{1+G(s)H(s)}=\lim_{s\to 0}\frac{A}{1+G(s)H(s)}$$

$$\Rightarrow e_{ss}=\frac{A}{1+\lim_{s\to 0}G(s)H(s)}=\frac{A}{1+K_P}$$

Where, $K_P=\lim_{s\to 0}G(s)H(s)$

Ramp Input:

$$R(s)=\frac{A}{s^2}$$

$$e_{ss}=\lim_{s\to 0}\frac{s\left(\frac{A}{s^2}\right)}{1+G(s)H(s)}=\lim_{s\to 0}\frac{A}{s\left[1+G(s)H(s)\right]}$$

$$\Rightarrow e_{ss}=\lim_{s\to 0}\frac{A}{s+sG(s)H(s)}$$

$$\Rightarrow e_{ss}=\frac{A}{\lim_{s\to 0}sG(s)H(s)}=\frac{A}{K_V}$$

Where, $K_V=\lim_{s\to 0}sG(s)H(s)$

Parabolic Input:

$$R(s)=\frac{A}{s^3}$$

$$e_{ss}=\lim_{s\to 0}\frac{s\left(\frac{A}{s^3}\right)}{1+G(s)H(s)}=\lim_{s\to 0}\frac{A}{s^2\left[1+G(s)H(s)\right]}$$

$$\Rightarrow e_{ss}=\lim_{s\to 0}\frac{A}{s^2+s^2G(s)H(s)}$$

$$\Rightarrow e_{ss}=\frac{A}{\lim_{s\to 0}s^2G(s)H(s)}=\frac{A}{K_A}$$

Where, $K_A=\lim_{s\to 0}s^2G(s)H(s)$

Types of input and steady-state error are summarized as follows.

Error Constant	Equation	Steady-state error (e_{ss})
Position Error Constant (K_P)	$K_P=\lim_{s\to 0}G(s)H(s)$	$e_{ss}=\frac{A}{1+K_P}$
Velocity Error Constant (K_V)	$K_V=\lim_{s\to 0}sG(s)H(s)$	$e_{ss}=\frac{A}{K_V}$
Acceleration Error Constant (K_A)	$K_A=\lim_{s\to 0}s^2G(s)H(s)$	$e_{ss}=\frac{A}{K_A}$

Where, Kp,Kv and Ka are position error constant, velocity error constant and acceleration error constant respectively.

EXAMPLE:

Let us find the steady state error for an input signal

$$r(t) = \left(5 + 2t + \frac{t^2}{2}\right) u(t)$$

of unity negative feedback control system with

$$G(s) = \frac{5(s+4)}{s^2(s+1)(s+20)}$$

Solution: The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s\to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s\to 0} sG(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s\to 0} s^2G(s) = 1$	$e_{ss3} = \frac{1}{k_a} = 1$

We will get the overall steady state error, by adding the above three steady state errors.

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$$

$$\Rightarrow e_{ss} = 0 + 0 + 1 = 1$$

Therefore, we got the steady state error ess as 1.

5.4.1 Types of open-loop transfer function G(s)H(s)and Steady-state error:

Static Error coefficient Method:

The general form of G(s)H(s) is

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)...(1+T_ns)}{s^j(1+T_as)(1+T_bs)...(1+T_ms)}$$

Here, j = no. of poles at origin (s = 0)
or, type of the system given by eq. is j.

Type 0

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)...(1+T_ns)}{(1+T_as)(1+T_bs)...(1+T_ms)}$$

Here, $K_p = \lim_{s\to 0} G(s)H(s) = K$

Therefore, $e_{ss} = \frac{A}{1+K}$

Type 1

$$G(s)H(s)=\frac{K(1+T_1s)(1+T_2s)...(1+T_ns)}{s(1+T_as)(1+T_bs)...(1+T_ms)}$$

Here, $K_V=\lim_{s\to 0} sG(s)H(s)=K$

Therefore, $e_{ss}=\frac{A}{K}$

Type 2

$$G(s)H(s)=\frac{K(1+T_1s)(1+T_2s)...(1+T_ns)}{s^2(1+T_as)(1+T_bs)...(1+T_ms)}$$

Here, $K_A=\lim_{s\to 0} s^2G(s)H(s)=K$

Therefore, $e_{ss}=\frac{A}{K}$

Steady-state error and error constant for different types of input are summarized as follows.

Type	Step input		Ramp input		Parabolic input	
	K_P	e_{ss}	K_V	e_{ss}	K_A	e_{ss}
Type 0	K	$\frac{A}{1+K}$	0	∞	0	∞
Type 1	∞	0	K	$\frac{A}{K}$	0	∞
Type 2	∞	0	∞	0	K	$\frac{A}{K}$

The static error coefficient method has following advantages:

- Can provide time variation of error
- Simple calculation

But, the static error coefficient method has following demerits:

- Applicable only to stable system
- Applicable only to three standard input signals
- Cannot give exact value of error. It gives only mathematical value i.e. 0 or ∞

PROBLEMS: Find the error co-efficient for the system.

$$G(s)H(s) = \frac{S+3}{S(1+0.60S)(1+0.35S)}$$

Solution: Given that, $G(s)H(s) = \frac{S+3}{S(1+0.60S)(1+0.35S)}$

a. $K_p = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{S+3}{S(1+0.60S)(1+0.35S)} = \infty$

b. $K_v = \lim_{S\to 0} G(s)H(s)\ S = \lim_{S\to 0} \frac{S(S+3)}{S(1+0.60S)(1+0.35S)} = 0$

c. $K_a = \lim_{S\to 0} S^2G(s)H(s) = \lim_{S\to 0} \frac{S^2(S+3)}{S(1+0.60S)(1+0.35S)} == \lim_{S\to 0} \frac{S(S+3)}{(1+0.60S)(1+0.35S)} = 0$

PROBLEMS: Find the error co-efficient for the system

$G(s) = \frac{5}{S^2+3S+5}$ & $H(s) = 0.6$

Solution: Given that, G(s) = $\frac{S}{S^2+3S+5}$ & $H(s) = 0.6$

$$\therefore G(s)H(s) = \frac{5}{S^2+3S+5} \times 0.6 = \frac{3}{S^2+3S+5}$$

a. $K_p = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{3}{S^2+3S+5} = \frac{3}{5}$ $or\ 0.6$

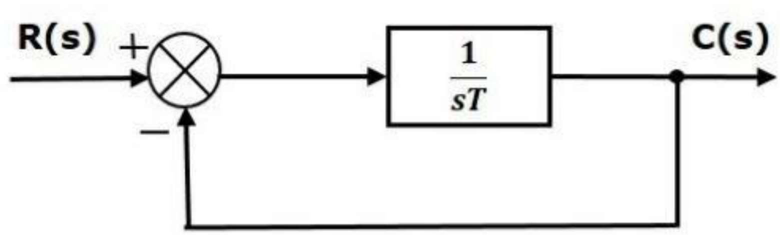
b. $K_v = \lim_{S\to 0} G(s)H(s)\,S = \lim_{S\to 0} \frac{3S}{S^2+3S+5} = 0$

c. $K_a = \lim_{S\to 0} S^2 G(s)H(s) = \lim_{S\to 0} \frac{3S^2}{S^2+3S+5} = 0$

5.5 PARAMETERS OF FIRST ORDER SYSTEM& SECOND ORDER SYSTEM:

5.5.1 FIRST ORDER SYSTEM:

Consider a 1st order system with unity feedback



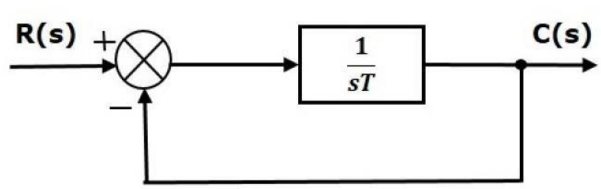
$$G(s) = \frac{1}{sT}, H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{sT}}{1+\frac{1}{sT}.1} = \frac{\frac{1}{sT}}{\frac{(sT+1)}{sT}} = \frac{1}{sT+1}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{1}{sT+1}$$

TIME RESPONSE OF THE FIRST ORDER SYSTEM

The time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, $\frac{1}{sT}$ is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Substitute, G(s) = $\frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = = \frac{\frac{1}{sT}}{1+\frac{1}{sT}} = \frac{1}{sT+1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the first order system.We can re-write the above equation as,

$C(s) = \frac{1}{sT+1} R(s)$

Where,
C is the Laplace transform of the output signal c
R is the Laplace transform of the input signal **r** and **T** is the time constant.
Follow these steps to get the response output of the first order system in the time domain.
Take the Laplace transform of the input signal **r**(t)

IMPULSE RESPONSE OF FIRST ORDER SYSTEM

Consider the unit impulse signal as an input to the first order system.

$$\text{So}, r(t) = \delta(t)$$

Apply Laplace transform on both the sides.

$$R(s) = 1$$

Consider the equation,

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Substitute, R(s) = 1 in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

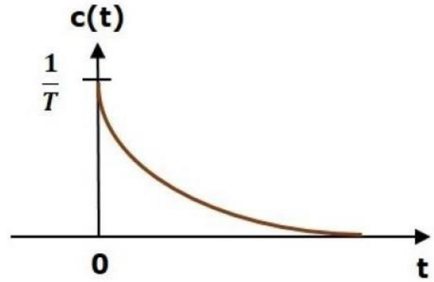
Rearrange the above equation in one of the standard forms of Laplace transforms.

$$C(s) = \frac{1}{T\left(s+\frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s+\frac{1}{T}}\right)$$

Apply inverse Laplace transform on both sides.

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

The unit impulse response is shown in the following figure.



The unit impulse response, c is an exponential decaying signal for positive values of ‘t’ and it is zero for negative values of ‘t’.

STEP RESPONSE OF FIRST ORDER SYSTEM

Consider the unit step signalas an input to first order system.

So, r(t) = u(t)

Apply Laplace transform on both the sides.

R(s) = $\frac{1}{S}$

Consider the equation, C(S) = $\frac{1}{1+Ts}\ R(s)$

Substitute, R(s) = $\frac{1}{S}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\,(sT+1)}$$

Do partial fractions of C

$$C(s) = \frac{1}{s\,(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\,(sT+1)} = \frac{A\,(sT+1)+Bs}{s\,(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$1 = A\,(sT+1) + Bs$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$0 = T + B \Rightarrow B = -T$

Substitute, A = 1 and B = −T in partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

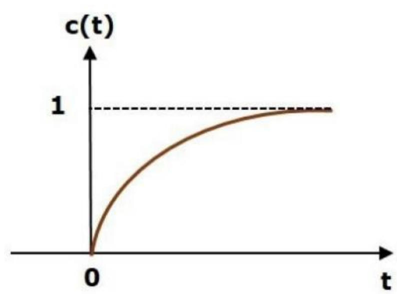
<u>FOR UNIT STEP RESPONSE</u>

Here r(t) = u(t)

R(s) = $\frac{1}{S}$

C(s) = $\frac{1}{1+Ts}\ R(s) = \frac{1}{s(1+Ts)} = \frac{\frac{1}{T}}{s(s+\frac{1}{T})} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$

C(t) = 1 - e^{-at}



The value of the **unit step response, c** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

RAMP RESPONSE OF FIRST ORDER SYSTEM

Consider the **unit ramp signal** as an input to the first order system.
So, r(t) = tu(t)

Apply Laplace transform on both the sides.

R(s) = $\frac{1}{S^2}$

Consider the equation, $\frac{1}{sT+1} R(s)$

Substitute, R(s) = $\frac{1}{S^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1.
Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Similarly, substitute B = −T and equate the coefficient of S^2 terms on both the sides.
You will get C = T^2
Substitute A = 1, B = −T and C = T^2in the partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right)u(t)$$

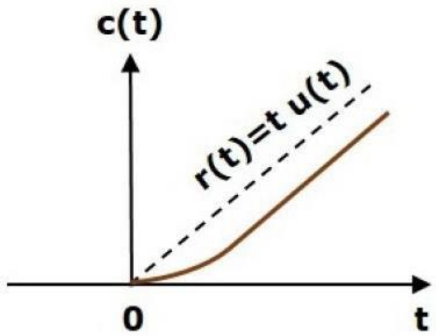
The **unit ramp response**, c has both the transient and the steady state terms.
The transient term in the unit ramp response is

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit ramp response is

$$c_{ss}(t) = (t - T)u(t)$$

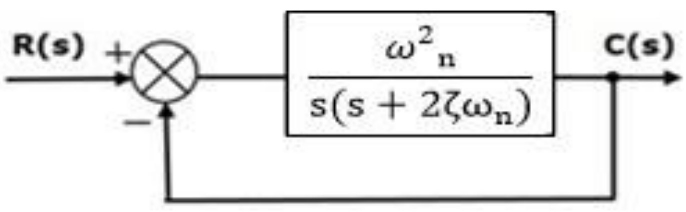
The following figure shows the unit ramp response.



The **unit ramp response**, c follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

5.5.2 SECOND-ORDER SYSTEM

The block diagram of a 2nd order system is



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

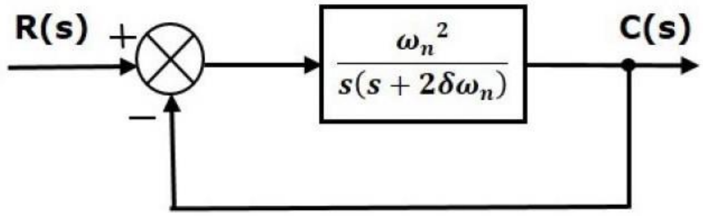
$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s(s+2\zeta\omega_n)},\ H(s) = 1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega^2{}_n/s(s+2\zeta\omega_n)}{1 + \frac{\omega_n^2}{s(s+2\zeta\omega_n)}}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega^2{}_n}$$

TIME RESPONSE OF SECOND ORDER SYSTEM

The time response of second order system. Consider the following block diagram of closed loop control system. Here, an open loop transfer function, is connected with a unity negative $\frac{\omega_n^2}{s(s+2\delta\omega_n)}$ feedback.



We know the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, G(s) = $\frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the second order system.

The characteristic equation is

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2-1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2-1}$$

The two roots are imaginary when δ = 0.
The two roots are real and equal when δ = 1.
The two roots are real but not equal when δ > 1.
The two roots are complex conjugate when 0 < δ < 1.
We can write equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}\right)R(s)$$

Where,
C is the Laplace transform of the output signal, c
R is the Laplace transform of the input signal, r
ωnis the natural frequency
δ is the damping ratio.
Follow these steps to get the response output of the second order system in the time domain.
Take Laplace transform of the input signal, r(t).
Consider the equation,

$$C(s)=\left(\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}\right)R(s)$$

Substitute R(s) value in the above equation.Do partial fractions of C(s) if required.
Apply inverse Laplace transform to C(s) .

STEP RESPONSE OF SECOND ORDER SYSTEM

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is, R(s) = $\frac{1}{s}$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

Case 1: δ = 0

Substitute, **δ = 0** in the transfer function.

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+\omega_n^2}$$

$$\Rightarrow C(s)=\left(\frac{\omega_n^2}{s^2+\omega_n^2}\right)R(s)$$

Substitute, R(s) = $\frac{1}{s}$ in the above equation.

$$C(s)=\left(\frac{\omega_n^2}{s^2+\omega_n^2}\right)\left(\frac{1}{s}\right)=\frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t)=(1-\cos(\omega_n t))\,u(t)$$

Case 2: δ = 1

Substitute, /delta = 1 in the transfer function.

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$$

$$\Rightarrow C(s)=\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)R(s)$$

Substitute, R(s) = $\frac{1}{s}$ in the above equation.

$$C(s)=\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)\left(\frac{1}{s}\right)=\frac{\omega_n^2}{s(s+\omega_n)^2}$$

Do partial fractions of C(s).

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as 1,−1 and $-\omega_n$respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right) R(s)$$

Substitute, R(s) = $\frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of C(s).

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs+C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as 1, −1 and − 2δωn respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s+2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s)=\frac{1}{s}-\frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2}-\frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute,

$$\omega_n\sqrt{1-\delta^2} \text{ as } \omega_d$$

In the above equation.

$$C(s)=\frac{1}{s}-\frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+\omega_d^2}-\frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2+\omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t)=\left(1-e^{-\delta\omega_n t}\cos(\omega_d t)-\frac{\delta}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin(\omega_d t)\right)u(t)$$

$$c(t)=\left(1-\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t)+\delta\sin(\omega_d t)\right)\right)u(t)$$

If $\sqrt{1-\delta^2}=\sin(\theta)$, then 'δ' will be $\cos\theta$. Substitute these values in the above equation.

$$c(t)=\left(1-\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t)+\cos(\theta)\sin(\omega_d t))\right)u(t)$$

$$\Rightarrow c(t)=\left(1-\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t+\theta)\right)u(t)$$

So, the unit step response of the second order system is having damped oscillations when 'δ' lies between zero and one.

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows –

$$s^2+2\delta\omega_n s+\omega_n^2=\left\{s^2+2(s)(\delta\omega_n)+(\delta\omega_n)^2\right\}+\omega_n^2-(\delta\omega_n)^2$$

$$=(s+\delta\omega_n)^2-\omega_n^2\left(\delta^2-1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{(s+\delta\omega_n)^2-\omega_n^2(\delta^2-1)}$$

$$\Rightarrow C(s)=\left(\frac{\omega_n^2}{(s+\delta\omega_n)^2-\omega_n^2(\delta^2-1)}\right)R(s)$$

Substitute, R(s) = $\frac{1}{S}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2-(\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of C(s).

$$C(s) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$= \frac{A}{s} + \frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}} + \frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

After simplifying, you will get the values of A, B and C as 1,

$\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively.

Substitute these values in above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\left(\frac{1}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}}\right)$$

$$-\left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)\left(\frac{1}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}\right)$$

Apply inverse Laplace transform on both the sides.

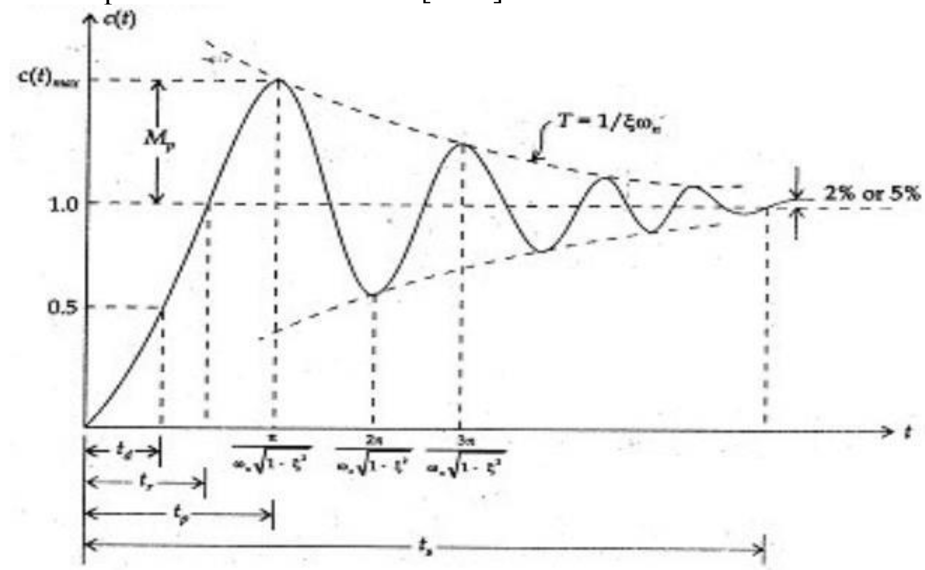
$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right)u(t)$$

Since it is over damped, the unit step response of the second order system when $\delta > 1$ will never reach step input in the steady state.

5.6 DERIVATION OF TIME RESPONSE SPECIFICATIONS:

(DELAY TIME, RISE TIME, PEAK TIME, SETTING TIME, PEAK OVER SHOOT)

1. Delay time, Td
2. Rise time, Tr
3. Peak time, Tp
4. Peak overshoot, Mp
5. Settling time, Ts



(1) Delay time, Td: It is the time required to reach 50% of output

$$y(t_d) = \frac{1}{2} = 1 - \frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_d + \varphi)$$

$$\Rightarrow t_d = \frac{1+0.7\zeta}{w_n}$$

(2) Rise time, Tr: The time required by the system response to reach from 10% to 90% of the final value for over-damped case, from 0% to 100% of the final value for under-damped case and from 5% to 95% of the critically value for over-damped case.

$$y(t_r) = 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \varphi)$$

$$\Rightarrow \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \varphi) = 0$$

$$\Rightarrow \omega_d t_r + \varphi = \pi$$

$$\Rightarrow t_r = \frac{\pi - \varphi}{w_d}$$

(3) Peak time, Tp: The time required by the system response to reach the first maximum value.

$$\frac{dy(t_p)}{dt}=0$$

$$\Rightarrow \frac{d\left[1-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\varphi)\right]}{dt}=0$$

$$\Rightarrow \frac{d\left[-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\varphi)\right]}{dt}=0$$

$$\Rightarrow w_d t_p+\varphi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}=n\pi+\varphi\text{ ; where } n=1,2,3,...$$

For n=1,

$$\Rightarrow w_d t_p=n\pi$$

$$\Rightarrow t_p=\frac{n\pi}{w_d}$$

(4) Peak overshoot, Mp: It is the time required to reach 50% of output.

$$M_p(\%)=100\times\frac{y(t_p)-1}{1}$$

$$\Rightarrow M_p(\%)=100\times\left[1-\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r+\varphi)-1\right]$$

$$\Rightarrow M_p(\%)=100\times\left[-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\varphi)\right]=100\times\left[-\frac{e^{-\zeta\omega_n\frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\varphi)\right]$$

$$\Rightarrow M_p(\%)=100\times\left[-\frac{e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin(\omega_d\frac{\pi\zeta}{\sqrt{1-\zeta^2}}+\varphi)\right]=100\times\left[-\frac{e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin(\pi+\varphi)\right]$$

$$\Rightarrow M_p(\%)=100\times\left[\frac{e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\varphi\right]=100\times\left[\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2}\right]$$

$$\Rightarrow M_p(\%)=100\times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

(5) Settling time, Ts: It is the time taken by the system response to settle down and stay with in$\pm$2% or $\pm$5% its final value. For $\pm$2% error band.

$$t_s = \frac{4}{\zeta w_n}$$

For ±5% error band, $t_s = \frac{3}{\zeta w_n}$

Sl. No.	Time Specifications	
	Type	**Formula**
1	Delay time	$t_d = \frac{1+0.7\zeta}{w_n}$
2	Rise time	$t_r = \frac{\pi - \varphi}{w_d}$
3	Peak time	$t_p = \frac{\pi}{w_d}$
4	Maximum overshoot	$M_p(\%) = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$
5	Settling time	$t_s = \frac{4}{\zeta w_n}$

EXAMPLE5.6.1:

The open loop transfer function of a system with unity feedback is given by

$$G(s) = \frac{10}{(s+2)(s+5)}$$

Determine the damping ratio, undamped natural frequency of oscillation. What is the percentage overshoot of the response to a unit step input.

SOLUTION: Given that

$$G(s) = \frac{10}{(s+2)(s+5)}$$

H(s) = 1

Characteristic equation, 1+G(s)H(s) = 0

$$1 + \frac{10}{(s+2)(s+5)} = 0$$

$$\frac{(s+2)(s+5)+10}{(s+2)(s+5)} = 0$$

$$\frac{s^2+5s+2s+10+10}{(s^2+5s+2s+10)} = 0$$

$$s^2 + 7s + 20 = 0$$

Compare with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

We get,

$$\omega_n^2 = 20$$
$$2\xi\omega_n = 7$$
$$\therefore \omega_n = \sqrt{20} = 4.472 rad/\sec$$
$$2 * \xi * 4.472 = 7$$
$$\xi = 0.7826$$

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = e^{-\frac{\pi * 0.7826}{\sqrt{1-(0.7826)^2}}} * 100 = 1.92\%$$

EXAMPLE 5.6.2:

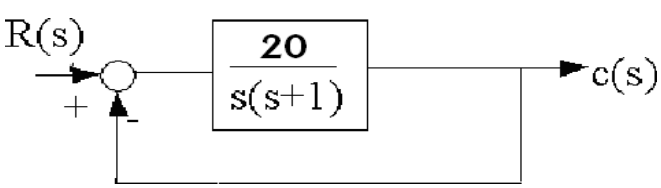
A feedback system is described by the following transfer function

$$G(s) = \frac{12}{s^2 + 4s + 16}$$
$$H(s) = Ks$$

The damping factor of the system is 0.8. determine the overshoot of the system and value of 'K'.

EXAMPLE 5.6.3:

Consider the system shown in Fig. 5.6.3.. To improve the performance of the system a feedback is added to this system, which results in Fig. 5.6.4. Determine the value of K so that the damping ratio of the new system is 0.4. Compare the overshoot, rise time, peak time and settling time and the nominal value of the systems shown in Fig. 5.6.3 and Fig. 5.6.4.



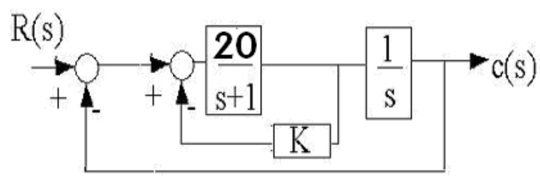


Fig. 5.6.3Fig.5.6.4

Solution: For Figure 5.6.3,

$$G(s) = \frac{20}{S(S+1)}$$
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$
$$= \frac{\frac{20}{S(S+1)}}{1 + \frac{20}{S(S+1)}}$$

$$= \frac{\frac{20}{S(S+1)}}{\frac{S(S+1)+20}{S(S+1)}}$$

$$= \frac{\frac{20}{S(S+1)}}{\frac{S^2+S+20}{S(S+1)}} = \frac{20}{S^2+S+20}$$

Here, $\omega_n^2 = 20$ and $2\zeta\omega_n = 1$

$\omega_n = \sqrt{20}$ rad/s and $\zeta = \frac{1}{2\omega_n} = \frac{1}{2\times\sqrt{20}} = 0.112$

For Figure 5.6.4,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\frac{20}{S(S+1+20K)}}{1+\frac{20}{S(S+1+20K)}}$$

$$= \frac{\frac{20}{S(S+1+20K)}}{\frac{S(S+1+20K)+20}{S(S+1+20K)}}$$

$$= \frac{20}{S^2+(1+20K)S+20}$$

Here, $\omega_n^2 = 20$ and $2\zeta\omega_n = 1+20\text{K}$

$\omega_n = \sqrt{20}$ rad/s

But, given that $\zeta = \frac{1+20K}{2\omega_n} = \frac{1+20K}{2\sqrt{20}} = 0.4$

$\Rightarrow K = 0.128$

Transient characteristics of Figures 5.6.3 and 5.6.4

CharacteristicS	Figure :5.6.3	Figure : 5.6.4
Overshoot, M_p	70%	25%
Rise time, t_r, sec	0.38	0.48
Peak time, t_p, sec	0.71	0.77
Settling time (2%), sec	8	2.24
Steady-state value, c_∞	1.0	1.0

SHORT QUESTIONS WITH ANSWER

Q1.What is peak time ?

Ans- If the signal is over damped, then rise time is counted as the time required by the response to rise from 10% to 90% of its final value. Peak time (t_p) is simply the time required by response to reach its first peak i.e. the peak of first cycle of oscillation, or first overshoot.

Q2. What is settling time?

Ans- In control theory the settling time of a dynamical system such as an amplifier or other output device is the time elapsed from the application of an ideal instantaneous step input to the time at which the amplifier output has entered and remained within a specified error band.

Q3. What is time response ?

Ans- If the output of control system for an input varies with respect to time, then it is called the time response of the control system. The time response consists of two parts. Transient response. Steady state response.

Q4. What is rise time in control system ?

Ans - Rise time (t_r) is the time required to reach at final value by a under damped time response signal during its first cycle of oscillation. If the signal is over damped, then rise time is counted as the time required by the response to rise from 10% to 90% of its final value.

LONG QUESTIONS:

Q1. Derivation of time response specification for delay time, rise time, peak time, setting time, peak over shoot of a second order system.

Q2.What are the different type of input and derive steady state error for step, ramp and parabolic input.

CHAPTER - 6
FEEDBACK CHARACTERISTICS OF CONTROL SYSTEM

6.1 EFFECT OF PARAMETER VARIATION IN OPEN LOOP SYSTEM & CLOSED LOOP SYSTEMS:

Effect of parameter variation:

Feedback reduces error, reduces the sensitivity at the system to parameter variation.
Parameter may very due to some change in condition and it's variation effect the performance of the system.So it is necessary to make the system insensitive to parameter variation.

Effect of parameter variation on overall gain of a degenerative Feedback Control system:

The overall gain or transfer function of a degenerative feedback control system depends upon these parameters i.e. (i) variation in parameters of plant, and (ii) variation in parameter of feedback system and (ii) disturbance signals.

The term sensitivity is a measure of the effectiveness of feedback on reducing the influence of any of the above described parameters. For an example, it is used to describe the relative variations in the overall Transfer function of a system T(s) due to variation in G(s).

$$\text{Sensitivity} = \frac{\%\ \text{change in output (Ts)}}{\%\ \text{change in input (Gs)}}$$

Effect of variation in G(s) on T(s) of a degenerative Feedback Control system:

In an open-loop system,

C(S) = G(S)R(S)

Let, due to parameter variation in plant G(s) changes to [G(s) + ΔG(s)] such that |G(s)| >> |ΔG(s)|. The output of the open-loop system then changes to

$$C(s) + \Delta C(s) = \left[G(s) + \Delta G(s)\right]R(s)$$

$$\Rightarrow C(s) + \Delta C(s) = G(s)R(s) + \Delta G(s)R(s)$$

$$\Rightarrow \Delta C(s) = \Delta G(s)R(s)$$

In an closed-loop system,

$$C(s) = \frac{G(s)}{1 + G(s)H(s)}R(s)$$

Let, due to parameter variation in plant G(s) changes to [G(s) + ΔG(s)] such that |G(s)| >> |ΔG(s)|. The output of the open-loop system then changes to

$$C(s) + \Delta C(s) = \frac{\left[G(s) + \Delta G(s)\right]}{1 + \left[G(s) + \Delta G(s)\right]H(s)}R(s)$$

$$\Rightarrow C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s) + \Delta G(s)H(s)}R(s)$$

Since, |G(s)| >> |ΔG(s)|, then G(S)H(S) +Δ G(S)H(S). Therefore, Δ G(S)H(S). is neglected. Now,

$$C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)}R(s)$$

$$\Rightarrow C(s) + \Delta C(s) = \frac{G(s)}{1 + G(s)H(s)}R(s) + \frac{\Delta G(s)}{1 + G(s)H(s)}R(s)$$

$$\text{Or} \quad \Delta C(s) = \frac{\Delta G(s)}{1+G(s)H(s)} R(s)$$

Comparing eq. (1) and (2), it is clear that $\Delta C_{(open\ loop)} = (1 + GH)\ \Delta C_{(closed\ loop)}$.
This concept can be reproved using sensitivity. Sensitivity on T(s) due to variation in G(s) is given by

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{\partial T}{\partial G} \times \frac{G}{T}$$

For open-loop system,

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{\partial G}{\partial G} \times \frac{G}{G} = 1$$

For closed-loop system,

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{(1+GH)-GH}{(1+GH)^2} \times \frac{G}{G/(1+GH)} = \frac{1}{(1+GH)}$$

Therefore, it is proved that $S_{G\,(open\ loop)}^T = (1+GH) S_{G\,(closed\ loop)}^T$. Hence, the effect of parameter variation in case of closed loop system is reduced by a factor of $\frac{1}{(1+GH)}$.

Effect of variation in H(s) on T(s) of a degenerative Feedback Control system:
This concept can be reproved using sensitivity. Sensitivity on T(s) due to variation in H(s) is given by

$$S_H^T = \frac{\partial T/T}{\partial H/H} = \frac{\partial T}{\partial H} \times \frac{H}{T}$$

$$S_H^T = \frac{\partial T}{\partial H} \times \frac{H}{T} = G\left[\frac{-G}{(1+GH)^2}\right] \times \frac{H}{G/(1+GH)} = \frac{-GH}{(1+GH)}$$

For higher value of GH, sensitivity S_H^T approaches unity. Therefore, change in H affectsdirectly the system output.

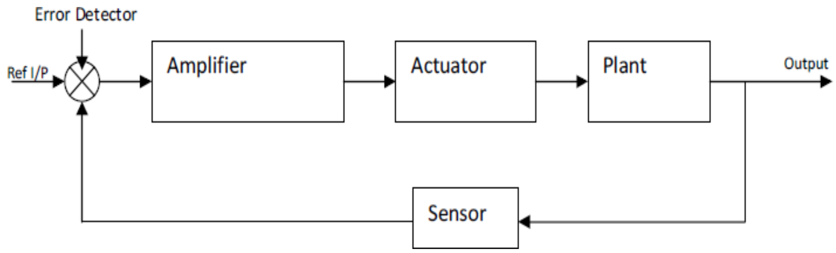
6.2 INTRODUCTION TO BASIC CONTROL ACTION & BASIC MODES OF FEEDBACK:

6.2.1 Introduction to basic Control action:

It is the value of controlled variable, compare the actual value to the desired value (reference i/p) and its deviation & produces a control signal that will reduce the deviation to zero or to a smallest possible value and produces the control signal is called mode of control or basic control action. Example: – Mechanical, Hydraulic, Pneumatic or Electromechanical.

Basic Control action and response of Control Systems:

An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value. The automatic controller produces the control signal is called the control action. The Fig. below is a block diagram of an industrial control system, which consists of an automatic controller, an actuator, a plant and a sensor (measuring element).



The controller detects the actuating error signal, which is usually at a low power level, and amplifies it to a sufficiently high level. The output of the controller is fed to an actuator such as pneumatic motor or valve, hydraulic motor or electric motor. The actuator is the device that produces the input to the plant according to the control signal so that the output signal will approach the reference input signal.

The sensor or measuring element is device that converts the output variable into another suitable variable such as a displacement, pressure or voltage that can be used to compare the output to the reference input signal. This element is in the feedback path of the closed-loop system. The set point of the controller must be converted to a reference input with the same units as feedback signal from sensor.

6.2.2 BASIC MODES OF FEEDBACK

PROPORTIONAL:

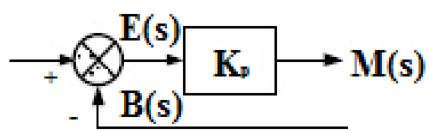
In Proportional control action, there is a continuous linear relation between the o/p of the controller m(manipulated variable) and actuating error signal e(deviation**).**

Mathematically,

$m(t) = K_p e(t)$

or in Laplace Transform of

$M(s) = K_p \frac{M(s)}{E(s)}$



Where K_p is known as proportional gain

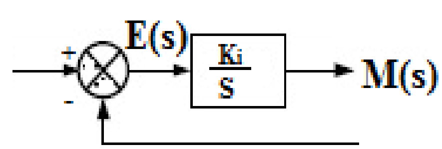
Basically it is an amplifier with adjustable gain.

INTEGRAL:

The o/p of the controller is changed at a rate which is proportional to the actuating error signal e(t).

Mathematically,

$\frac{d}{dt}m(t) = K_i\, e(t)$ .... (1)



Where K_i is a constant.

Equation (1) can be written as,

$m(t) = K_i \int e(t) + m(0)$

where m(0) = control o/p at t = 0

Laplace Transform of equation(1)

$SM(s) = K_i E(s)$

or $\frac{M(s)}{E(s)} = \frac{K_i}{s}$

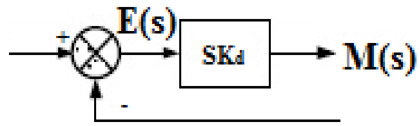
The o/p of the controller is ramp(positive), for zero error, there is no change in the o/p of the controller and negative error the o/p of the controller is negative ramp. It is also known as “Reset control”.

DERIVATIVE:

The o/p of the controller depends on the rate of change of actuating error signal e(t).

Mathematically,

$m(t) = K_d \frac{d}{dt} e(t)$



where, K_d= derivative gain constant

Laplace transform of equation

$M(s) = K_d SE(s)$

or $\frac{M(s)}{E(s)} = sK_d$

When the error is zero or constant, the o/p of the controller will be zero. This type of controller cannot be used alone. Its gain should be small. It is also known as rate control.

OR

6.2 INTRODUCTION TO BASIC CONTROL ACTION & BASIC MODES OF FEEDBACK CONTROL: PROPORTIONAL, INTEGRAL AND DERIVATIVE

Proportional Controller

The proportional controller produces an output, which is proportional to error signal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

Therefore, the transfer function of the proportional controller is K_P .

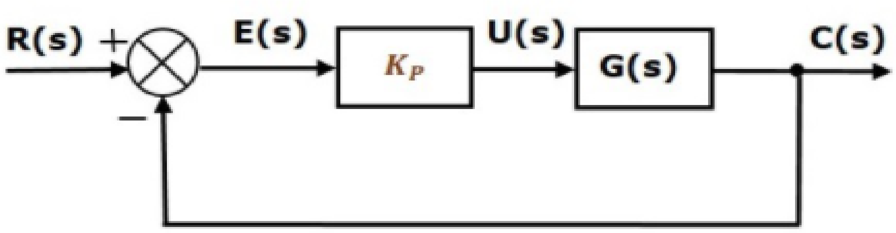
Where,

U(s) is the Laplace transform of the actuating signal u(t)

E(s) is the Laplace transform of the error signal e(t)

K_P is the proportionality constant

The block diagram of the unity negative feedback closed loop control system along with the proportional controller is shown in the following figure.



The proportional controller is used to change the transient response as per the requirement

Derivative Controller

The derivative controller produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides.

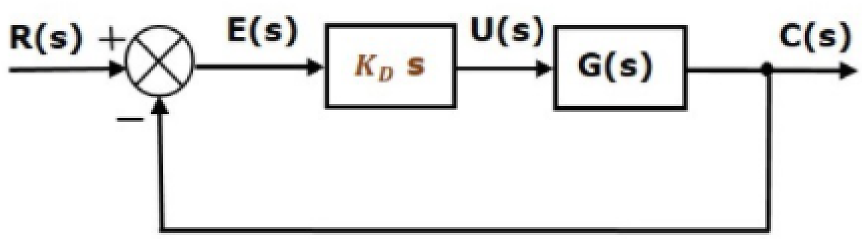
$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore, the transfer function of the derivative controller is $K_D s$.

Where, K_D is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure.



The derivative controller is used to make the unstable control system into a stable one.

The integral controller produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t)dt$$

Apply Laplace transform on both the sides -

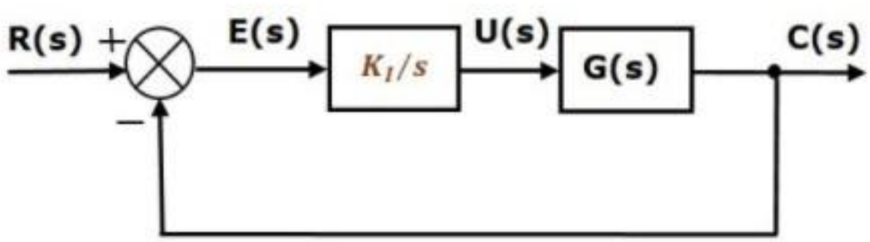
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where, K_I is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure.



The integral controller is used to decrease the steady state error.

Let us now discuss about the combination of basic controllers.

Proportional Derivative (PD) Controller

The proportional derivative controller produces an output, which is the combination of the outputs of proportional and derivative controllers.

$$u(t) = K_P e(t) + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

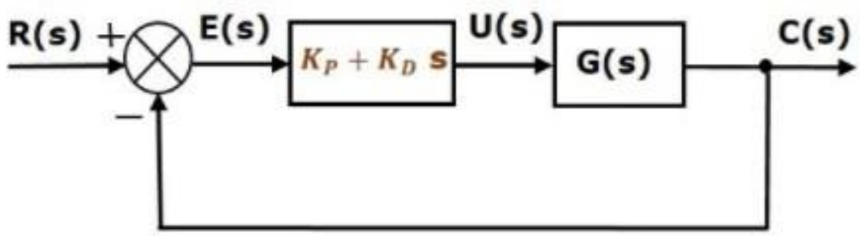
Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore, the transfer function of the proportional derivative controller is $K_P + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure.



The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

Proportional Integral (PI) Controller

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt$$

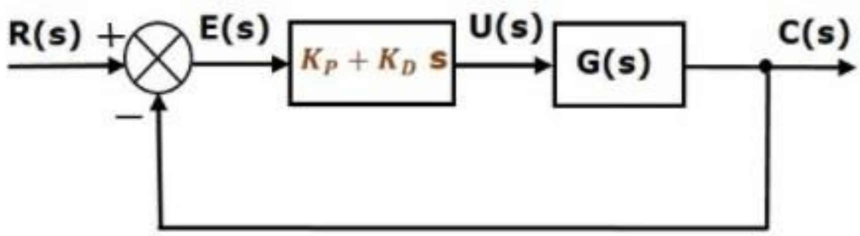
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore, the transfer function of the proportional derivative controller is $K_P + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure.



The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

Proportional Integral (PI) Controller

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt$$

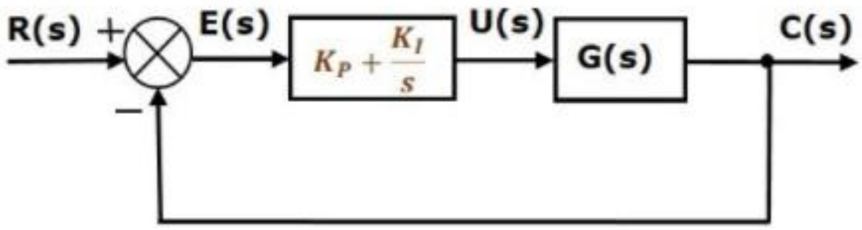
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is $K_P + \frac{K_I}{s}$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral controller is shown in the following figure.



The proportional integral controller is used to decrease the steady state error without affecting the stability of the control system.

Proportional Integral Derivative (PID) Controller

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

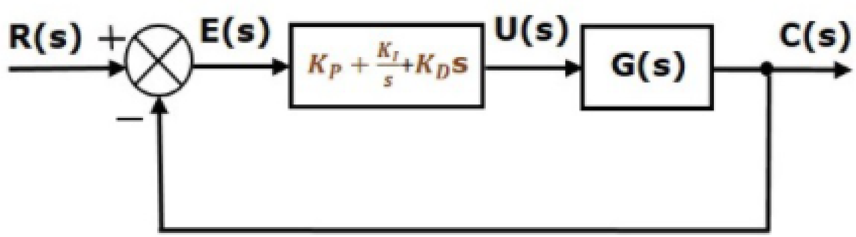
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P + \frac{K_I}{s} + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.



The proportional integral derivative controller is used to improve the stability of the control system and to decrease steady state error.

6.3 EFFECT OF FEEDBACK ON OVERAL GAIN, STABILITY

6.3.1 ***Effect of feedback on Gain:***

The overall transfer function in an open-loop system is

$G(S)\frac{C(S)}{R(S)}$

Closed loop, $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S).H(S)}$

Hence, the gain is reduced by a factor of $\frac{1}{1+G(S)H(S)}$

6.3.2 ***Effect of feedback on Stability:***

$$G(S)\frac{K}{(S+t)} \text{ (Open loop system)}$$

So the pole is located at s = - t

For closed loop system,

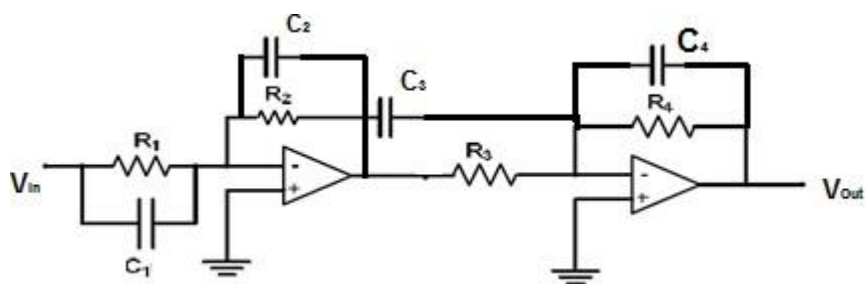
$$\frac{C(S)}{R(S)} = \frac{K}{S+(t+k)}$$

So the pole is located at –(t+k)

Hence feedback control the time response by adjusting the location of poles.

The stability depends on the location of pole. Hence we can say feedback effects the stability.

6.4 REALISATION OF CONTROLLER WITH OPAMP:



It's shows the op-amp circuit realization of a two stage phase-lead controller.

The i/p Transfer function of the circuit is

$Gc(s) = \frac{V_{out}(s)}{V_{in}(s)}$

$= \left(\frac{S+\frac{1}{R_1C}}{S+\frac{1}{R_2C}}\right)\left(\frac{S+\frac{1}{R_3C}}{S+\frac{1}{R_4C}}\right)$

or $Gc(s) = \frac{1}{a_{1a2}}\left(\frac{1+a_1T_1S}{1+T_1S}\right)\left(\frac{1+a_2T_2S}{1+T_2S}\right)$

Where, $a_1 = R_1/R_2$

$$a_2 = R_3/R_4$$

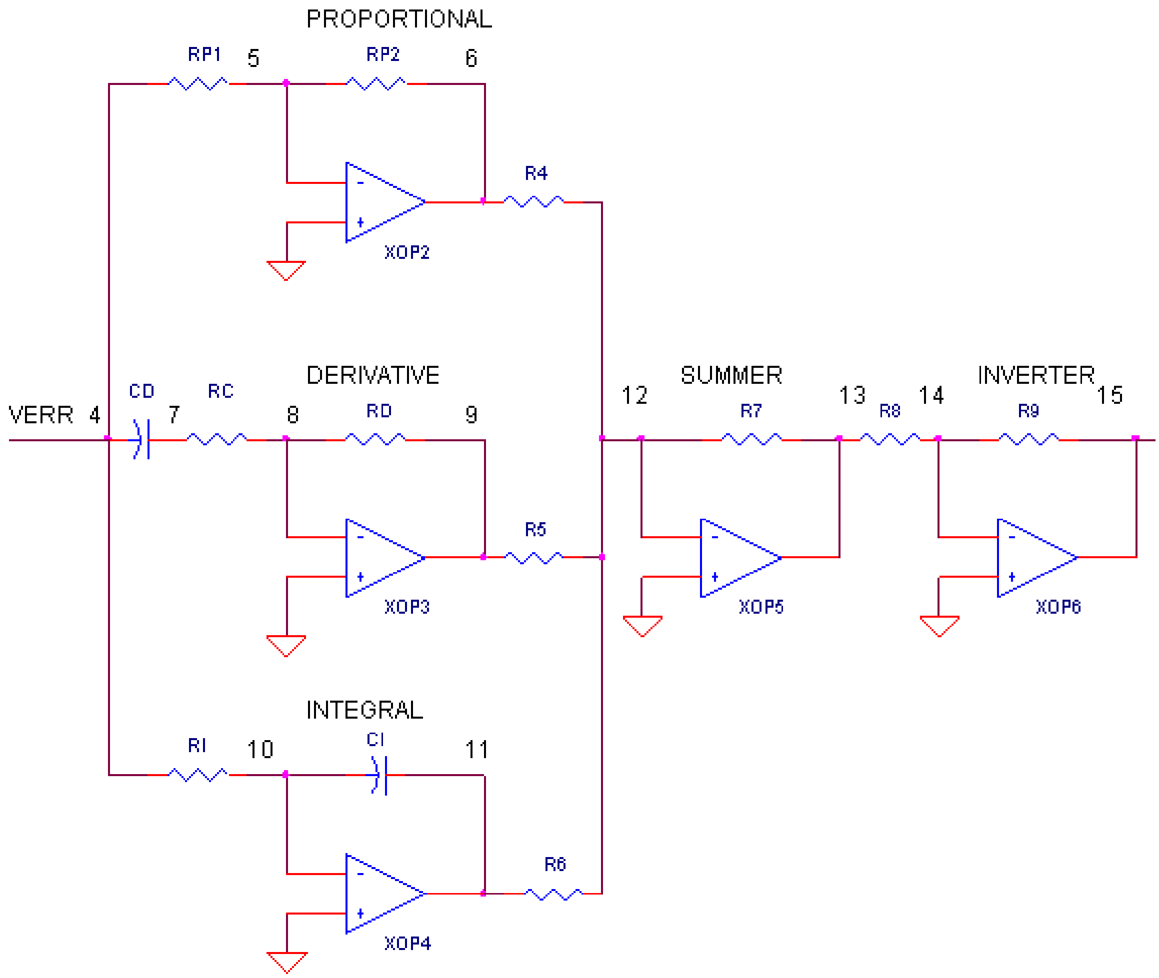
$$T_1 = R_2C$$

$T_2 = R_4C$

The design of multistage phase lead controller in time-domain becomes more as there are more poles and zero to be placed.For a two-stage controller, we can choose the parameters of 1st stage of a two stage controller so that a portion of phase margin requirement is satisfied and the 2nd stage fulfils the remaining requirement.

OR

Realization of Controllers (P, PI,PD,PID) with OPAMP



SHORT QUESTIONS WITH ANSWER

Q1. What are the different controllers?

Ans- Proportional, Derivative, Integral and Combination of Proportional and integral controllers (PI Controller) Proportional and derivative controllers (PD Controller) Proportional integral derivative control (PID Controller).

Q2.What is Proportional control system ?

Ans- Proportional control, in engineering and process control, is a type of linear feedback control system in which a correction is applied to the controlled variable which is proportional to the difference between the desired value (setpoint, SP) and the measured value (process variable).

Q3. What happens when a derivative controller applied to a Control system ?

Ans- When derivative control is applied, the controller senses the rate of change of the error signal and contributes a component of the output signal that is proportional to a derivative of the error signal.

LONG QUESTIONS

Q1.What is the effect of feedback on overall gain and stability?

Q2. State different type of controller with block diagram and represent them mathematically.

CHAPTER - 7
STABILITY CONCEPT & ROOT LOCUS METHOD

7.1.0 CONCEPT OF STABILITY:

Stability is a very important characteristic of the transient performance of a system. Any working system is designed considering its stability. Therefore, all instruments are stable with in a boundary of parameter variations.

A linear time invariant (LTI) system is stable if the following two conditions are satisfied.

(i) **Notion-1:** When the system is excited by a bounded input, output is also bounded.

A SISO system is given by

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}$$

So, $c(t) = \alpha^{-1}\left[G(s)R(s)\right]$

Using convolution integral method

$$c(t) = \int_0^\infty g(\tau) r(t-\tau) d\tau$$

$g(\tau) = \alpha^{-1} G(s) =$ impulse response of the system

Taking absolute value in both sides,

$$|c(t)| = \left|\int_0^\infty g(\tau) r(t-\tau) d\tau\right|$$

Since, the absolute value of integral is not greater than the integral of absolute value of the integrand.

$$|c(t)| \le \int_0^\infty |g(\tau) r(t-\tau) d\tau|$$

$$\Rightarrow |c(t)| \le \int_0^\infty |g(\tau) r(t-\tau)| d\tau$$

$$\Rightarrow |c(t)| \le \int_0^\infty |g(\tau)||r(t-\tau)| d\tau$$

Let, r(t) and c(t) are bounded as follows.

$$|r(t)| \le M_1 < \infty$$

$$|c(t)| \le M_2 < \infty$$

Then,

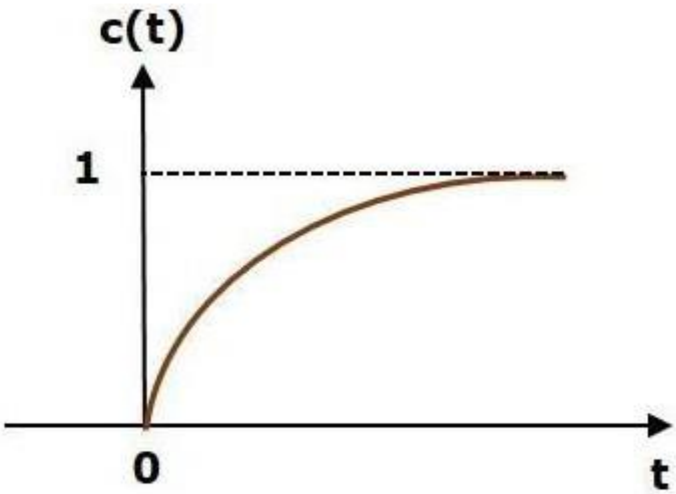
$$|c(t)| \le M_1 \int_0^\infty |g(\tau)| d\tau \le M_2$$

Hence, first notion of stability is satisfied if $\int_0^\infty |g(\tau)| d\tau$ is finite or integrable.

(ii) **Notion-2:** In the absence of the input, the output tends towards zero irrespective of initial conditions. This type of stability is called asymptotic stability.

7.1. EFFECT OF LOCATION OF POLES ON STABILITY

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A **stable system** produces a bounded output for a given bounded input.
The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of **t** including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systems based on Stability

We can classify the systems based on stability as follows.

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely Stable System

If the system is stable for all the range of system component values, then it is known as the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of **'s' plane**. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as **conditionally stable system**.

Marginally Stable System

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis

OR

7.1 EFFECT OF LOCATION OF POLES ON STABILITY:

Location of poles has direct effect on stability.
The entire S-Plane is divided in to three categories.

- LHP (Left Half Plane)
- Jω-axis
- RHP (Right Half Plane)

LHP POLES:

- On real axis & Simple
- On real axis & Multiple
- Complex conjugate & Simple
    - For all these conditions the system is stable.

Jω-AXIS:

- Complex & Simple
- Complex & Multiple
- At origin & Simple
- At origin & Multiple
    - In this case the system is always unstable.

RHP POLES:

- On real axis & simple
- On real axis & multiple
- Complex conjugate & simple

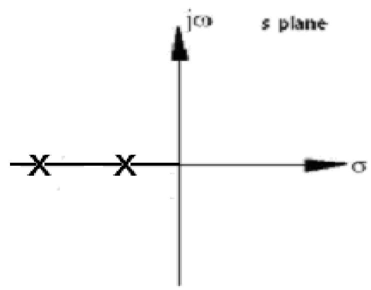
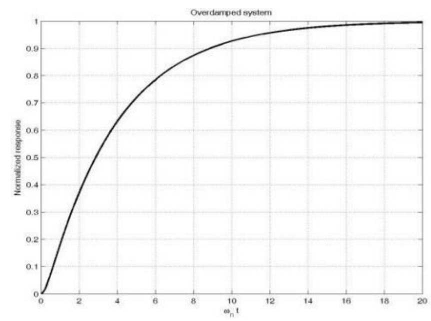
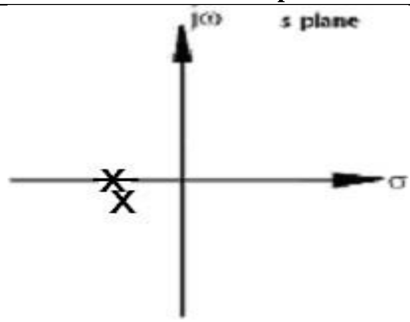
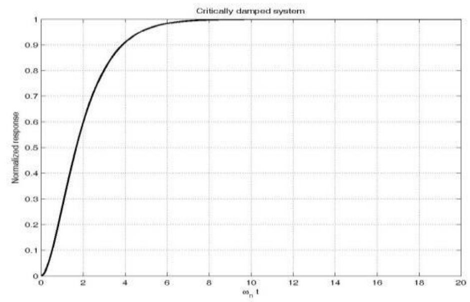
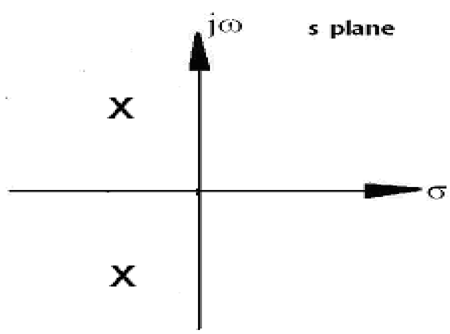
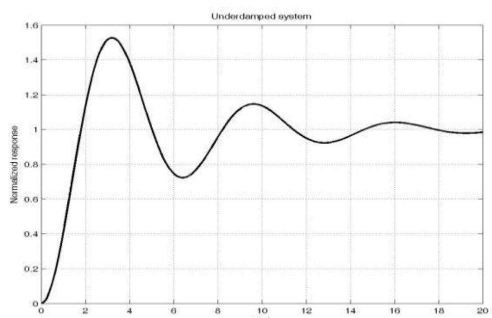
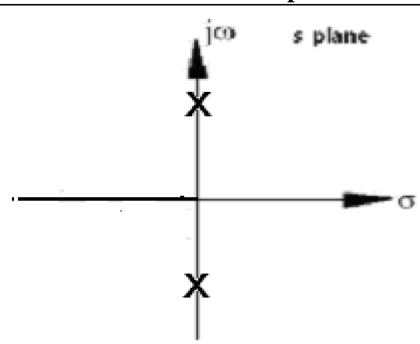
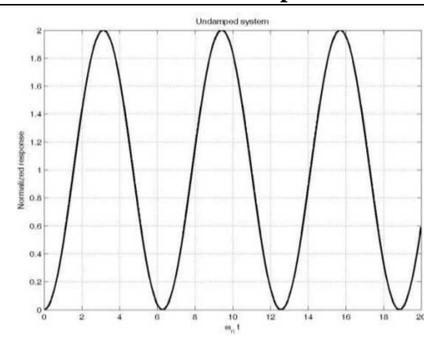
- In this case the system is always unstable.

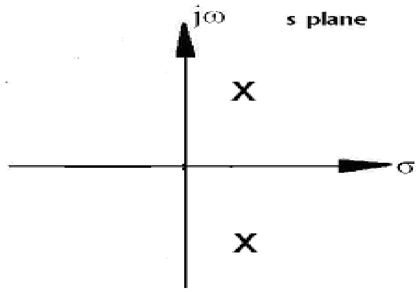
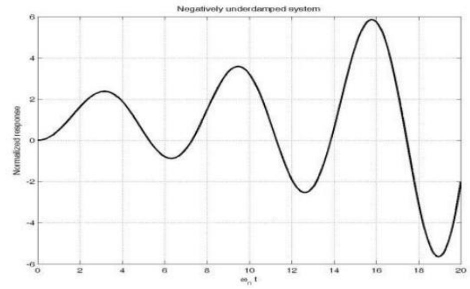
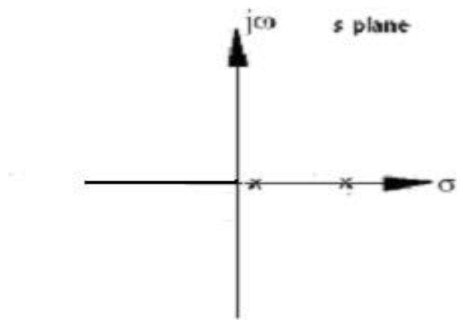
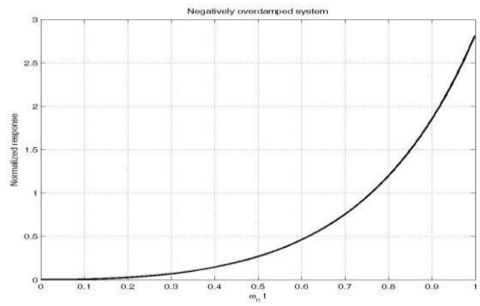
EXAMPLE:

1. Determine whether the system is stable, Unstable or Marginary stable.

- **-2,-5 → Stable (LHP)**
- **5,-7 → Unstable (RHP)**
- **-2,0 → Stable (LHP)**
- **-2+j,-2-j → Stable (LHP)**
- **-2+j4,-2-j4,-2 → Stable (LHP)**
- **-2+j2,-2j2,-2j,+2j → Marginary Stable**.
- Marginary Stable because the poles aren’t repeat& have a zero real part.

7.1.1 EFFECT OF LOCATION OF POLES ON STABILITY:

Pole-zero map	Normalized response
Over-damped close-loop poles	**Over-damped close-loop poles**
	
Critically damped close-loop poles	**Critically damped close-loop poles**
Pole-zero map	**Normalized response**
	
Under-damped close-loop poles	**Under-damped close-loop poles**
Pole-zero map	**Normalized response**
	
Un-damped close-loop poles	**Un-damped close-loop poles**
Pole-zero map	**Normalized response**
	

Negative Under-damped close-loop poles	Negative Under-damped close-loop poles
Pole-zero map	**Normalized response**
	
Negative Over-damped close-loop poles	**Negative Over-damped close-loop poles**
Pole-zero map	**Normalized response**
	

CLOSED-LOOP POLES ON THE IMAGINARY AXIS:

Closed-loop can be located by replace the denominator of the close-loop response with s=jω.

Example:

1. Determine the close-loop poles on the imaginary axis of a system given below.

$$G(s) = \frac{K}{s(s+1)}$$

Solution:

Characteristics equation, $s^2 + s + K = 0$

Replacing $s = j\omega$

$(j\omega)^2+(j\omega)+K = 0$

$(K-\omega^2)+j\omega = 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \sqrt{K}$ and $\omega = 0$

Therefore, Closed-loop poles do not cross the imaginary axis.

Example:

2. Determinethe Close-loop poles on the imaginary axis of a system given below.

   $(j\omega)^3+6(j\omega)^2+8(j\omega)+K = 0$

Solution: Characteristics equation,

$(j\omega)^3+6(j\omega)^2+8(j\omega)+K = 0$

$\Rightarrow (K - 6\omega)^2+j(8\omega - \omega^3) = 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \pm\sqrt{8}$ rad/s and $K = 6\omega^2 = 48$

Therefore, Close-loop poles cross the imaginary axis for K>48.

7.2 ROUTH-HURWITZ'S STABILITY CRITERION:

General form of characteristics equation,

$$B(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0 = 0$$
$$\Rightarrow (s - r_1)(s - r_2) \cdots (s - r_n) = 0$$

Where, r_i = Roots of the characteristics equation

Necessary condition of stability:

Coefficients of the characteristic polynomial must be positive.

Example 7.2.1:

Consider a third order polynomial $B(s) = s^3 + 3s^2 + 16s + 130$. Although the coefficients of the above polynomial are positive, determine the roots and hence prove that the rule about coefficients being positive is only a necessary condition for the roots to be in the left s-plane.

Solution: Characteristics equation, $B(s) = s^3 + 3s^2 + 16s + 130 = 0$

By using Newton-Raphson's method r_1 = -5 and $r_{2,3} = 1 \pm j5$

Therefore, from the above example, the condition that coefficients of a polynomial should be positive for all its roots to be in the left s-plane is only a necessary condition.

7.2 ROUTH-HURWITZ'S STABILITY CRITERION:

It determines the poles of the characteristics equation w.r.t the left half and right half of the S-Plane without solving the equation.

The Transfer function of any close loop system is given by

$$\frac{C(S)}{R(S)} = \frac{a_0 S^m + a_1 S^{m-1} + \ldots\ldots\ldots\ldots a_m}{b_0 S^n + b_1 S^{n-1} + \ldots\ldots\ldots\ldots a_n}$$

SYSTEM STABILITY:

This means that the system must be stable at all times during operation. Stability may be used to define the usefulness of the system. Stability studies include absolute & relative stability. It is the quality of stable or unstable performance.

The stability study is based on the properties of the TF. In the analysis, the characteristic equation is of importance to describe the transient response of the system. From the roots of the characteristic equation, some of the conclusions drawn will be as follows,

(1) When all the roots of the characteristic equation lie in the left half of the S-plane, the system response due to initial condition will decrease to zero at time t = ∞ . Thus the system will be termed as stable.

(2) When one or more roots lie on the imaginary axis & there are no roots on the RHS of S-plane, the response will be oscillatory without damping. Such a system will be termed as critically stable.

(3) When one or more roots lie on the RHS of S-plane, the response will exponentially increase in magnitude; there by the system will be Unstable.

SOME OF THE DEFINITIONS OF STABILITY ARE:

(1) A system is stable, if its o/p is bounded for any bounded i/p.

(2) A system is stable, if it's response to a bounded disturbing signal vanishes ultimately as time "t" approaches infinity.

(3) A system is un stable, if it's response to a bounded disturbing signal results in an o/p of infinite amplitude or an Oscillatory signal.
(4) If the o/p response to a bounded i/p signal results in constant amplitude or constant amplitude oscillations, then the system may be stable or unstable under some limited constraints. Such a system is called Limitedly Stable system.
(5) If a system response is stable for a limited range of variation of its parameters, it is called Conditionally Stable System.
(6) If a system response is stable for all variation of its parameters, it is called Absolutely Stable system.

7.2 STABILITY CRITERION:

In general, a system before being put in to use has to be tested for its stability. Routh-Hurwitz stability criteria may be used. This criterion is used to know about the absolute stability.
As per Routh-Hurwitz criteria, the necessary conditions for a system to be stable are,
(1) None of the co-efficient" of the Characteristic equation should be missing or zero.
(2) All the co-efficient" should be real & should have the same sign.
A sufficient condition for a system to be stable is that each & every term of the 1st column of the Routh array must be positive or should have the same sign. Routh array can be obtained as follows. In this criterion the co-efficient are arranged in an array known as Routh's array.
General form of characteristics equation,

Sufficient condition of stability:

Method I (using determinants)
The coefficients of the characteristics equation are represented by determinant form as follows

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & a_{n-1} & a_{n-3} & \cdots \\ & & & \end{vmatrix}$$

Here, the determinant decreases by two along the row by one down the column. For stability, the following conditions must satisfy.

$$\Delta_1 = a_{n-1} > 0, \Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} > 0 \cdots$$

Method II (using arrays)
The coefficients of the characteristics equation are represented by array form as follows.

$$\mathbf{a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-3} \;\ldots\ldots\ldots\ldots a_n = 0}$$

S^n	a_0	a_2	a_4	a_6
S^{n-1}	a_1	a_3	a_5	a_7
S^{n-2}	b_1	b_3	b_5	0
S^{n-3}	c_1	c_3	c_5	0
S^{n-4} :	d_1	d_3	0	0 :
S^0	a_n	0	0	0

Where,

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_2}, \quad b_3 = \frac{a_1a_4 - a_0a_5}{a_1}, \quad b_5 = \frac{a_1a_6 - a_0a_7}{a_1}, \quad c_1 = \frac{b_2a_3 - a_1b_3}{b_1},$$

$$c_3 = \frac{b_1a_5 - a_1b_5}{a_2}, \quad c_5 = \frac{b_1a_7 - 0}{b_1} = a_7, \quad d_1 = \frac{c_1b_3 - b_1c_3}{c_1}, \quad d_3 = \frac{c_1b_3 - b_1c_5}{c_1}$$

OR

$$\begin{array}{c|ccc} s^n & a_n & a_{n-2} & a_{n-4} \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\ \vdots & & & \end{array}$$

Where,

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}}$$

$$b_{n-3} = \frac{(a_{n-1})(a_{n-4}) - a_n(a_{n-5})}{a_{n-1}}$$

$$c_{n-1} = \frac{(b_{n-1})(a_{n-3}) - a_{n-1}(b_{n-3})}{b_{n-1}}$$

For stability, the following conditions must satisfy.
The number of roots of B(s) with positive real parts is equal to the number of sign changes $\mathbf{a_n}$, $\mathbf{a_{n-1}}$, $\mathbf{b_{n-1}}$, $\mathbf{c_{n-1}}$,etc.

Similarly we can evaluate rest of the elements:

The following are the limitations of Routh-Hurwitz stability criteria,
(1) It is valid only if the Characteristic equation is algebraic.
(2) If any co-efficient of the Characteristic equation is complex or contains power of "e", this criterion cannot be applied.
(3) It gives information about how many roots are lying in the RHS of S-plane; values of the roots are not available. Also it cannot distinguish between real & complex roots.

Special cases in Routh-Hurwitz criteria:

(1) When the 1st term in a row is zero, but all other terms are non-zeroes then substitute a small positive number for "ϵ"zero & proceed to evaluate the rest of the elements. When the 1st column term is zero, it means that there is an imaginary root.

Example:1

Find stability of the following system given $G(S) = \frac{K}{S(S+1)}$ and H(S) = 1 using Routh-Hurwitz stability criterion.

Solution: In the system, $\frac{G(S)}{1+G(S)H(S)} = \frac{\frac{K}{S(S+1)}}{1+\frac{K}{S(S+1)}} = \frac{K}{S^2+S+K}$

Method-I, Characteristics equation, $B(S) = s^2 + s + K$

$\Delta_1 = 1$

Here, $\Delta_2 = \begin{vmatrix} 1 & 0 \\ 1 & K \end{vmatrix} = K$

For stability, $\Delta_1 > 0$,

$\Delta_2 > 0$

The system is always stable for K>0.

Method-II,

Characteristics equation, $B(S) = s^2 + s + K = 0$

Here, Routh array is

$$\begin{array}{c|cc} s^2 & 1 & K \\ s^1 & 1 & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array. Therefore, the system is always stable
for K>0.

Example:2

Find stability of the following system given by $G(S) = \frac{K}{S(S+2)(S+4)}$ and H(S) = 1 using Routh-Hurwitz stability criterion.

Solution: $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{\frac{K}{S(S+2)(S+4)}}{1+\frac{K}{S(S+2)(S+4)}} = \frac{K}{S^3+6S^2+8S+K} = 0$

Characteristics equation is $s^3 + 6s^2 + 8s + K = 0$

And Routh's array

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \frac{48-K}{6} & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array if K <48. Therefore, the system is always stable for 0 <K <48.

Example:3

Find stability of the following system given by $B(s) = s^3 + 5s^2 + 10s + 3$using Routh-Hurwitzstability criterion.

Solution:

In this problem, given Characteristics equation is$B(s) = s^3 + 5s^2 + 10s + 3 = 0$,and Routh's array is

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 5 & 3 \\ s^1 & 9.4 & 0 \\ s^0 & 3 & \end{array}$$

There are no sign changes in first column elements of this array. Therefore, the system is always stable.

Example:4

Find stability of the following system given by $B(s) = s^3 + 2s^2 + 3s + 10$ using Routh-Hurwitzstability criterion.

Solution:In this problem, given characteristics equation is $B(s) = s^3 + 2s^2 + 3s + 10 = 0$ and Routh's array is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 2 & 10 \\ s^1 & -2 & 0 \\ s^0 & 10 & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Example:5

Examine stability of the following system given by

$B(s) = s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^3 & 0 & 2.5 & \\ s^2 & \infty & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used.

Method-1

Replace 0 by ε(very small number) and complete the array with ε.

(ii) Examine the sign change by taking $\varepsilon \to 0$

Now, Routh's array becomes

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^3 & \varepsilon & 2.5 & 0 \\ s^2 & \dfrac{5-8\varepsilon}{\varepsilon} & 1 & 0 \\ s^1 & \dfrac{2.5\left(\dfrac{5-8\varepsilon}{\varepsilon}\right)-\varepsilon}{\dfrac{5-8\varepsilon}{\varepsilon}} & & \\ s^0 & 1 & & \end{array}$$

Now putting $\varepsilon \to 0$, Routh's arr

$$\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 2 & 8 & 1 \\
s^3 & \varepsilon & 2.5 & 0 \\
s^2 & \dfrac{5-8\varepsilon}{\varepsilon} & 1 & 0 \\
s^1 & \dfrac{2.5\left(\dfrac{5-8\varepsilon}{\varepsilon}\right)-\varepsilon}{\dfrac{5-8\varepsilon}{\varepsilon}} & & \\
s^0 & 1 & &
\end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Method-2

Replace s by $\frac{1}{Z}$. The system characteristic equation $B(s) = s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1 = 0$ becomes

$$\frac{1}{Z^5}+\frac{2}{Z^4}+\frac{4}{Z^3}+\frac{8}{Z^2}+\frac{3}{Z}+1=0$$

$$\Rightarrow Z^5+3Z^4+8Z^3+4Z^2+2Z+1=0$$

Now, Routh's array becomes

$$\begin{array}{c|ccc}
s^5 & 1 & 8 & 2 \\
s^4 & 3 & 4 & 1 \\
s^3 & 6.67 & 1.67 & 0 \\
s^2 & 3.25 & 1 & 0 \\
s^1 & -0.385 & 0 & 0 \\
s^0 & 1 & 0 & 0
\end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Example:6

Examine stability of the following system given by $B(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 4s + 8$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 4 \\
s^4 & 2 & 4 & 8 \\
s^3 & 0 & 0 & 0 \\
s^2 & & & \\
s^1 & & & \\
s^0 & & &
\end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used. The auxiliary equation is

$A(s)=2s^4+4s^2+8$

$\Rightarrow\frac{dA(s)}{ds}=8s^3+8s$

Now, the array is rewritten as follows.

s^5	1	2	4
s^4	2	4	8
s^3	8	8	0
s^2	2	8	0
s^1	−24	0	
s^0	8		

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Example:7 Examine stability of the following system given by$B(s) = s^4 + 5s^3 + 2s^2 + 3s + 1$ using Routh-Hurwitz stability criterion. Find the number of roots in the right half of the s-plane.

Solution:

In this problem, Routh's array is

s^4	1	2	2
s^3	5	3	0
s^2	1.4	2	
s^1	−4.14	0	
s^0	2		

There are two sign changes in first column elements of this array. Therefore, the system is unstable. There are two poles in the right half of the s-plane.

- **Advantages of Routh-Hurwitz stability**
  - Stability can be judged without solving the characteristic equation
  - Less calculation time
  - The number of roots in RHP can be found in case of unstable condition
  - Range of value of K for system stability can be calculated
  - Intersection point with the jw-axis can be calculated
  - Frequency of oscillation at steady-state is calculated
- **Advantages of Routh-Hurwitz stability:**
  - It is valid for only real coefficient of the characteristic equation
  - Unable to give exact locations of closed-loop poles
  - Does not suggest methods for stabilizing an unstable system
  - Applicable only to the linear system

7.3 ROOT LOCUS:

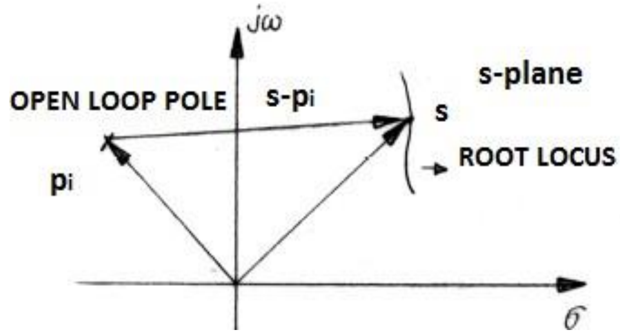
Definition:The locus of all the closed-loop poles for various values of the open-loop gain K is called root locus. The root-locus method is developed by W.R. Evans in 1954. It helps to visualize the variouspossibilities of transient response of stable systems.

Closed-loop response function$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$

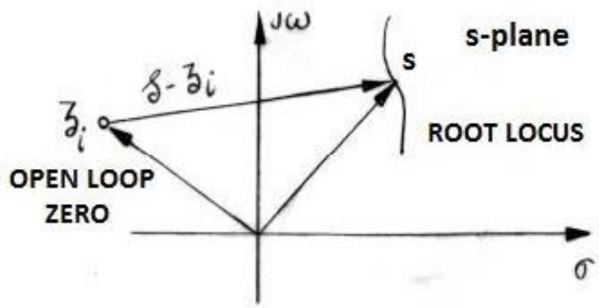
Characteristic equation

$$1+G(s)H(s)=1+\frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}=0$$

Vector from open-loop pole to the root-locus



Vector from open-loop zero to the root-locus



BEHAVIORS OF CLOSED-LOOP POLES:

Closed-loop poles negative and real	Exponential decay	Stable
Closed-loop poles complex with negative real parts	Decaying and oscillatory	Stable
Closed-loop poles positive and real	Exponential increase	Unstable
Closed-loop poles complex with positive real parts	Exponential and oscillatory increase	Unstable

7.3 BASIS FOR CONSTRUCTION:

Construction steps:

1. Determine the number of open-loop poles and zeros
2. Mark open-loop poles and zeros on the s-plane
3. Determine parts of the root-locus on the real axis
4. Determine breakaway and break-in points
5. Draw asymptotes to the root-locus
6. Determine angles of departure
7. Determine angles of arrival
8. Determine points on the root-locus crossing imaginary axis
9. Obtain additional points and complete the root-locus

7.3 RULES FOR THE CONSTRUCTION OF ROOT LOCUS:

(1) The root locus is symmetrical about the real axis.

(2) The no. of branches terminating on '∞' equals the no. of open-loop pole-zeroes.
(3) Each branch of the root locus originates from an open-loop pole at 'K = 0'& terminates At open-loop zero corresponding to 'K = ∞'.
(4) A point on the real axis lies on the locus, if the no. of open-loop poles & zeroes on the real axis to the right of this point is odd.
(5) The root locus branches that tend to '∞', do so along the straight line.
Asymptotes making angle with the real axis is given by

$$\theta = \frac{n \times 180^0}{P-Z}$$

Where, n=1,3,5,…………………

P = No. of poles & Z =No. of zeroes.

(6) The asymptotes cross the real axis at a point known as Centroid. i.e.,

$$i.e.,\ \sigma = \frac{\sum poles - \sum zeroes}{P-Z}$$

(7) The break away or the break in points [Saddle points] of the root locus or determined from the roots of the equation $\frac{dk}{ds} = 0$.

(8) The intersection of the root locus branches with the imaginary axis can be determined by the use of Routh-Hurwitz criteria or by putting 's = jω' in the characteristic equation & equating the real part and imaginary to zero. To solve for 'ω' & 'K' i.e., the value of 'ω' is intersection point on the imaginary axis & 'K' is the value of gain at the intersection point.

(9) The angle of departure from a complex open-loop pole is (θ_d) given by, $\theta_d = 180^0 + \angle GH^1$

Starting points:

Characteristics equation of a closed-loop system

$$1 + G(s)H(s) = 1 + \frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} = 0$$

For K=0,

$$\Rightarrow \frac{(s-p_1)(s-p_2)\ldots(s-p_n) + K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} = 0$$

$$\Rightarrow (s-p_1)(s-p_2)\ldots(s-p_n) = 0$$

Open-loop poles are also closed-loop poles for K=0. A root-locus starts from every open-loop pole.

Ending points:

Characteristics equation of a closed-loop system

$$1 + G(s)H(s) = 1 + \frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} = 0$$

For K=∞,

$$1 << \frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

$$\Rightarrow (s-z_1)(s-z_2)\ldots(s-z_m) = 0$$

Root-locus ends at an open-loop zero or at infinity.

7.4 ROOT LOCUS METHOD OF DESIGN (SIMPLE PROBLEM)

Problem 7.4.1:

Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \frac{K}{S(S+1)}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=2

Number of open-loop zeros m=0

Open-loop poles: s=0 and s=-1

Step 2: Mark open-loop poles and zeros on the s-plane

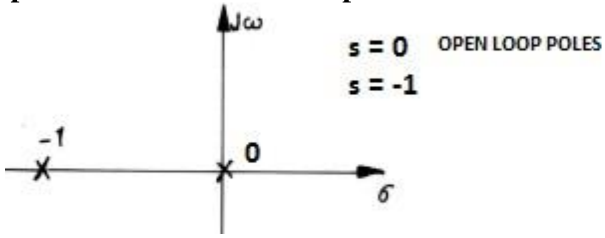


Fig.7.4.1 (Step-2)

Step 3: Determine parts of the root-locus on the real axis

Test points on the positive real axis

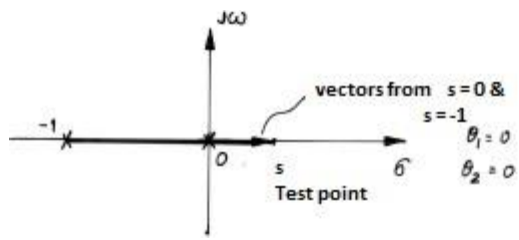


Fig.7.4.1 (Step-3)

Test points in between the open-lop poles

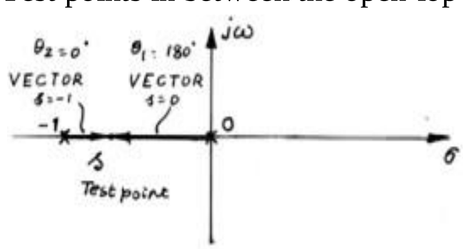
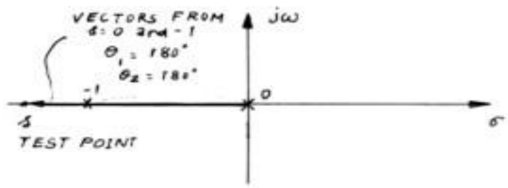


Fig.7.4.1 (Step-3)



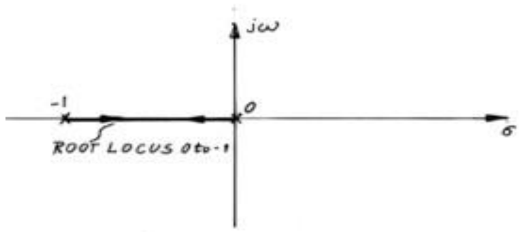


Fig.7.4.1 (Step-3)

Step 4: Determine breakaway and break-in point

Characteristic equation, K = -s(s+1)

$\frac{dK}{dS}$=-2s+1=0

breakaway point as **σb**= - 0.5

Gain at the breakaway point

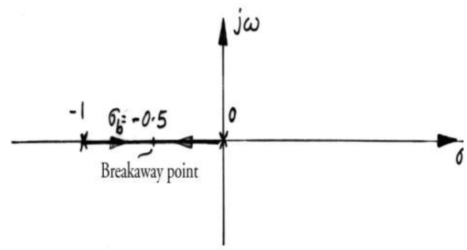


Fig.7.4.1 (Step-4)

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$\theta_c = \theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{2}$$

$$\theta_c = 90^0 \; k = 0$$

$$\theta_c = 270^0 \; k = 1$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \ldots p_n) - (z_1 + z_2 + \ldots z_m)}{(n-m)} = \frac{0-1}{2} = -0.5$$

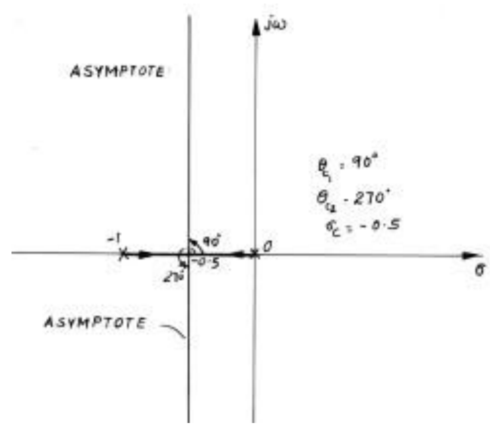


Fig.7.4.1 (Step-5)

Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$$1 + GH = 1 + \frac{K}{s(s+1)} = s^2 + s + K = 0$$

$$B(j\omega) = (j\omega)^2 + (j\omega) + K = (K - \omega^2) + j\omega$$

$$K - \omega^2 = 0 \Rightarrow j\omega = 0$$

The root-locus does not cross the imaginary axis for any value of K>0

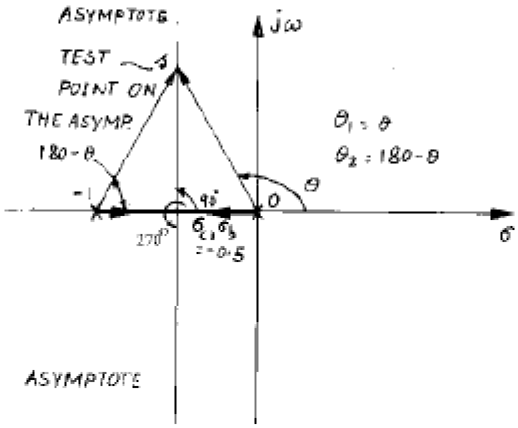


Fig.7.4.1 (Step-8.1)

Here,

$$s = \frac{-1 \pm \sqrt{1-4K}}{2}$$

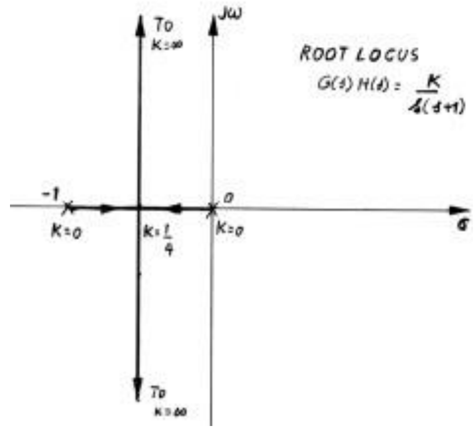


Fig.7.4.1 (Step-8.2)

PROBLEM 7.4.2:

Draw the root-locus of the feedback system whose open-loop transfer function is given by

$G(s)H(s) = \frac{K}{S(S+2)(S+4)}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=3

Number of open-loop zeros m=0

Open-loop poles: s=0, s=-2 and s=-4

Step 2: Mark open-loop poles and zeros on the s-plane

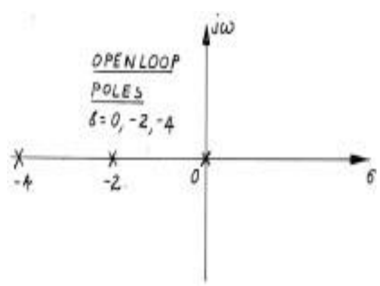


Fig.7.4.2 (Step-2)

Step 3: Determine parts of the root-locus on the real axis

Test points on the positive real axis

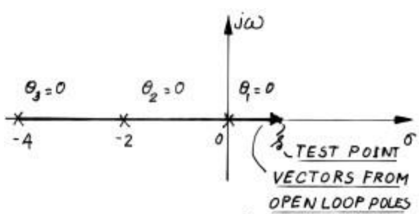


Fig.7.4.2 (Step-3)

Test points in between the open-lop poles

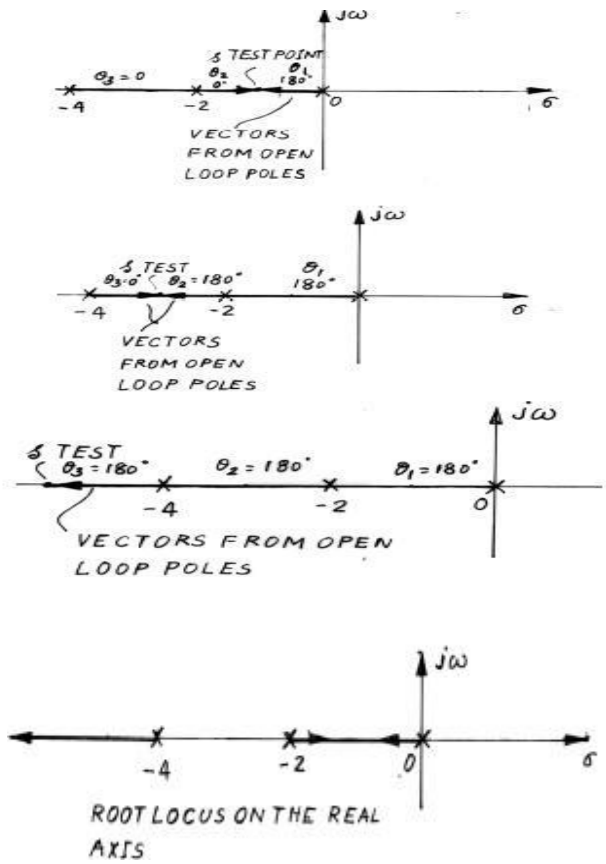


Fig.7.4.2 (Step-3)

Step 4: Determine breakaway and break-in point

Characteristic equation,

K = -s(s+2)(s+4)

$$\frac{dK}{ds} = -(s+2)(s+4) - s(s+4) = 0$$

Breakaway point as σb=-0.85 and –3.15

σb = –3.15 is not on the root-locus and therefore not a breakaway or break-in point

Gain at the breakaway point

$K_b = |-0.85-0||-0.855-(-2)||-0.85-(-4)| = 3.079$

1	6	8	K
	-0.85	-4.378	-3.079
1	5.15	3.622	K-3.079=0

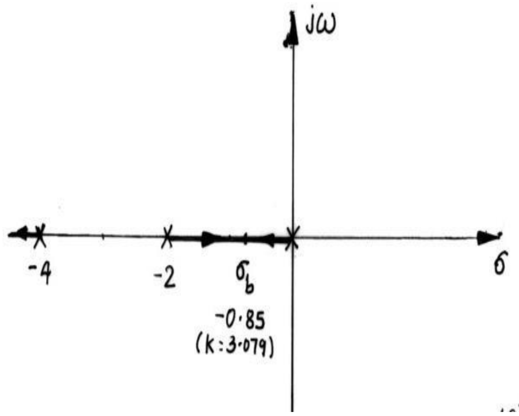


Fig.7.4.2 (Step-4)

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{3}$$

$$\theta_c = 60^0 \;\; k = 0$$

$$\theta_c = 180^0 \;\; k = 1$$

$$\theta_c = 300^0 \;\; k = 2$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \ldots p_n) - (z_1 + z_2 + \ldots z_m)}{(n-m)} = \frac{0-2-4}{3} = -2$$

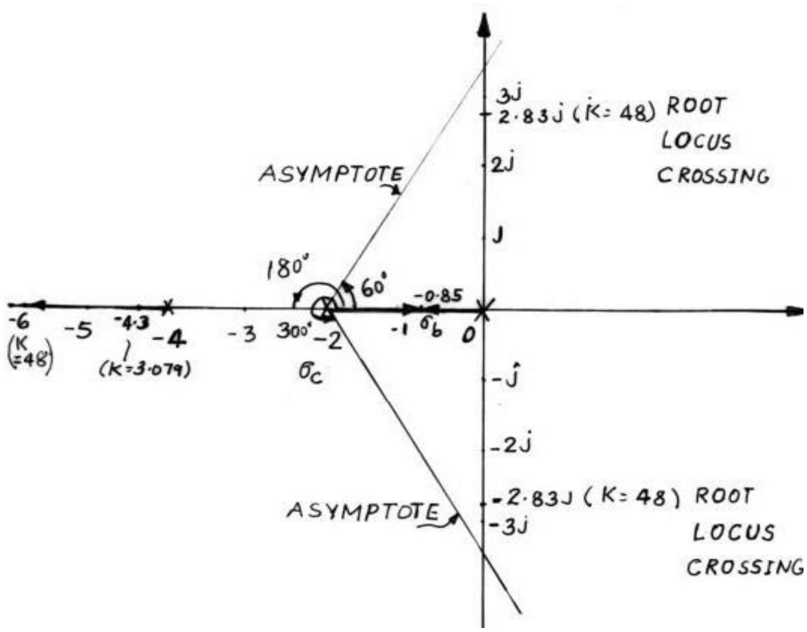


Fig.7.4.2 (Step-5)

Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

Step 8: Determine points on the root-locus crossing imaginary axis

$$1+GH = 1+\frac{K}{s(s+2)(s+4)} = s^3+6s^2+8s+K=0$$

$$B(j\omega) = (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = (K-6\omega^2) + j(8\omega-\omega^3) = 0$$

When imaginary-part is zero, then $\omega = \pm\sqrt{8}$

$\Rightarrow s = \pm j\sqrt{8}$

$K = 6\omega^2 = 48$

The root-locus does not cross the imaginary axis for any value of K>48.

1	6	8	48
	+j2.828	-8+j16.97	-48
1	6+j2.828	J16.97	0
1	6+j2.828	J16.97	
	-j2.828	-j16.97	
1	6	0	

Therefore, closed-loop pole on the real axis for K=48 at s = -6

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles	Remarks
1	-4.309	3.07	-0.85,-0.85	Already computed
2	-4.50	5.625	-0.75±j0.829	
3	-5.00	15	-0.5±j1.6583	
4	-5.50	28.875	-0.25±j2.2776	
5	-6.00	48	±j2.8284	Already computed
6	-6.5	73.125	0.25±j3.448	

Determine the gain corresponding to s=-4.5

K=|-4.5-(-4)||-4.5-(-2)||-4.5-0|= 5.625

$$s^3+6s^2+8s+K=0$$

1	6	8	K
	-4.5	-6.75	-5.625
1	1.5	1.25	K-5.625=0

$$(s^2+1.5s+1.25)=0$$

$$s_{2,3} = -0.75 \pm j0.829$$

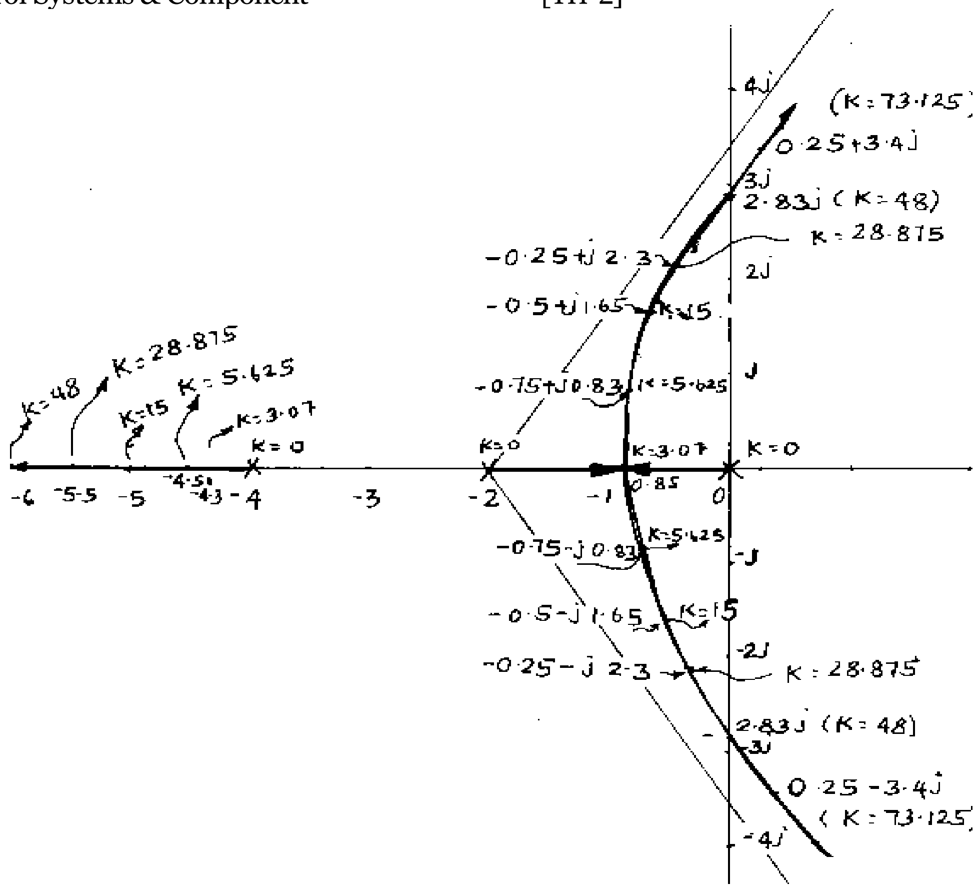


Fig.7.4.2 (Step-8)

SHORT QUESTIONS WITH ANSWER

Q1. What do you mean Root locus ?

Ans : The root locus is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs. In this chapter, let us discuss how to construct (draw) the root locus.

Q2. What is Routh–Hurwitz stability criterion ?

Ans : In control system theory, the Routh–Hurwitz stability criterion is a mathematical test that is a necessary and sufficient condition for the stability of a linear time invariant (LTI) control system.

Q3.Wht is the need of root locus ini control system ?

Ans : In control theory and stability theory, root locus analysis is a graphical method for examining how the roots of a system change with variation of a certain system parameter, commonly a gain within a feedback system.

LONG QUESTIONS

Q1. Write down the rules for constructing a root locus ?

Q2. What is Routh–Hurwitz stability criterion, explain with 7th order characteristic equation?

Q3. What happens if first element or any row of Routh–Hurwitz table become zero?

CHAPTER - 8
FREQUENCY RESPONSE ANALYSIS & BODE PLOT

8.1 FREQUENCY RESPONSE:

This is defined as the steady-state response of a system due to a sinusoidal input.

Here,

$$G(s)=\frac{C(s)}{R(s)}=\frac{N(s)}{(s+a)(s+b)(s+c)...}$$

$$\Rightarrow C(s)=\frac{N(s)R(s)}{(s+a)(s+b)(s+c)...} \quad ....(1\&2)$$

Let, r (t) = Asinωt , then

$$R(s)=\frac{A\omega}{s^2+\omega^2} \quad ....(3)$$

Using eq. (3) in eq. (2),

$$C(s)=\frac{N(s)}{(s+a)(s+b)(s+c)...}\left[\frac{A\omega}{s^2+\omega^2}\right]$$

$$\Rightarrow C(s)=\frac{A_1}{s+a}+\frac{A_1}{s+b}+\frac{A_1}{s+c}+...+\frac{B_1}{s+j\omega}+\frac{B_2}{s-j\omega}$$

In time domain, eq (5) becomes

$$c(t)=A_1e^{-at}+A_2e^{-bt}+A_3e^{-ct}+...+B_1e^{-j\omega t}+B_2e^{j\omega t}$$

The term with Ai terms are decaying components. So, they tend to zero as time tends to infinity.
Then, eq. (5) becomes

$$C_{ss}(t)=B_1e^{-j\omega t}+B_2e^{j\omega t}$$

Where,

$$B_1=\left.\frac{A\omega G(s)}{s-j\omega}\right|_{s=-j\omega}=\frac{-A}{2j}\left|G(-j\omega)\right|e^{j\angle G(-j\omega)}$$

$$B_2=\left.\frac{A\omega G(s)}{s+j\omega}\right|_{s=j\omega}=\frac{A}{2j}\left|G(j\omega)\right|e^{j\angle G(j\omega)}$$

Since, $\left|G(j\omega)\right|=\left|G(-j\omega)\right|$ and $\angle G(-j\omega)=\angle G(j\omega)=\phi$

$$c(t)=\frac{-A}{2j}\left|G(j\omega)\right|e^{-j(\omega t+\phi)}+\frac{A}{2j}\left|G(j\omega)\right|e^{j(\omega t+\phi)}$$

$$\Rightarrow c(t)=-A\left|G(j\omega)\right|e^{-j\omega t}\left[\frac{e^{j\phi}-e^{-j\phi}}{2j}\right]$$

$$\Rightarrow c(t)=A\left|G(j\omega)\right|\sin(\omega t+\phi)$$

Where, $B(\omega)=A\left|G(\right.$

$$\Rightarrow c(t)=B(\omega)\sin(\omega t+\phi)$$

Therefore, the steady-state response of the system for a sinusoidal input of magnitude A and frequency ωis a sinusoidal output with a magnitude B(ω), frequency ωand phase shift φ.
The following plots are used in frequency response.

1. Polar plot
2. Bode plot
3. Magnitude versus phase angle plot

8.1.1 RELATIONSHIP BETWEEN TIME AND FREQUENCY RESPONSE:

For a second order system:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Putting $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega}$$

$$\Rightarrow \frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\left(\frac{\omega}{\omega_n}\right)}$$

Let, $u = \frac{\omega}{\omega_n}$, then

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1 - u^2) + j2\zeta u}$$

Now,

$$M(j\omega) = |M(j\omega)| \angle M(j\omega)$$

Where,

$$|M(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

$$\theta = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$$

Now,

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

$$\omega_b = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$PM = -180^0 + \varphi$$

Where, $\varphi = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^2+1} - 2\zeta^2}}$

8.2 METHODES OF FREQUENCY RESPONSE:

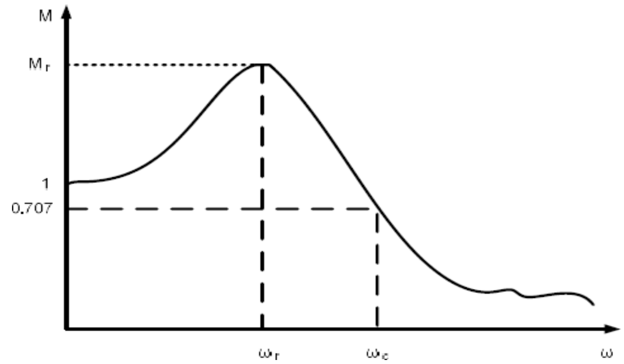
There are three types of methods in frequency response.They are

1. Bode Plot
2. Polar Plot
3. Nyquist Plot

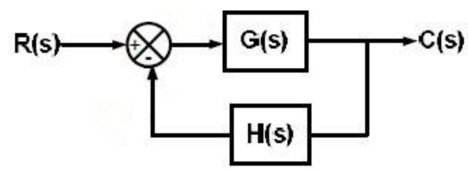
These methods are used in frequency domain to find out the stability of the system.

Definition of frequency domain specifications:

The frequency domain refers to the analysis of mathematical functions or signals with respect to frequency, rather than time.



(i) **Resonant peakM_r**: Maximum value of M (jω)when ωis varied from 0 to ∞(Minimum to Maximum as response peak ()).



Where, $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = M(s)$

$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$

(ii) **Response frequency (ω_r):** The frequency at which the peak resonance (M_r) is attained is known as response frequency.

(iii) **Cut-off frequency(ω_c)**: The frequency at which M (jω)has a value $\frac{1}{\sqrt{2}}$. It is the frequency at which the magnitude is 3dB below its zero frequency value.

(iv) **Band-width (ω_b)**: It is the range of frequencies in which the magnitude of a closed-loop system is $\frac{1}{\sqrt{2}}$times ofM_r.

(v) **Phase cross-over frequency**: The frequency at which phase plot crosses -180^0.

(vi) **Gain margin (GM)**: It is the increase in open-loop gain in dB required to drive the closed-loop system to the verge of instability.

(vii) **Gain cross-over frequency**: The frequency at which gain or magnitude plot crosses 0dB line.

(viii) **Phase margin (PM):** It is the increase in open-loop phase shift in degree required to drive the closed-loop system to the verge of instability.

8.3 POLAR PLOTS& STEPS FOR POLAR PLOTS:

8.3.1 Polar Plots:

- It is a graphical method of determining stability of feedback control systems by using the polar plot of their open-loop transfer functions.
- It is a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar plot coordinates as 'ω' is varied from zero to infinity.

- Therefore the locus of vectors |G(jω)|∠G(jω) as ω is varied from zero to infinity.

8.3.2 Steps for polar plot:

Step 1: Determine the transfer function G(s) of the system

Step 2: Put s = jω in the transfer function to obtain G(jω)

Step 3: At ω = ∞ calculate |G(jω)|. by $\lim_{\omega\to 0}|G(j\omega)|$ and $\lim_{\omega\to\infty}|G(j\omega)|$.

Step 4: Calculate the phase angle of G(jω) at ω = 0 and ω = ∞ by $\lim_{\omega\to 0}\angle G(j\omega)$ and$\lim_{\omega\to 0}\angle G(j\omega)$

Step 5: Rationalize the function G(jω) and separate the real and imaginary parts.

Step 6: Equate the imaginary part Im|G(jω)| to zero and determine the frequencies at which plots intersects the imaginary axis and calculate the value of G(jω) at the point of intersection by substituting the determined value of frequency in the rationalized expression of G(jω).

Step 7: Equate the real part Re|G(jω)| to zero and determine the frequencies at which plots intersects the imaginary axis and calculate the value of G(jω) at the point of intersection by substituting the determined value of frequency in the rationalized expression of G(jω).

Step 8: Sketch the polar plot with the help of above information.

Example 8.3.1:

Draw a polar plot of the open-loop transfer function for

$$G(s)H(s)=\frac{K}{s(s+1)}$$

Frequency response
$$G(j\omega)H(j\omega)=\frac{K}{j\omega(j\omega+1)}$$

Magnitude
$$|G(j\omega)H(\omega)|=\frac{K}{\omega\sqrt{1+\omega^2}}$$

Angle
$$\angle G(j\omega)H(j\omega)=-\frac{\pi}{2}-\tan^{-1}\omega$$

$$270^0<\angle G(j\omega)H(j\omega)<180^0$$

Magnitude and phase of the open-loop frequency transfer function

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0	∞	270
2	0.2	4.9029	259
3	0.4	2.3212	248
4	0.8	0.9761	231
5	1	0.7071	225
6	4	0.0606	194
7	10	0.01	186
8	50	0.0004	181
9	100	0.0001	181
10	200	≈0	≈180

Polar plot of the transfer function $\frac{K}{S(S+1)}$ and K=1

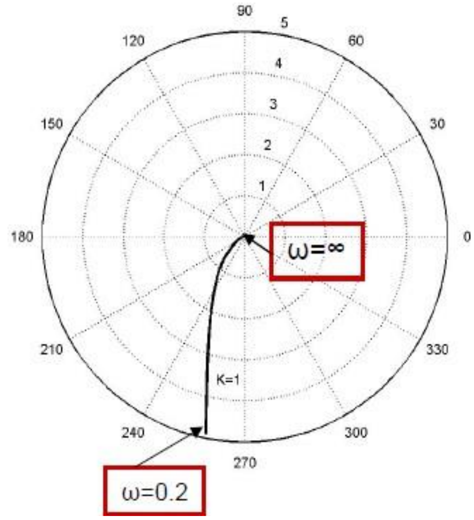


Fig. 8.3.1

Example 8.3.2:

Draw a polar plot of the open-loop transfer function for K=1, 10, 25, 55

$$GH = \frac{K}{S(S+2)(S+4)}$$

Solution:

Frequency response

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+4)}$$

Magnitude

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{\omega^2+4}\sqrt{\omega^2+16}}$$

Angle

$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$$

The lies in II and III quadrants as

$$90^0 < \angle G(j\omega)H(j\omega) < 270^0$$

Magnitude and phase of the open-loop frequency transfer function (K=1)

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0.1	1.2481	266
2	0.2	0.6211	261
4	0.4	0.3049	253
5	0.8	0.1423	237
6	1	0.1085	229
7	4	0.0099	162
8	10	0.0009	123
9	50	0	97

Polar plot of the transfer function $GH = \frac{K}{S(S+2)(S+4)}$ for K=1, 10, 25, 55

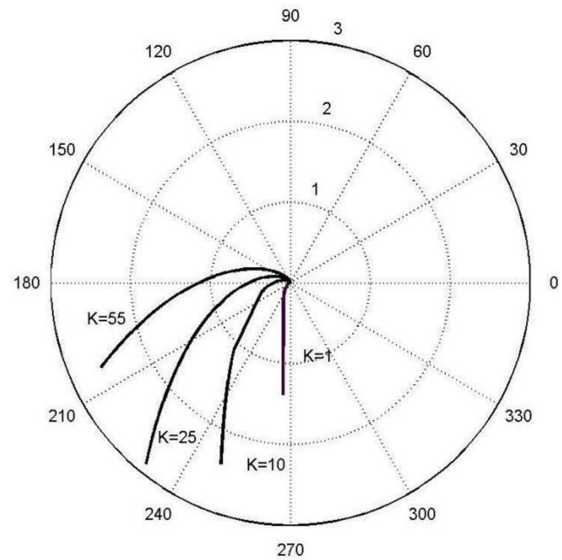


Fig. 8.3.2

Example 8.3.3:

Draw a polar plot of the open-loop transfer function G(s)H(s) = $\frac{K}{S^2(S+1)}$

Solution:

Frequency response

$$G(j\omega)H(j\omega) = \frac{K}{(j\omega)^2(j\omega+1)}$$

Magnitude

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega^2\sqrt{\omega^2+1}}$$

Angle

$$\angle G(j\omega)H(j\omega) = -180^0 - \tan^{-1}\omega$$

The lies in II quadrant only as

$$90^0 < \angle G(j\omega)H(j\omega) < 180^0$$

Magnitude and phase of the open-loop frequency transfer function (K=1)

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0.4	5.803	158
2	0.5	3.5777	153
4	0.8	1.2201	141
5	1	0.7071	135
6	2	0.1118	117
7	3	0.0351	108
8	4	0.0152	104
9	5	0.0078	101

Polar plot of the transfer function $\frac{K}{S^2(S+1)}$

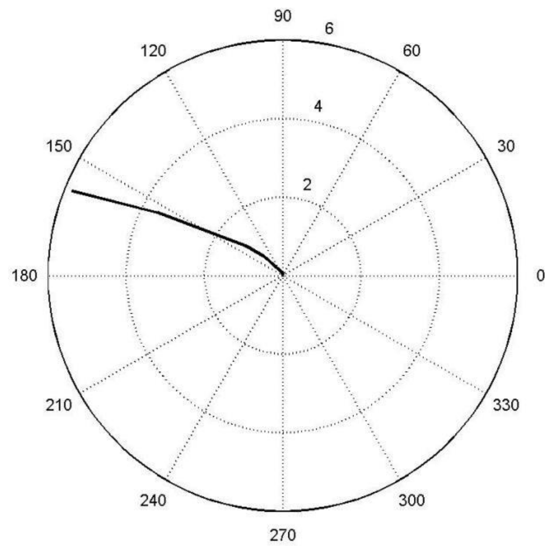


Fig. 8.3.3

OR

POLAR PLOT

1. TYPE 'ZERO' SYSTEM

$$G(s) = \frac{K}{(1+ST_1)(1+ST_2)}$$

Step 1 : Put $S = j\omega$

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$G(j\omega) = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step 2 : Taking the limit for the magnitude of $G(j\omega)$.

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} = K$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} = 0$$

Step 3 : Taking the limit for the phase angle of $G(j\omega)$

The frequency at which plot intersects the imaginary axis is $\frac{1}{\sqrt{T_1T_2}}$.

For positive values of frequencies the polar plot intersects the imaginary axis at $\omega = \infty$

$$\omega = \frac{1}{\sqrt{T_1T_2}} \quad \text{and} \quad \omega = \infty$$

Value of $G(j\omega)$ when

$$\omega = \frac{1}{\sqrt{T_1T_2}}$$

$$G(j\omega) = 0 - j\,\frac{K\frac{1}{\sqrt{T_1T_2}}(T_1+T_2)}{1+\frac{1}{T_1T_2}T_2^2+\frac{1}{T_1T_2}T_1^2+\frac{1}{T_1^2T_2^2}\cdot T_1^2T_2^2}$$

$$= -j\,\frac{K\frac{T_1+T_2}{\sqrt{T_1T_2}}}{2+\frac{T_2}{T_1}+\frac{T_1}{T_2}} = -j\,\frac{K\frac{T_1+T_2}{\sqrt{T_1T_2}}}{\frac{(T_1+T_2)^2}{T_1T_2}} = -j\,\frac{K\sqrt{T_1T_2}}{T_1+T_2}$$

2. TYPE 'ONE' SYSTEM

$$G(s) = \frac{K}{S(1+ST_1)(1+ST_2)}$$

Step 1 : Put $s = j\omega$

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+(j\omega T_2)}$$

$$G(j\omega) = \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\ \sqrt{1+(\omega T_2)^2}}\ \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step 2 : Taking the limit for the magnitude of $G(j\omega)$

$$\lim_{\omega\to 0}|G(j\omega)| = \lim_{\omega\to 0}\frac{K}{\omega\sqrt{1+(\omega T_1)^2}\ \sqrt{1+(\omega T_2{}^2)}} = \infty$$

$$\lim_{\omega\to \infty}|G(j\omega)| = \lim_{\omega\to \infty}\frac{K}{\omega\sqrt{1+(wT_1)^2}\ \sqrt{1+(wT_2)^2}} = 0$$

Step 3 : Taking the limit for the phase angle of $G(j\omega)$

$$\lim_{\omega\to 0}\angle G(j\omega) = \lim_{\omega\to 0}\angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -90^\circ$$

$$\lim_{\omega\to \infty}\angle G(j\omega) = \lim_{\omega\to \infty}\angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -270^\circ$$

Step 4 : Separating the real and imaginary parts

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{-\omega K(T_1+T_2)}{\omega+\omega^3(T_1{}^2+T_2{}^2+\omega^2T_1{}^2T_2{}^2)} + \frac{j(K\omega^2T_1T_2-K)}{\omega+\omega^3(T_1{}^2+T_2{}^2+\omega^{2}T_1{}^{2}T_2{}^{2})} \quad ...(A)$$

Equate the imaginary part equal to zero

$$\frac{K\omega^2 T_1 T_2 - K}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} = 0$$

$$\therefore \qquad \omega = \frac{1}{\sqrt{T_1 T_2}} = \frac{1}{\sqrt{T_1 T_2}} \ \& \ \omega = \pm \infty$$

The frequency at the point of intersection on real axis is $\frac{1}{\sqrt{T_1 T_2}}$. Now calculate the $G(j\omega)$ at this point.

Put $\omega = \frac{1}{\sqrt{T_1 T_2}}$ in equation (A)

Step 5 : Equate the real part to zero

$$\frac{-\omega K(T_1 + T_2)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1 T_2)} = 0$$

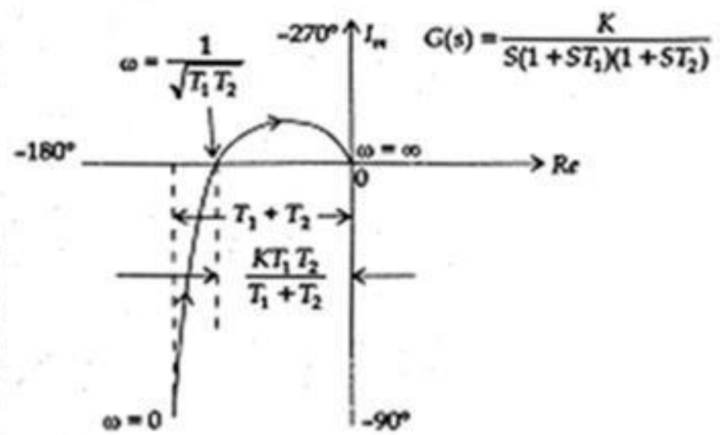
$$\therefore \qquad \omega = \infty$$

For positive values of frequencies the polar plot intersects the imaginary axis at $\omega = \infty$

$$\therefore \qquad G(j\omega) = 0 \angle -270^\circ$$

Polar plot is shown in Fig.

From the polar plot it is clear that in type one system the $j\omega$ term in denominator contributes -90° to the total phase angle. At $\omega = 0$, the magnitude is infinity and phase angle -90°. At $\omega = \infty$, the magnitude becomes zero and curve converges to origin. At low frequency, the polar plot is asymptotic to a line parallel to negative imaginary axis.



3. TYPE 'TWO' SYSTEM

$$G(s) = \frac{K}{s^2 (1 + sT_1)}$$

Put $s = j\omega$

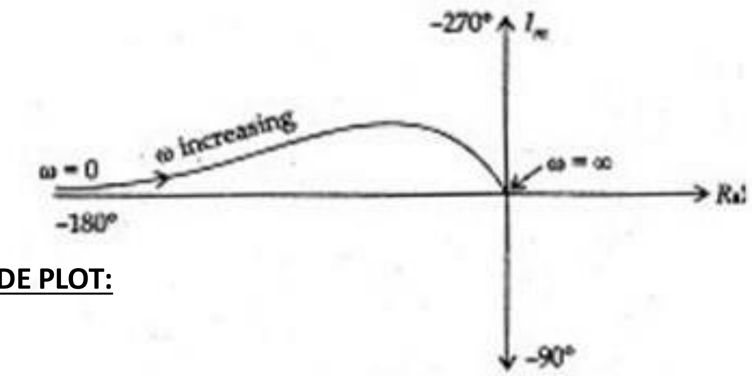
$$G(j\omega) = \frac{K}{(j\omega)^2 (1 + j\omega T_1)} = \frac{K}{-\omega^2 \sqrt{1 + (\omega T_1)^2}} \angle -180^\circ - \tan^{-1} \omega t_1$$

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{-\omega^2 \sqrt{1 + (\omega T_1)^2}} = \infty$$

$$\lim_{\omega\to\infty}|G(j\omega)| = \lim_{\omega\to\infty}\frac{K}{-\omega^2\sqrt{1+(\omega T_1)^2}}=0$$

$$\lim_{\omega\to 0}\angle -180^\circ-\tan^{-1}\omega T_1 = -180^\circ$$

$$\lim_{\omega\to\infty}\angle G(j\omega) = \lim_{\omega\to\infty}\angle -180^\circ-\tan^{-1}\omega T_1 = -270^\circ$$



8.4 BODE PLOT& STEPS FOR BODE PLOT:

8.4.1 Bode Plot:

It consist of

(i) The plot of the logarithm of the magnitude in dB of a sinusoidal transfer function vs frequency in logarithmic scale.

(ii) The plot of phase angle vs frequency in logarithmic scale.

8.4.2 Steps for Bode Plot:

Step 1: To identify the corner frequency

Step 2: Draw the asymptotic magnitude plot. The slope will change at each corner frequency by +20db/dec. for zero and -20db/dec. for pole. For complex conjugate pole and zero the slope will change by $\mp$ 0db/decade.

Step 3: (a) For type 0 system draw a line upto 1st (lowest) corner frequency having 0 dbdec. slope.

(b) For type 1 system draw a line having slope -20db/dec.upto$\omega = K$. Mark 1st (lowest) corner frequency.

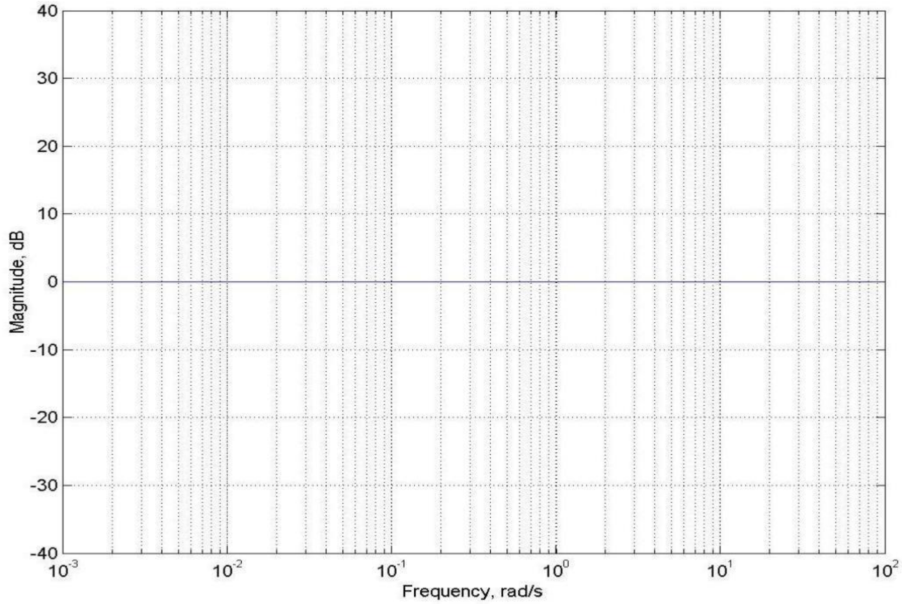
(c) For type 2 system draw the line having slope -40db/dec.upto$\omega = \sqrt{K}$andso on. Mark 1st corner frequency.

Step 4: Draw a line upto 2nd corner frequency by adding the slope of next pole or zero to the previous slope and so on.

Step 5: Calculate phase angle for different values of ω from the equation and join all points.

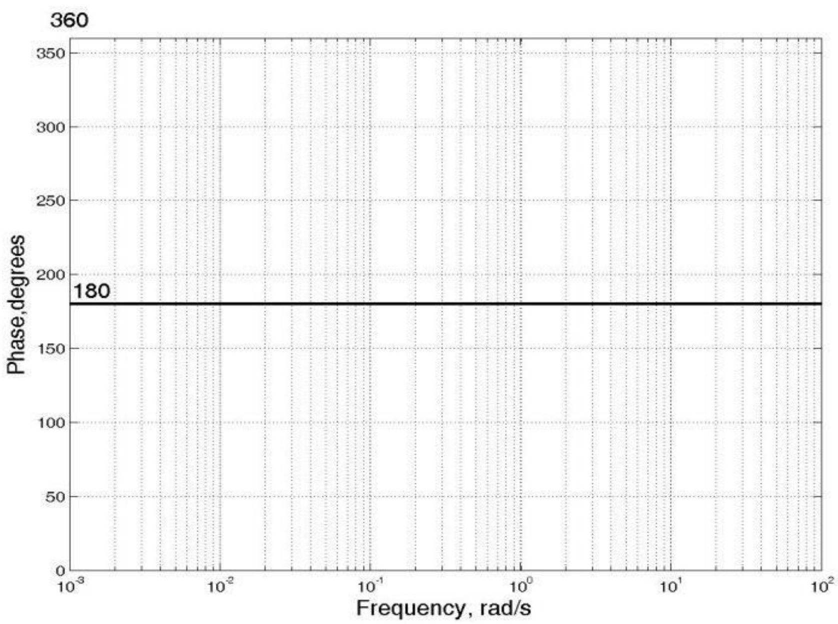
8.4.3 Magnitude plot and phase plot on a semi-log paper:

Magnitude plot on a semi-log paper

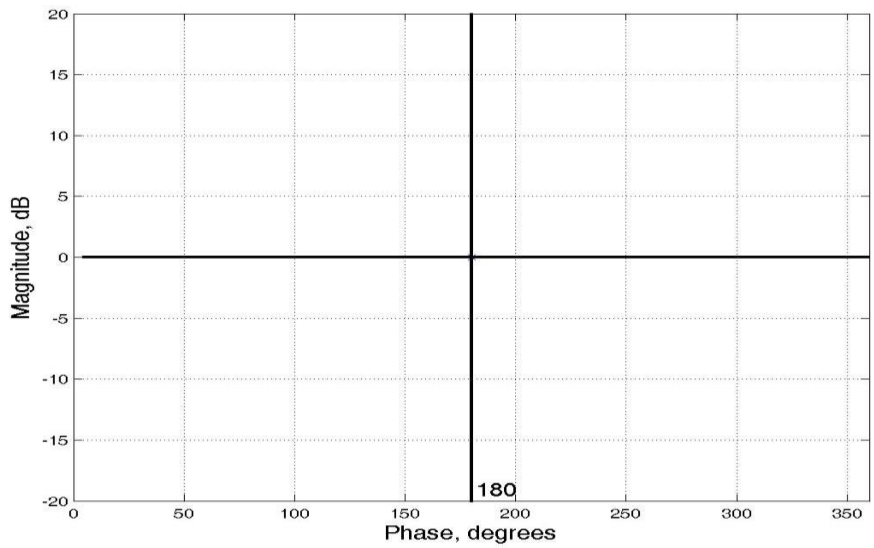


$M = 20\log|G(j\omega)H(j\omega)| dB$

Phase plot on a semi-log paper



Magnitude versus phase Bode plot Nichols plot:



Basic frequency response factors:

No	Laplace term	Frequency response	Type of factor
1	K	K	Constant
2	s	$j\omega$	Derivative factor
3	1/s	$1/j\omega$	Integral factor
4	$\tau s+1$	$(1+j\omega\tau)$	First order derivative factor
5	$1/(\tau s+1)$	$1/(1+j\omega\tau)$	First order integral factor
6	$s^2+2\varsigma\omega_n s+\omega_n^2$	$\omega_n^2-\omega^2+j2\varsigma\omega_n\omega$	Second order derivative factor
7	$\dfrac{1}{s^2+2\varsigma\omega_n s+\omega_n^2}$	$\dfrac{1}{\omega_n^2-\omega^2+j2\varsigma\omega_n\omega}$	Second order integral factor

Derivative factor: magnitude:

$$M = 20\log|j\omega| = 20\log\omega \text{ dB}$$

$$\angle j\omega = 90^0$$

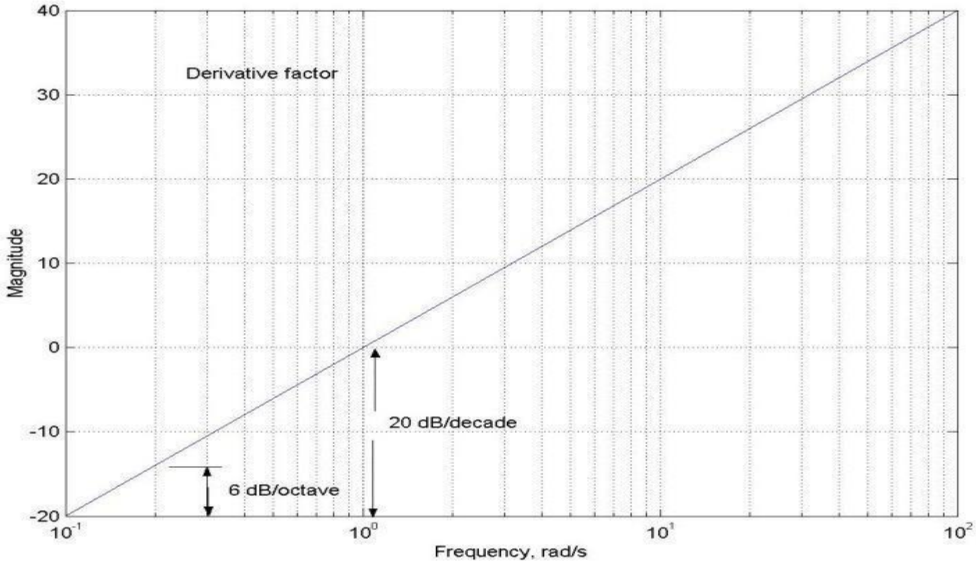
$$\Delta M = 20\log\omega_2 - 20\log\omega_1 = 20\log\frac{\omega_2}{\omega_1} \text{ dB/decade}$$

$$\Delta M = 20\log 10 = 20 \text{ dB/decade}$$

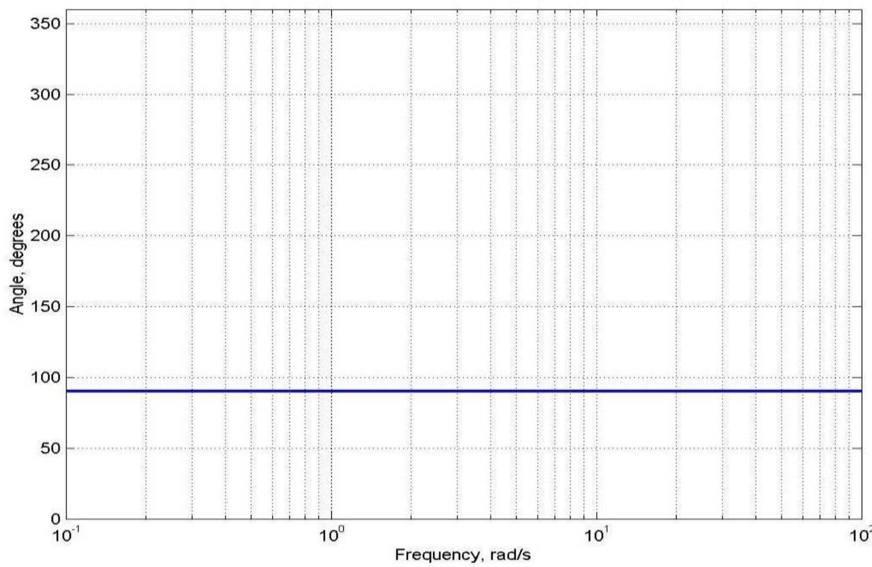
$$\Delta M = 20\log 2 \approx 6 \text{ dB/octave}$$

Magnitude variation of a derivative factor for various multiples of the initial frequency:

$\frac{\omega_2}{\omega_1}$	1	2	3	4	5	6	7	8	9	10
ΔM dB	0	6	10	12	14	16	17	18	19	20



Derivative Factor: (phase)



Derivative factor

	Frequency, rad/s				
	0.1	1	10	30	100
Magnitude, dB	-20	0	20	30	40
Phase, degrees	90	90	90	90	90

Integral factor: magnitude

$$M = 20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \text{ dB}$$

$$\angle j\omega = 270^0$$

$$\Delta M = -20\log\omega_2 + 20\log\omega_1 = -20\log\frac{\omega_2}{\omega_1} \text{ dB/decade}$$

$$\Delta M = -20\log 10 = -20 \text{ dB/decade}$$

$$\Delta M = 20\log 2 \approx -6 \text{ dB/octave}$$

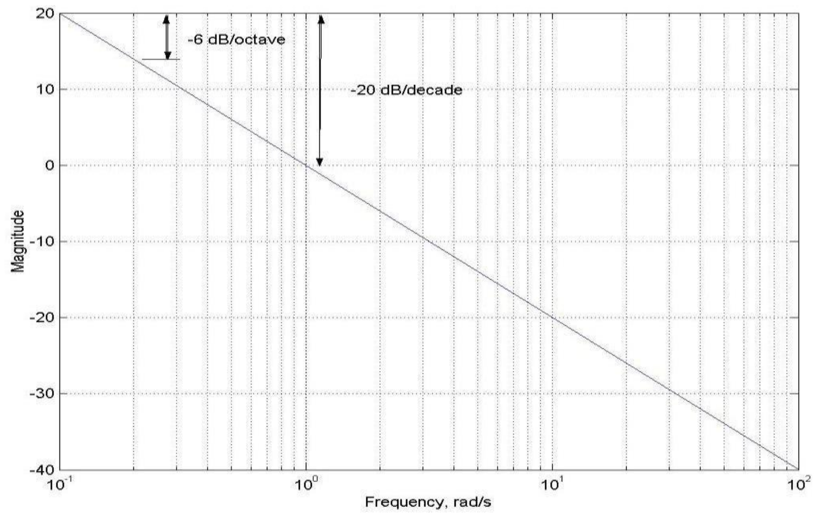


Table: Magnitude variation of an integral factor for various multiples of the initial frequency

$\frac{\omega_2}{\omega_1}$	1	2	3	4	5	6	7	8	9	10
ΔM, dB	0	-6	-10	-12	-14	-16	-17	-18	-19	-20

Integral factor: phase:

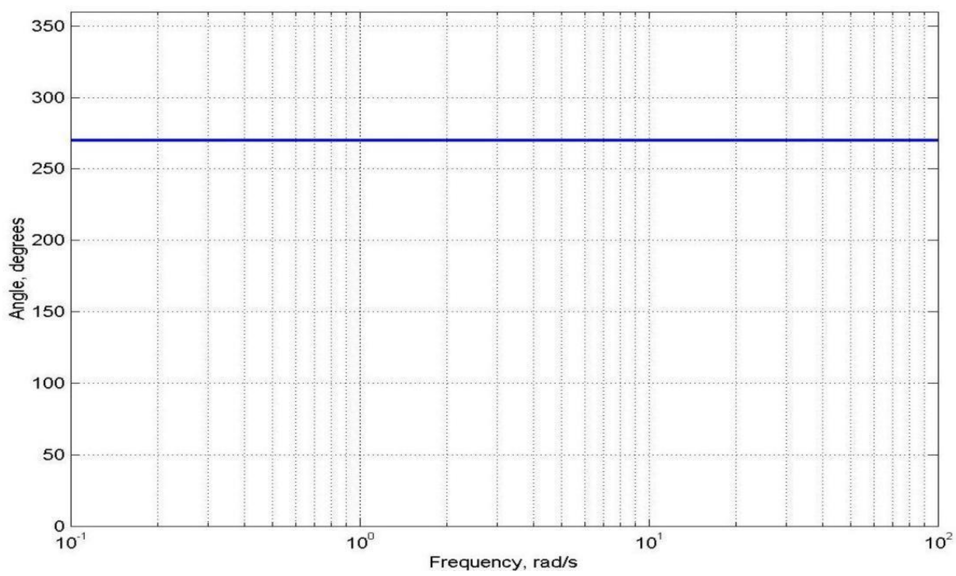


Table: Bode magnitude and phase of an integral factor

	Frequency, rad/s				
	0.1	1	10	20	100
Magnitude, dB	20	0	-20	-26	-40
Phase, degrees	270	270	270	270	270

Problem 8.4.1:

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$G(s)H(s) = \frac{1}{S(S+1)}$

Solution:

Applying S = jω

$G(j\omega)H(j\omega) = \frac{1}{j\omega(j\omega+1)}$

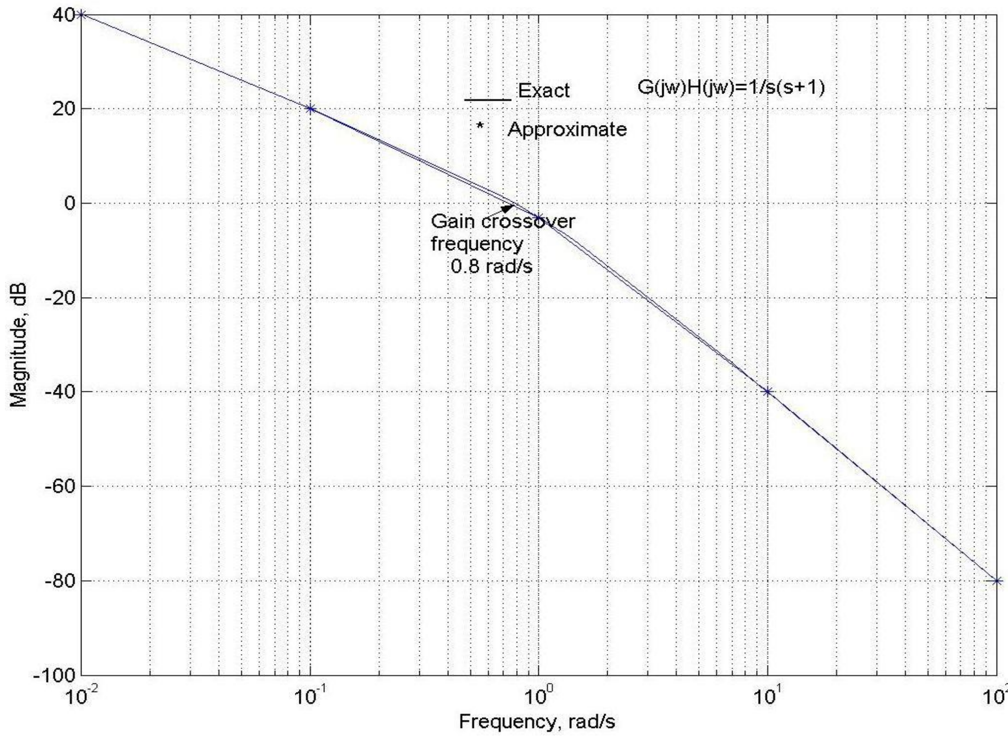
The above frequency response function has two factors: (1) Integral factor and (2) First order integral factor with a corner frequency of 1 rad/s

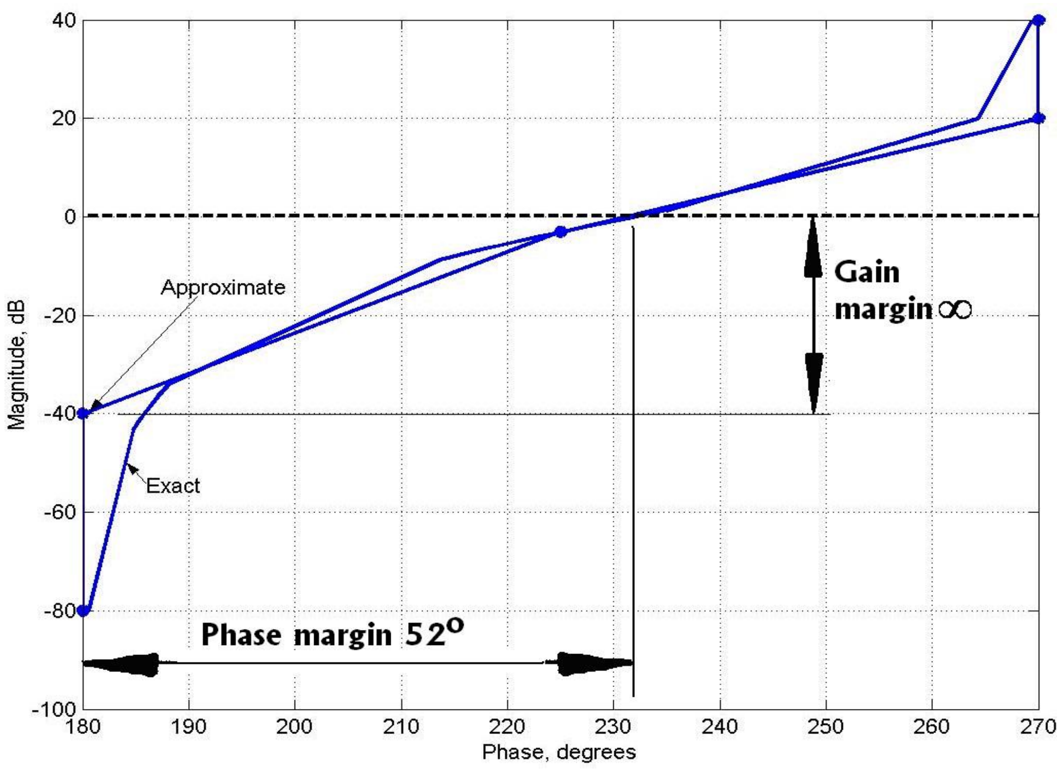
Bode magnitude of the transfer function:

	Frequency, radians/s				
	0.01	0.1	1	10	100
$20\log\frac{1}{j\omega}$ dB	40	20	0	-20	-40
$20\log\frac{1}{j\omega+1}$ dB	0	0	-3	-20	-40
Magnitude, dB	40	20	-3	-40	-80

ω_p= 100 rad/s

	Frequency, rad/s				
	0.01	0.1	1	10	100
$\angle\frac{1}{j\omega}$ degrees	270	270	270	270	270
$\angle\frac{1}{j\omega+1}$ degrees	360	360	315	270	270
Bode phase, degrees	270	270	225	180	180


Exact
G(jw)H(jw)=1/s(s+1)
* Approximate
Gain crossover frequency 0.8 rad/s
Magnitude, dB
Frequency, rad/s


Approximate
Exact
Gain margin ∞
Phase margin 52°
Magnitude, dB
Phase, degrees

Problem 8.4.2:
Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$G(s)H(s) = \frac{1}{S(S+2)(S+4)}$

Solution:

$G(j\omega)H(j\omega) = \frac{1}{8j\omega(\frac{j\omega}{2}+1)(\frac{j\omega}{4}+1)}$

The corner frequencies corresponding to first order integral factors are 2 rad/s and 4 rad/s.
Minimum frequency is chosen as 0.01 rad/s and maximum frequency 100 rad/s.
Table: Computation of Bode magnitude using asymptotic properties of the integral first-order term $\tau = \frac{1}{2}$

	x1	x2		x1	x10		x2	x1		x1	x2		x1	x10
Frequency, rad/s	2	4		2	20		20	10		20	40		10	100
Magnitude, dB	0	-6		0	-20		-20	-14		-20	-26		-14	-34

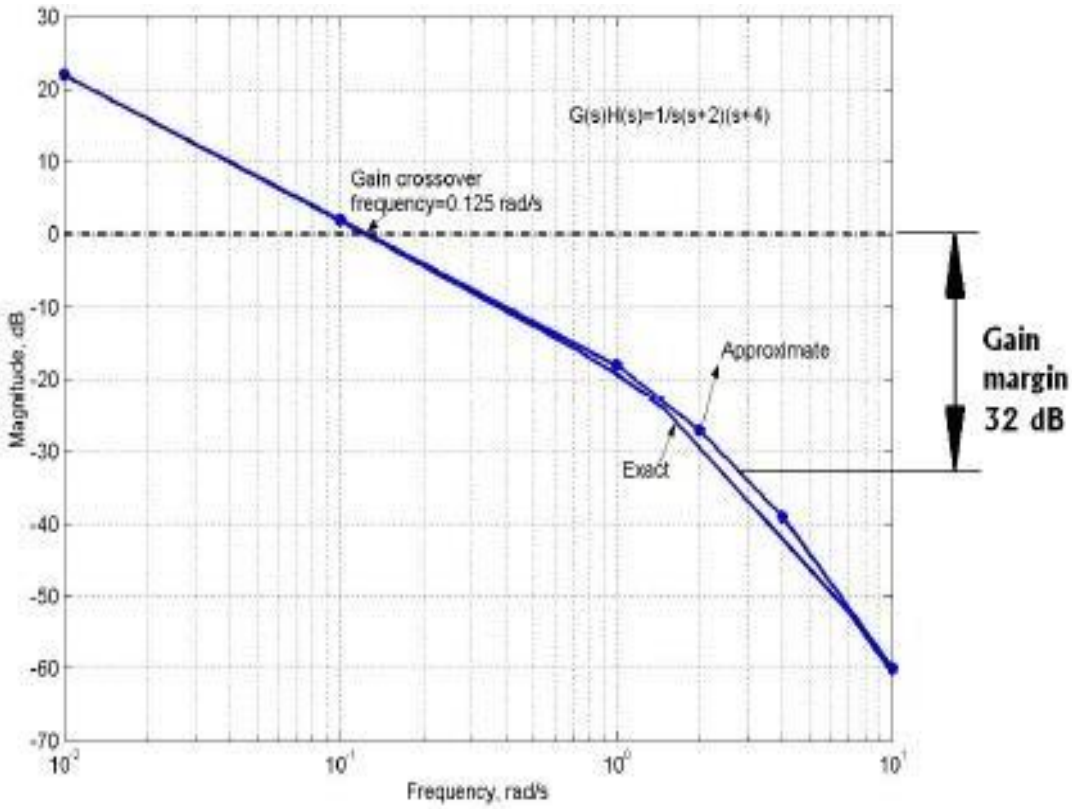
Table: Computation of Bode magnitude using asymptotic properties of the integral first-order Term $\tau = \frac{1}{4}$

	x1	x10		x2	x1		x2	x1		x1	x10
Frequency, rad/s	4	40		40	20		20	10		10	100
Magnitude, dB	0	-20		-20	-14		-14	-8		-8	-28

Table: Bode magnitude

	Frequency, rad/s										
Factor	0.01	0.1	0.2	0.4	1	2	4	10	20	40	100
$20\log\frac{1}{8}$	-18	-18	-18	-18	-18	-18	-18	-18	-18	-18	-18
$20\log\frac{1}{j\omega}$	40	20	14	8	0	-6	-12	-20	-26	-32	-40
$20\log\frac{1}{\frac{j\omega}{2}+1}$	0	0	0	0	-1	-3	-6	-14	-20	-26	-34
$20\log\frac{1}{\frac{j\omega}{4}+1}$	0	0	0	0	0	-1	-3	-8	-14	-20	-28
Bode magnitude, dB	22	2	-4	-10	-18	-28	-39	-60	-78	-96	-120

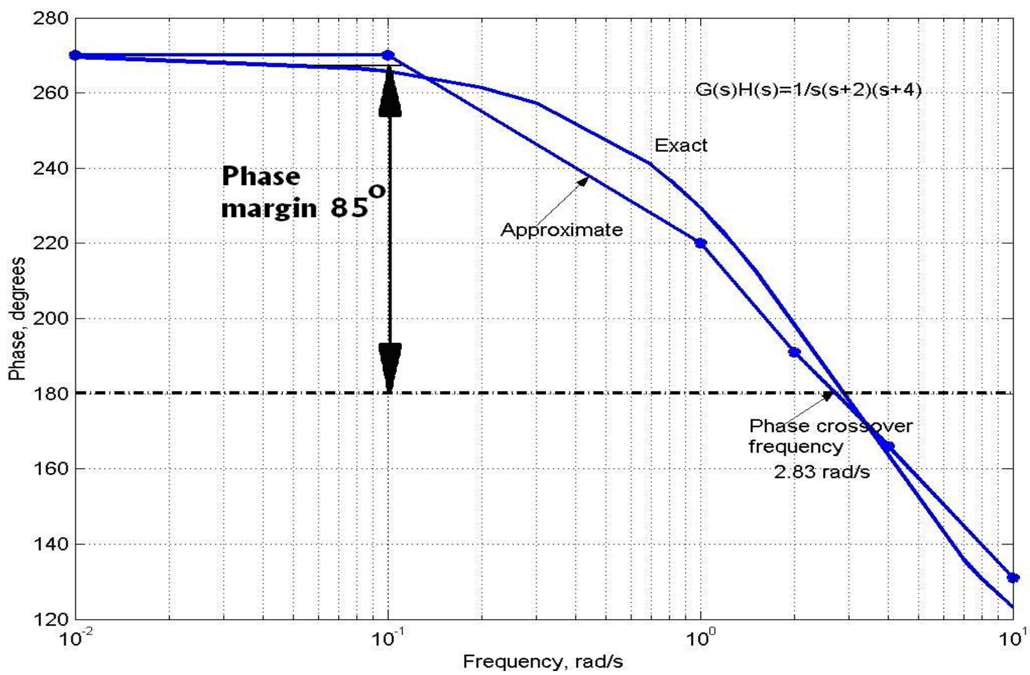
Bode magnitude:



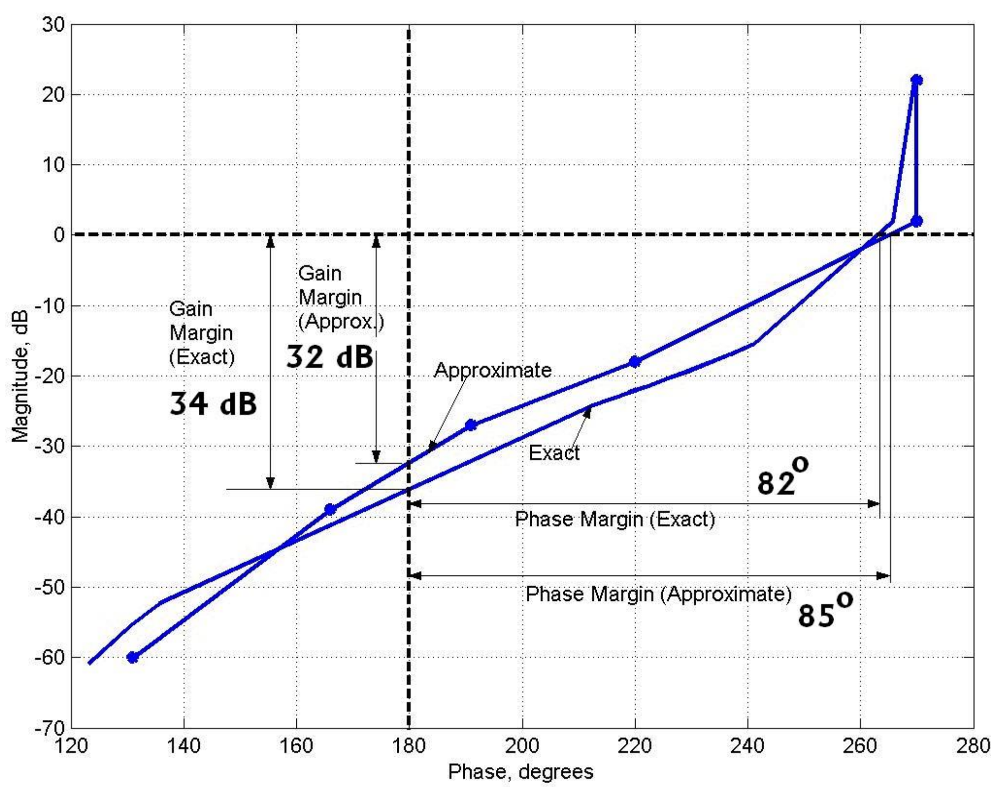
Bode phase:

	Frequency, rad/s										
Factor	0.01	0.1	0.2	0.4	1	2	4	10	20	40	100
$\angle \frac{1}{8}$	0	0	0	0	0	0	0	0	0	0	0
$\angle \frac{1}{j\omega}$	270	270	270	270	270	270	270	270	270	270	270
$\angle \frac{1}{\frac{j\omega}{2}+1}$	360	360	360	346	328	315	301	284	270	270	270
$\angle \frac{1}{\frac{j\omega}{4}+1}$	360	360	360	360	342	326	315	297	285	270	270
Phase degrees	270	270	270	256	220	191	166	131	105	90	90

Phase plot:


G(s)H(s)=1/s(s+2)(s+4)
Phase margin 85°
Exact
Approximate
Phase crossover frequency 2.83 rad/s
Phase, degrees
Frequency, rad/s

Bode plot


Gain Margin (Exact)
34 dB
Gain Margin (Approx.)
32 dB
Approximate
Exact
82°
Phase Margin (Exact)
Phase Margin (Approximate)
85°
Magnitude, dB
Phase, degrees

OR

BODE PLOT

The main advantage of using Bode plot is that multiplication of magnitudes can be converted to addition.

Consider open loop transfer function of a closed loop control system

$$G(s)\,H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots.}{s^N(1+sT_1)(1+sT_2)\ldots.}$$

Put $S = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b)\ldots\ldots.}{(j\omega)^N(1+j\omega T_1)(1+j\omega T_2)\ldots\ldots}$$

$$20\log_{10}|G(j\omega)\;H(j\omega)| = \left(20\log K + 20\log\sqrt{1+\omega^2T^2_a} + 20\log\sqrt{1+\omega^2T^2_b}\right)\ldots.)$$

$$-\left(20N\log\omega + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\sqrt{1+\omega^2T_2^2}\right)\ldots.)$$

Hence, in order to get $|G(j\omega)\,H(j\omega)|$ we will have to obtain the individual plots and adding individual components, the resultant can be obtained. Suppose, $H(s) = 1$.

Case 1. The Gain K

$$G(s) = K$$

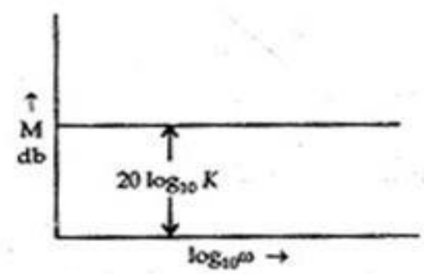
Put $s = j\omega$

$$G(j\omega) = K$$

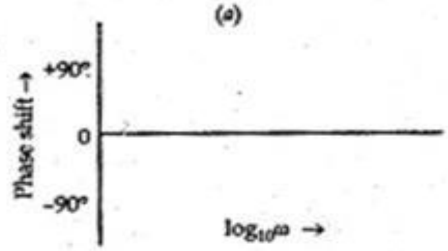
$$20\log_{10}|G(j\omega)| = 20\log_{10}K \qquad \ldots(4.1)$$

$$\text{Phase angle } \phi = \angle G(j\omega) = 0° \qquad \ldots(4.2)$$

From equations (4.1) and (4.2) it is clear that the magnitude is independent of $\log_{10}\omega$ and phase angle always zero. The plots are shown in Fig.



(a)



(b)

Case 2 : $G(s) = \dfrac{1}{S^N}$

Put $s = (j\omega)^N$

$\therefore$ $G(j\omega) = \dfrac{1}{(j\omega)^N}$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{1}{(j\omega)^N}$$

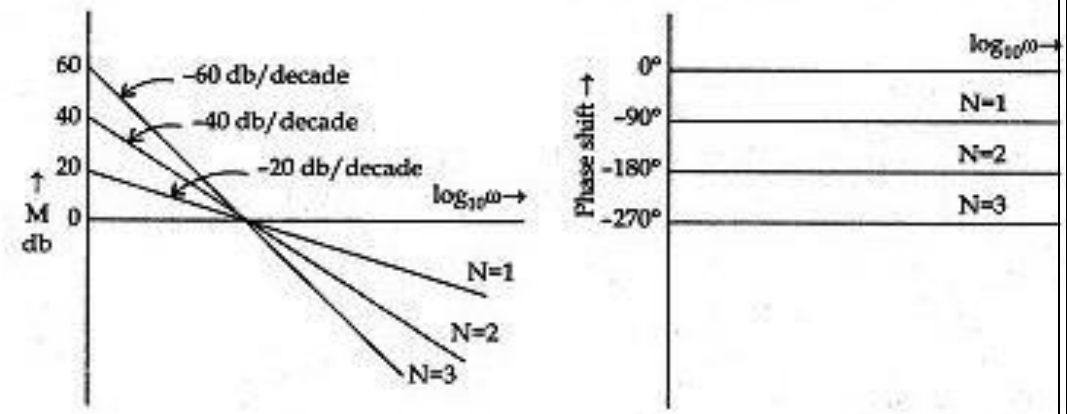
$$= 20\log_{10}(j\omega)^{-N}$$

$$= -20\,N\log_{10}(\omega)$$

$$\angle G(j\omega)^N = -90\,N°$$

where $N = 1,2,3,\ldots\ldots\ldots\ldots$

The plot M Vs $\log_{10}\omega$ is a straight line. For $N = 1$ the line has a slope of -20 db/decade and angle $-90°$. For $N = 2$, the slope of the line will be -40 db/decade and angle will be $-180°$ and so on.



Case 3 :

$$G(s) = S$$

Put

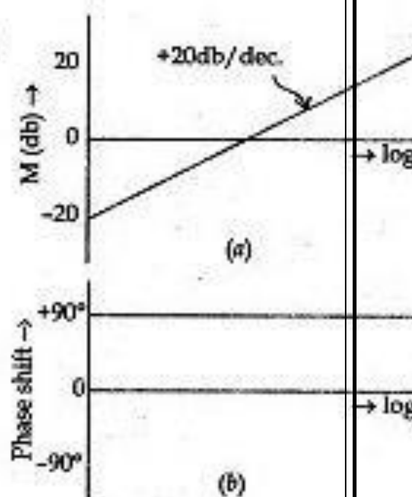
$$S = j\omega$$

$$G(j\omega) = j\omega$$

$$M = 20 \log_{10} |G(j\omega)| = 20 \log_{10} \omega$$

$$\angle G(j\omega) = +\ 90^\circ$$

The plot $M\ V_s \log_{10} \omega$ is a straight line having a slope of $+20\ db$/dec. and angular phase shift of $+90^\circ$.



Case 4 :

$$G(s) = \frac{1}{1+sT}$$

Put

$$s = (j\omega)$$

$\therefore$

$$G(j\omega) = \frac{1}{1+j\omega T}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$20 \log_{10} |G(j\omega)| = 20\log_{10}\left[\frac{1}{\sqrt{1+\omega^2T^2}}\right]$$

$$= 20\log_{10} 1 - 20\log_{10} \sqrt{1+\omega^2T^2}$$

$$= -20\log_{10} \sqrt{1+\omega^2T^2}$$

$\because \log_{10} 1 =$

Put the different values of ω, we will get $|G(j\omega)|$ consider following two cases.

(*a*) For $\omega T << 1$ (very low frequencies)

$$-20\log_{10} \sqrt{1+\omega^2T^2} = -20\log_{10} \sqrt{1} = 0$$

$\therefore$

$$M = 0 \text{ for } \omega T << 1 \text{ or } \omega \leq \frac{1}{T}$$

(*b*) For $\omega T >> 1$ (very high frequencies)

$$-20\log_{10} \sqrt{1+\omega^2T^2} = -20\log_{10} \sqrt{\omega^2T^2}$$

$$= -20 \log_{10} \omega T \text{ for } \omega >> 1/T$$

Hence, M Vs $\log_{10} \omega$ has two parts

(*i*) One part having $M = 0$ for $\omega << 1/T$

(*ii*) In other part M varies as a straight line with slope of – 20 db/decade for $\omega >> \frac{1}{T}$

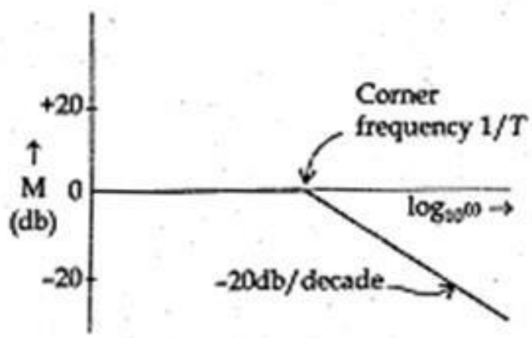
$$\omega = \frac{1}{T} \text{ is called break frequency or corner frequency}$$

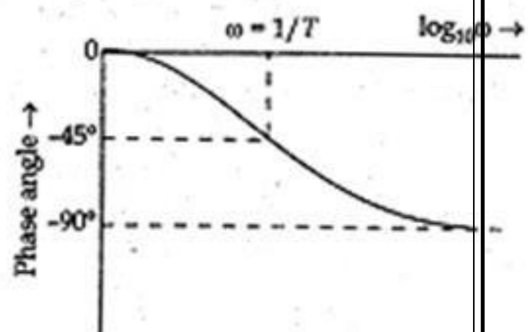
$$M = -20 \log_{10} \omega T = -20 (\log_{10} \omega + \log_{10} T)$$
$$M = -20 \log_{10} \omega - 20 \log_{10} T$$
$$= -20 \log_{10} \omega + 20 \log_{10} 1/T$$

The above two parts of the graph intersect 0 *db* axis is determined by equating the e to zero

$$0 = -20 \log_{10} \omega + 20 \log_{10} 1/T$$

$\therefore$ $\omega = 1/T$ is called break frequency.





Case 5 : $G(s) = (1 + sT)$

Put $s = j\omega$

$$G(j\omega) = (1 + j\omega T)$$
$$|G(j\omega)| = \sqrt{1 + \omega^2 T^2}$$
$$20 \log_{10} |G(j\omega)| = 20 \log_{10} \sqrt{1 + \omega^2 T^2}$$

(*i*) When $\omega T << 1$

$$M = 20 \log_{10} \sqrt{1} = 0\ db$$

(*ii*) When $\omega T >> 1$

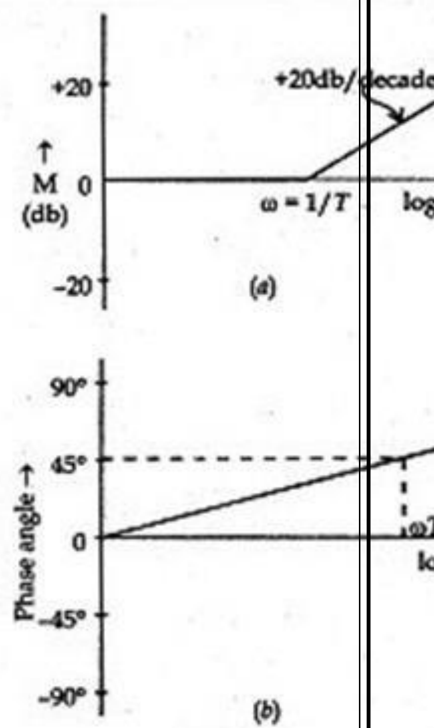
$$M = 20 \log_{10} \omega \mathrm{T}$$
$$M = 20 \log_{10} \omega T = 20 \log_{10} \frac{\omega}{1/T}$$
$$= 20 \log_{10} \omega - 20 \log_{10} \frac{1}{T}$$

Equate the above equation to zero

$$0 = 20 \log_{10} \omega - 20 \log_{10} \frac{1}{T}$$

1

Thus, the two parts of the graph intersects the '0' db axis at $\omega = \frac{1}{T}$. The second part is a straight line having the slope of + 20 db/decade.



Phase Angle Plot

$$\phi = \angle G(j\omega) = \tan^{-1}\omega T$$

(*i*) At very low frequencies ωT is very very small

$$\phi = \tan^{-1}(0) = 0°$$

(*ii*) At $\omega T = 1$

$$\phi = \tan^{-1}1 = 45°$$

(*iii*) At very high frequencies

$$\phi = \tan^{-1}(\infty) = 90°$$

Thus, the value of ϕ gradually changes from 0° to 90° as ω increases from 0 to very high values.

Case 6 : General second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Put $s = j\omega$

$$G(s) = \frac{\omega_n^2}{(jw)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

$$G(s) = \frac{w_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\left|\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}\right| = -20\log_{10}\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 4\zeta^2\left(\right.}$$

Suppose $\frac{\omega}{\omega_n} = u$

$\therefore \quad 20\log_{10}|G(j\omega)| = M = -20\log_{10}\sqrt{(1-u^2)^2 + 4\zeta^2u^2}$

Consider the two cases

1. $u << 1 \quad i.e\ \frac{\omega}{\omega_n} << 1$

$$M = -20\log_{10}\sqrt{1} = 0 \text{ db}$$

2. $u >> 1\ i.e\ \frac{\omega}{\omega_n} >> 1$

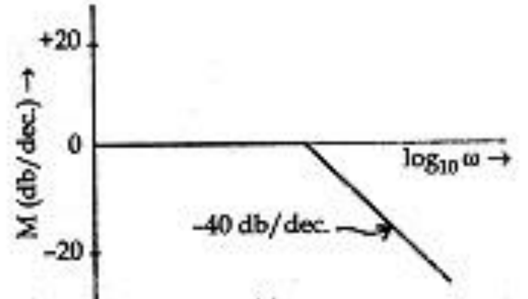
$$M = -20\log_{10}\sqrt{(u^2)^2} = -20\log_{10}u^2 = -40\log_{10}u$$

So, it is a straight line having slope of – 40 db/dec. and passing through the point u.

Therefore, the asymptotic plot consists of

(i) $M = 0$ $\quad u << 1$

(ii) $M = -40 \log_{10} u$ $\quad u >> 1$



(a)

Phase Angle Plot

$$\phi = \angle G(j\omega) = -\tan^{-1}\frac{2\xi u}{1-u^2}$$

(i) For small value of u, u^2 is small

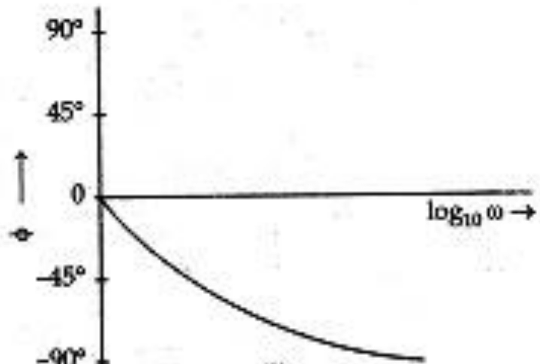
$$\therefore \quad \phi = -\tan^{-1} 2\xi u$$

(ii) For large value of u, $u^2 >> 1$

$$\therefore \quad \phi = +\tan^{-1}\frac{2\zeta}{u}$$

(iii) When $u = 1$

$\phi = -\tan^{-1}\infty = -90°$



(b)

Initial Slope of Bode Plot

Let $\quad G(s)\,H(s) = \frac{K}{S^N}$

Put $\quad s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{(j\omega)^N}$$

$$20\log_{10}|G(j\omega)\,H(j\omega)| = 20\log_{10}\left|\frac{K}{(jw)^N}\right| = 20\log_{10}K - 20\,N\log_{10}\omega$$

1. For $N = 0$ (Type zero system)

$20\log_0 |G(j\omega)\,H(j\omega)| = 20\log_{10} K.$

This is a straight line. The graph is shown in Fig.

2. For $N = 1$ (type one system)

Put $N = 1$ in equation

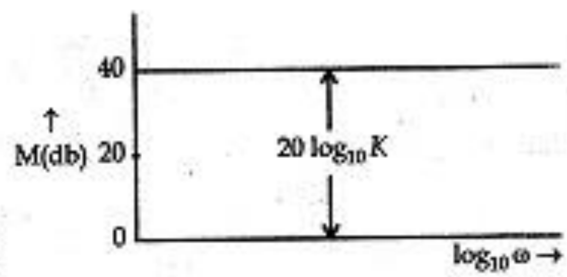
$20\log_{10}|G(j\omega)\,H(j\omega)| = 20\log_{10}K - 20\log_{10}\omega$

Intersection with 0 db axis

$$0 = 20\log_{10}K - 20\log_{10}\omega$$

$$\therefore \quad K = \omega$$

locate $\omega = K$ on 0 db axis and at this point draw a line of – 20 db/decade produce it till it intersect the y-axis that will be the starting point on Bode plot.



3. For $N = 2$ (type two system)

Put $N = 2$ in equation (4.4)

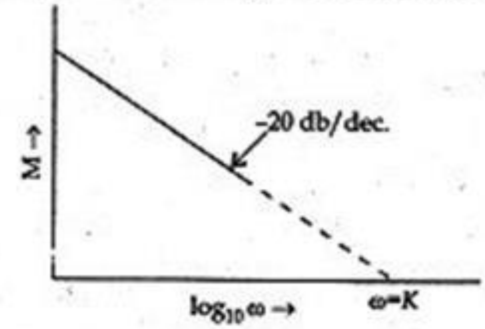
$$20\log_{10}|G(j\omega)\,H(j\omega)| = 20\log_{10}K - 20.\,2\log_{10}\omega$$
$$= 20\log_{10}K - 40\log_{10}\omega$$

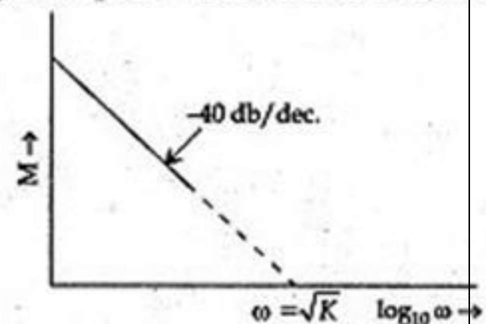
Intersection with 0 db axis

$$0 = 20\log_{10}K - 40\log_{10}\omega$$

$$20 \log_{10} K = 40 \log_{10} \omega$$
$$20 \log_{10} K = 20 \log_{10} \omega^2$$
$$\omega^2 = K$$
$$\omega = \sqrt{K}$$

Hence, graph intersect the 0 db axis at $\omega = \sqrt{K}$. Locate $\omega = \sqrt{K}$ on 0 db axis and draw a li
db/dec. and produce it to the y-axis. Graph having the slope of −40 db/decade is shown in Fi

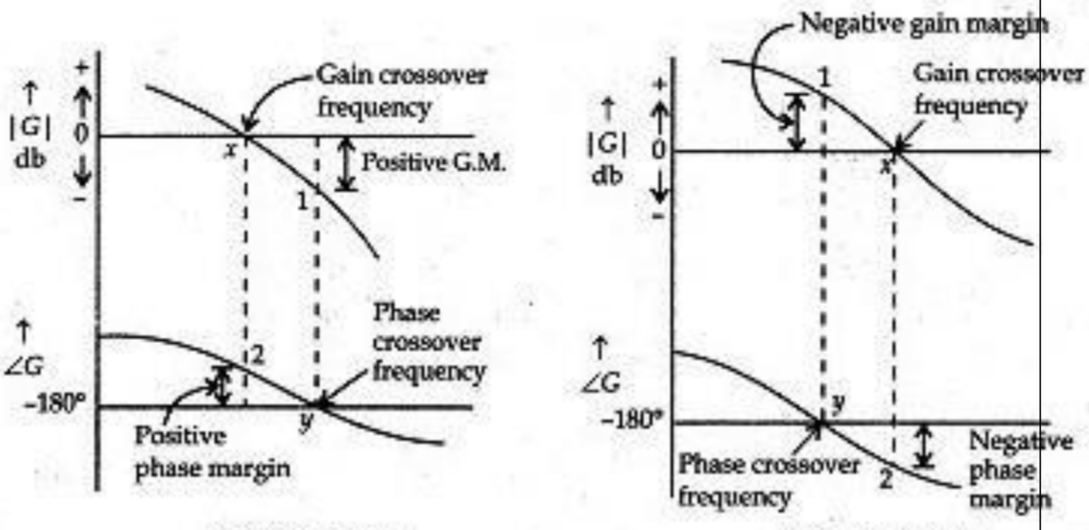




Table

Type of the System N	Initial Slope 0db Axis	Intesection with
0	0 db/decade	Parallel to 0 axis
1	− 20 db/dec.	$= K$
2	− 40 db/dec.	$= \sqrt{K}$
3	− 60 db/dec.	$= K^{1/3}$
⋮	⋮	⋮
N	−20N db/dec.	$K^{1/N}$

PHASE MARGIN & GAIN MARGIN



(a) Stable system (b) Unstable system

Positive gain margin means the system is stable and negative gain margin means the syst unstable. For minimum phase system both phase margin and gain margin must be positive f system to be stable.

The point at which the magnitude curve crosses the 0 db line is the gain crossover frequ The phase crossover frequency is the point where the phase curve crosses the 180° line.

Gain Margin : Gain margin is defined as the margin in gain allowable by which gain c increased till system reaches on the verge of instability. Mathematically gain margin is de as the reciprocal of the magnitude of the $G(j\omega)\, H(j\omega)$ at phase cross over frequency.

$$\therefore \qquad \text{G.M.} = \frac{1}{|G(j\omega)\, H(j\omega)|_{\omega = \omega_{c_2}}}$$

where ω_{c_2} = phase crossover frequency.

Generally, G.M. is expressed in decibels

8.6 NYQUIST PLOTS NYQUISTABILITY CRITERION:

8.6.1 Definition Nyquist Criterion:

Nyquist criterion is a graphical method of determining stability of feedback control systems by using the Nyquist plot of their open-loop transfer functions.

Feedback transfer function:

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)}$$

Poles and zeros of theopen-loop transfer function

$$G(s)H(s) = \frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

$$1 + G(s)H(s) = \frac{(s-p_1)(s-p_2)\ldots(s-p_n) + K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

Number of closed-loop poles - Number of zeros of 1+GH = N umber of open-loop poles

$$1 + G(s)H(s) = \frac{(s-z_{c_1})(s-z_{c_2})(s-z_{c_n})}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

These are also poles of the close-loop transfer function

Magnitude

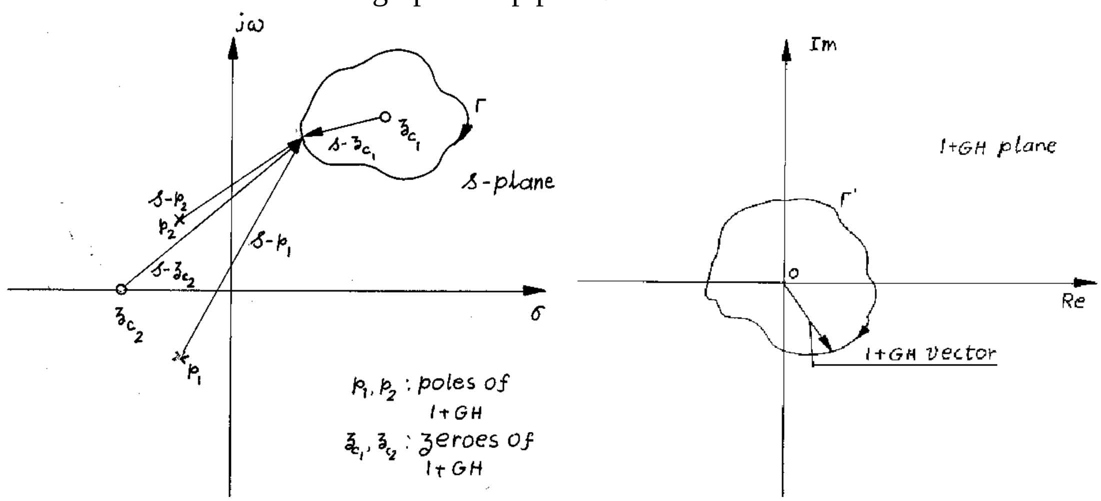
$$|1 + G(s)H(s)| = \frac{|s-z_{c_1}||s-z_{c_2}|\ldots|s-z_{c_n}|}{|(s-p_1)||(s-p_2)|\ldots|(s-p_n)|}.$$

Angle

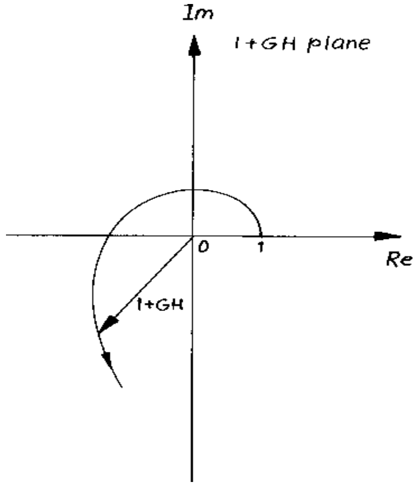
$$\angle 1 + G(s)H(s) = \frac{\angle s-z_{c_1}\angle s-z_{c_2}\angle s-z_{c_n}}{\angle(s-p_1)\angle(s-p_2)\angle(s-p_n)}$$

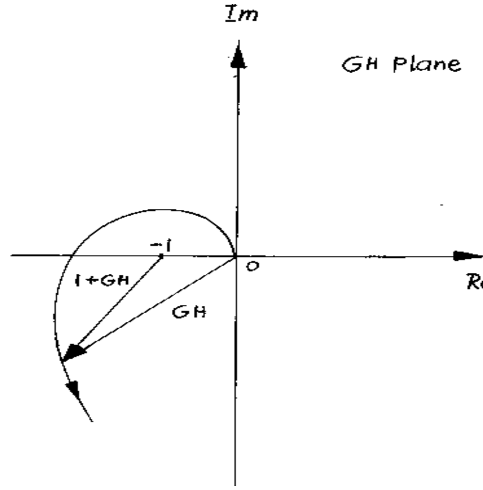
The s-plane to 1+GH plane mappingphase angle of the 1+G(s)H(s) vector, corresponding to a point onthe s-plane is the difference between the sum of the phase of all vectors drawn from zeros of 1+GH(close loop poles) and open loops on the s plane.

If this point s is moved along a closed contour enclosing any or all of theabove zeros and poles, only the phase of the vector of each of the enclosed zeros or open-loop poles will change by 360^0. The directionwill be in the same sense of the contour enclosing zeros and in the opposite sense for the contour enclosing open-loop poles.

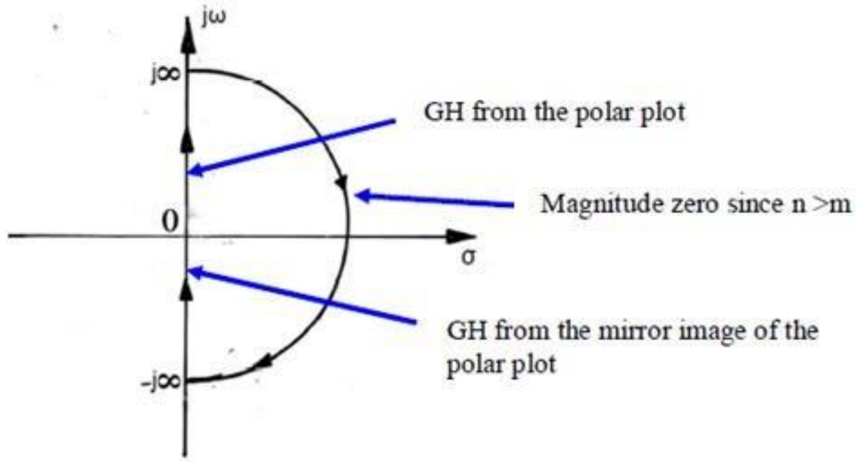


8.6.2 Principle of argument:

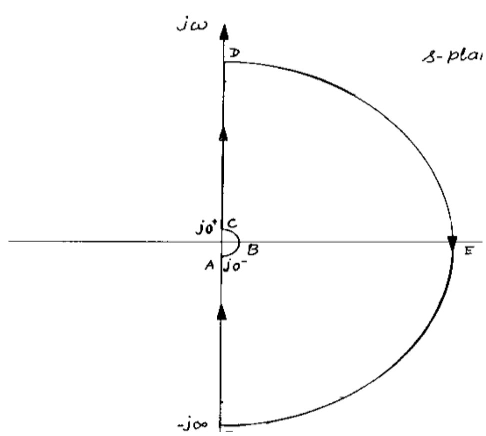




When a closed contour in the s-plane encloses a certain number of poles and zeros of 1+G(s)H(s) inthe clockwise direction, the number of encirclements of the origin by the corresponding contour in the G(s)H(s)plane will encircle the point (-1,0) a number of times given by the difference between the number of its zeros and poles of 1+G(s)H(s) it enclosed on the s-plane.



Modified contour on the s-plane for checking the existence of closed-loop poles $s = \varepsilon e^{j\beta}$



Magnitude of GH remains the same along the contour Phase of βchanges from 270 to 90 degrees.

Gain Margin and Phase Margin:

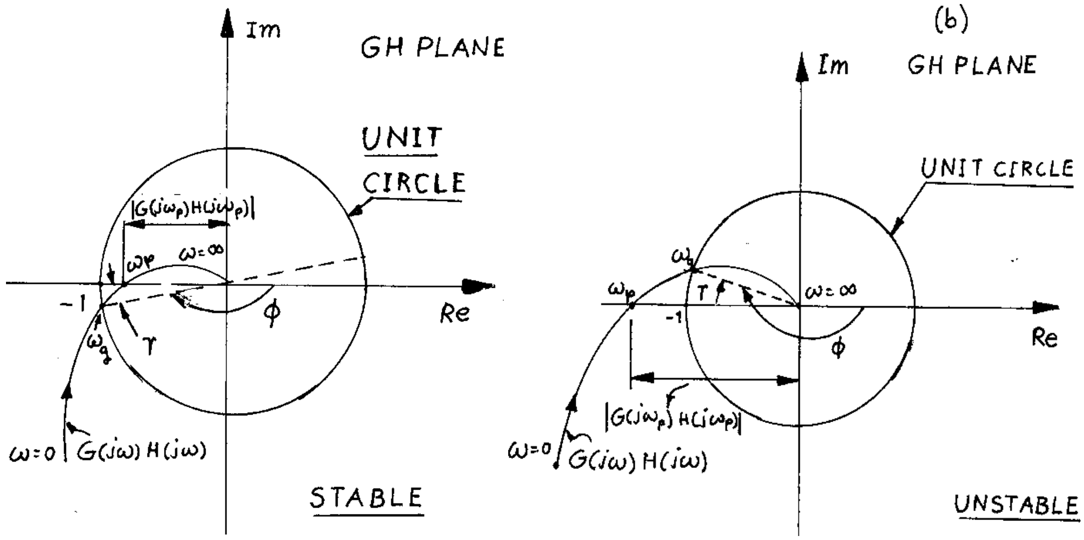
Phase crossover frequency ωp is the frequency at which the open-loop transfer function has a phase of 180^0. The gain crossover frequency ωg is the frequency at which the open-loop transfer function has aunit gain.

Gain margin

$$M = -20\log\left|G(j\omega_p)H(j\omega_p)\right|$$

Phase margin

$$\gamma = \angle G(j\omega_g)H(j\omega_g) - 180^o$$



Procedure:

- Locate open-loop poles on the s-plane
- Draw the closed contour and avoid open-loop poles on the imaginary axis
- Count the number of open-loop poles enclosed in the above contour of step 2, say P
- Plot G(jω)H(jω) and its reflection on the GH plane and map part of the small semi-circle detour on the s-plane around poles (if any) on the imaginary axis.
- Once the entire s-plane contour is mapped on to the GH plane, count the number of encirclements of the point (-1,0) and its direction. Clockwise encirclement is considered positive, say N.
- The number of closed-loop poles in the right-half s-plane is given by Z=N+P. if Z >0, the system is unstable.
- Determine gain margin, phase margin, and critical value of open-loop gain.

Problem 8.6.1:

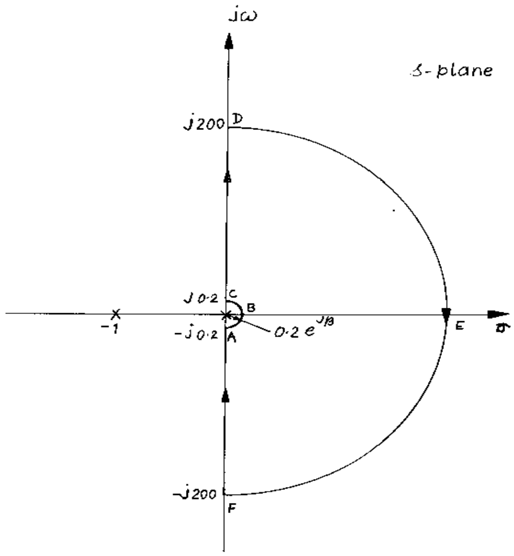
Using Nyquist criterion, determine the stability of a feedback system whose open-loop transfer function is given by G(s)H(s) = $\frac{K}{S(S+1)}$

Solution:

Step 1 Locate open-loop poles on the s-plane. Open-loop poles are at s=0 and –1. Let K=1

Step 2 Draw the closed contour on the s-plane to check the existence of closed-loop poles in the righthalf s-plane.

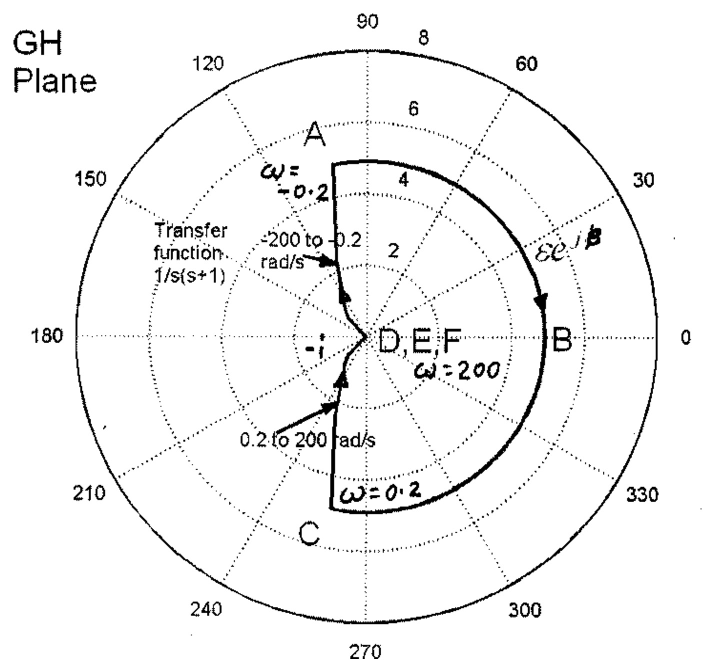
Open-loop poles and s-plane contour



$$\left|G(j\omega)H(\omega)\right| = \frac{1}{\omega\sqrt{1+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\omega$$

No.	Frequency, rad/s		Magnitude	Phase, degrees	β, s-plane, deg	β, GH plane, deg
1	0.2	Positive frequencies	4.9029	259	270	101
2	0.4		2.3212	248	280	91
3	0.8		0.9761	231	290	80
4	1		0.7071	225	300	69
5	4		0.0606	194	310	58
6	10		0.01	186	320	46
7	50		0.0004	181	330	35
8	100		0.0001	181	340	23
9	200		0	180	350	12
10	-200	Negative frequencies	0	180	0	0
11	-100		0.0001	179	10	348
12	-50		0.0004	179	20	337
13	-10		0.01	174	30	325
14	-4		0.0606	166	40	314
15	-1		0.7071	135	50	302
16	-0.8		0.9761	129	60	291
17	-0.4		2.3212	112	70	280
18	-0.2		4.9029	101	80	269



The above system is stable. Here, phase crossover frequency is very large (infinity) and gain crossover frequency 0.786 rad/s. Phase angle corresponding to gain crossover frequency = 232^0 and Phase margin is 52^0.

Problem 8.6.2:

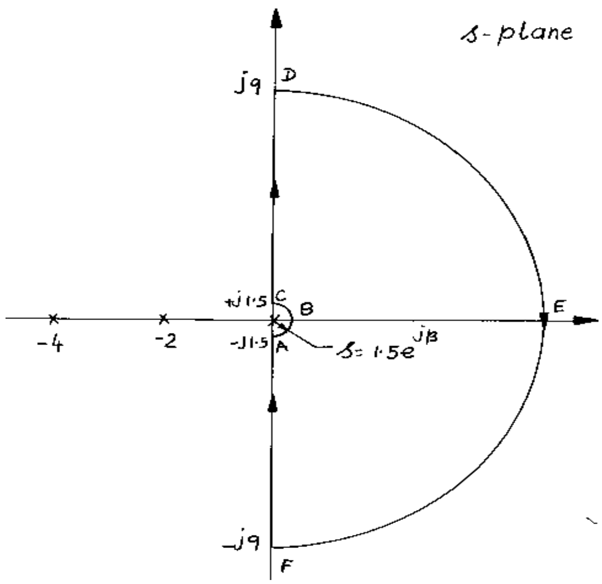
Using Nyquist criterion, determine the stability of a feedback system whose open-loop transfer function is given byG(s)H(s) $\frac{55}{S(S+2)(S+4)}$

Solution:

Step 1 Locate open-loop poles on the s-plane. Open-loop poles are at s=0, -2 and –4. Let K=1

Step 2 Draw the closed contour on the s-plane to check the existence of closed-loop poles in the right half s-plane.

Open-loop poles and s-plane contour

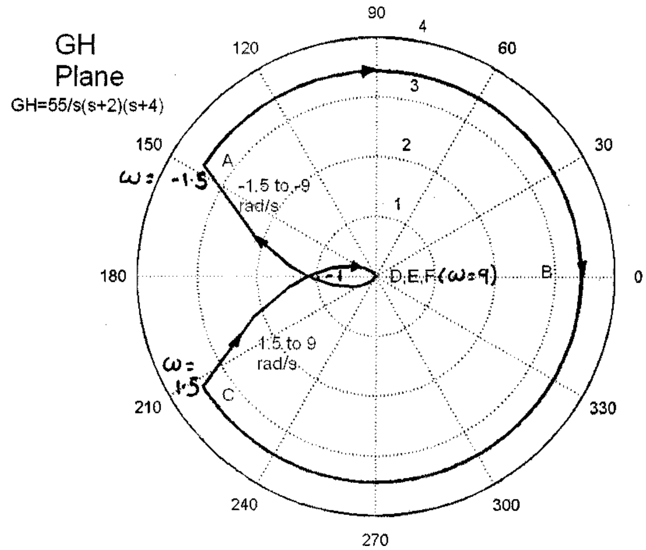


The number of open-loop pole enclosed, P is zero

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{\omega^2+4}\sqrt{\omega^2+16}}$$

$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$$

No.	Frequency		Magnitude	Phase, degrees	β, s-plane, deg
1	1.5	Positive frequencies	3.4332	213	270
2	2		2.1741	198	280
3	2.5		1.4568	187	290
4	2.83		1.1446	180	300
5	3		1.017	177	310
6	3.5		0.7334	169	320
7	4.5		0.4122	156	330
8	5		0.319	150	340
9	5.5		0.2513	146	350
10	6		0.201	142	0
11	7		0.1339	136	10
12	8		0.0932	131	20
13	9		0.0673	126	30
14	-9	Negative frequencies	0.0673	234	40
15	-8		0.0932	229	50
16	-7		0.1339	224	60
17	-6		0.201	218	70
18	-5.5		0.2513	214	80
19	-5		0.319	210	90
20	-4.5		0.4122	204	0
21	-3.5		0.7334	191	343
22	-3		1.017	183	326
23	-2.83		1.1446	180	309
24	-2.5		1.4568	173	292
25	-2		2.1741	162	276
26	-1.5		3.4332	147	259



Here, Z=N+P=2.
Hence, the above system is unstable.
Again,
Phase crossover frequency 2.83 rad/s
The gain at which the system becomes marginally stable, K*= 55 /1.1446 = 48
Gain margin

$$M = -20\log\left|G(j\omega_p)H(j\omega_p)\right|$$
$$= -20\log\left|1.1446\right| = -1.17\,\text{dB}$$

Gain crossover frequency =3 rad/s and the corresponding angle of GH=177^0
Phase margin=177-180=-3^0

OR NYQUIST PLOTS NYQUISTABILITY CRITERION

NYQUIST CRITERION

The characteristic equation is given by

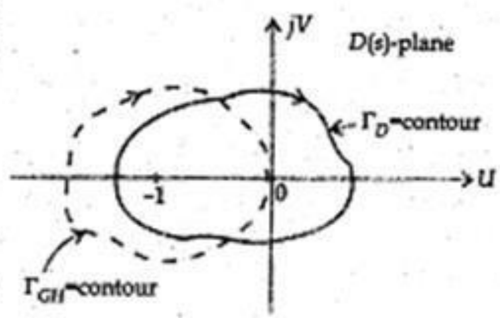
$$D(s) = 1 + G(s)\,H(s)$$

The zeros of $D(s)$ are the roots of the characteristic equation. For a feedback system the necessary and sufficient condition is that all zeros of $1 + G(s)\,H(s)$ that is the roots of the characteristic equation must have negative real part *i.e.,* they must lie in the left half of s-plane. In order to determine the presence of zeros in right half of s-plane we choose a contour as shown in Fig. called Nyquist contour. Let there are 'Z' zeros and 'P' poles in the right half of s-plane . If this contour is mapped in $D(s)$ plane as Γ_D then Γ_D enclosesthe origin N times (where $N = Z\text{-}P$) in clockwise. Hence the system is unstable because the clockwise encirclement is possible only when there are zeros of $D(s)$ in right half of s-plane.

A feedback system (close loop system) is stable if and only if there is no zeros of $D(s)$ in the right half of s-plane. *i.e.* $Z = 0$

$$\therefore \quad N = -P$$

Therefore, for a closed loop system to be stable, the number of counter clockwise encirclement of the origin of $D(s)$ plane by Γ_D should equal the number of right half s-plane poles of $D(s)$ which are the poles of open loop transfer function $G(s)\,H(s)$.



Since $\quad D(s) = 1 + G(s)\,H(s)$

or $\quad G(s)\,H(s) = D(s) - 1$

The contour Γ_D in $D(s)$ plane can be mapped in $G(s)$ $H(s)$ plane. Γ_{GH} by shifting horizontally to the left by one unit. Thus the encirclement of the origin by the contour Γ_D is equivalent to the encirclement of the point $(-1+j0)$ by the contour Γ_{GH} as shown in Fig.

In most single loop feedback system $G(s)\,H(s)$ has no poles in the right half plane *i.e,* $P = 0$ then closed loop system is stable if $N = P = 0$.

So, we can say that A closed loop system with $P = 0$ is stable if the net encirclement of the origin of $D(s)$ plane by Γ_D contour is zero.

Now, we can state the Nyquist stability criterion as follows:

A feedback system or closed loop system is stable if the contour Γ_{GH} of the open loop transfer function $G(s)\,H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1 + j0)$ in counterclockwise direction and the number of counterclockwise encirclements about the $(-1 + j0)$ equals the number of poles of $G(s)\,H(s)$ in the right half of s-plane *i.e.,* with positive real parts.

In common case of open loop stable system, the closed loop system is stable if the contour Γ_{GH} of $G(s)\,H(s)$ does not pass through or does not encircle $(-1 + j0)$ point, *i.e.,* net encirclement is zero.

GENERAL CONSTRUCTION RULES OF THE NYQUIST PATH

Consider the Fig.

Table

Path ab	$s = j\omega$	$0 < \omega < \omega_o$
Path bc	$s = \lim_{P \to 0} (j\omega_o + Pe^{j\theta})$	$-90° \le \theta \le 90°$
Path cd	$s = j\omega$	$\omega_o \le \omega \le \infty$
Path def	$s = \lim_{R \to \infty} Re^{j\theta}$	$-90° \le \theta \le 90°$
Path fg	$s = j\omega$	$-\infty < \omega < -\omega_o$
Path gh	$s = \lim_{P \to 0} (j\omega_o + Pe^{j\theta})$	$-90° \le \theta \le 90°$
Path hi	$s = j\omega$	$-\omega_o \le \omega \le 0$
Path ija	$s = \lim_{P \to 0} Pe^{j\theta}$	$-90° \le \theta \le 90°$

Step 1 : Check $G(s)$ for poles on $j\omega$ axis and at the origin.

Step 2 : Using equation to equation sketch the image of the path $a - d$ in the $G(s)$-plane. If there are no poles on $j\omega$ axis equation need not be employed.

Step 3 : Draw the mirror image about the real axis of the sketch resulting fro step 2.

Step 4 : Use equation plot the image of path def. This path at infinity usually plot into a point in the $G(s)$ -plane.

Step 5 : Use equation plot the image of path *ija* (pole at origin)

Step 6 : Connect all curves drawn into the previous steps.

PROBLEM: **Determine the closed loop stability of a control system whose open lo transfer function is**

$$G(s)\,H(s) = \frac{K}{s(1+sT)} \qquad \text{(Type '1' syste}$$

Solution : Given that

$$G(s)\,H(s) = \frac{K}{s(1+sT)}$$

Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{j\omega(1+j\omega T)}$$

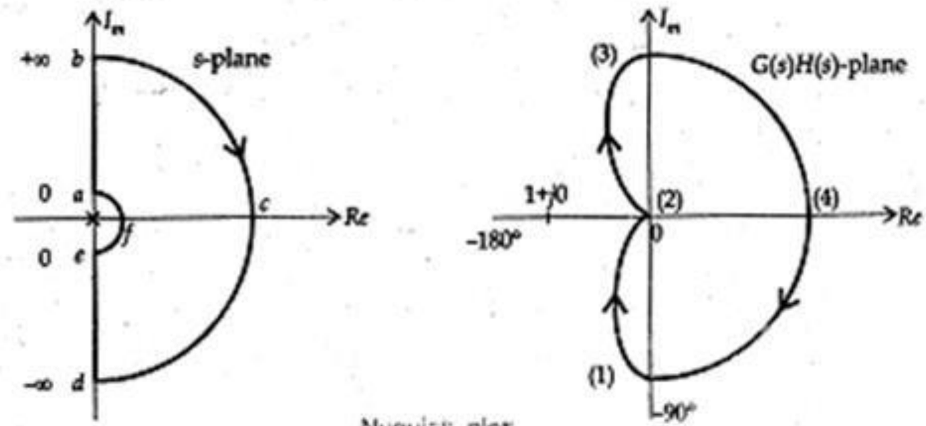
Rationalizing the equation and separating into real and imaginary parts.

$$G(j\omega)\,H(j\omega) = -\frac{KT}{1+\omega^2T^2} - j\frac{K}{\omega(1+\omega^2T^2)}$$

$$\lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega)H(j\omega) = -90°$$

$$\lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0$$



Nyquist plot

The polar plot will lie in third quadrant.

The Nyquist plot is shown in Fig. The part for $0 < \omega < +\infty$ is drawn (1) (2) and for $-\infty < \omega < 0$ is shown by the point (2), (3) which is the mirror image of (1), (2). The semicircular det around the origin in s-plane is mapped into a semicircular path of infinite radius representin change of phase from $+\pi/2$ to $-\pi/2$.

As the point $(-1 + jo)$ is not encircled by the plot, $N = 0$

$$N = 0 \qquad P = 0$$

$$\because \quad N = Z - P \qquad \therefore \quad Z = 0$$

The number of zeros or roots of the characteristic equation with positive real part is nil hence the closed loop system is stable.

EXAMPLE - **Sketch the Nyquist plot and determine the stability of a unity feedback control system.**

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)} \qquad \text{(Type O system)}$$

Solution : Given that :

$$G(s)\,H(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$|G(j\omega)\,H(j\omega)| = \frac{K}{\sqrt{1+\omega^2 T^2{}_1}\,\sqrt{1+\omega^2 T_2{}^2}}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

$$\lim_{\omega\to 0} |G(j(\omega)H(j\omega)| = \text{K}$$

$$\lim_{\omega\to 0} \angle G(j\omega)H(j\omega) = 0$$

$$\lim_{\omega\to\infty} |G(j\omega)H(j\omega)| = 0$$

$$\lim_{\omega\to\infty} \angle G(j\omega)H(j\omega) = -180^\circ$$

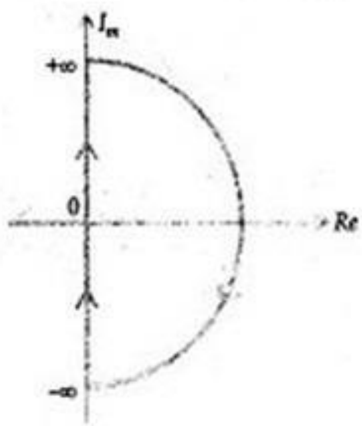
Rationalize the equation and separate the real and imaginary parts.

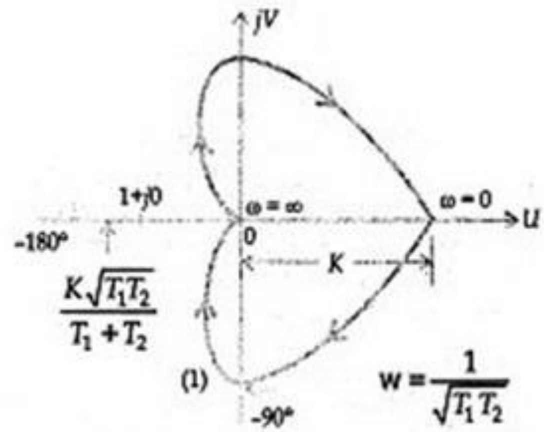
$$\frac{K}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{K(1-\omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} - j\frac{\omega(T_1+T_2)K}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$$

Equate the real part to zero, we get

$$\omega = \frac{1}{\sqrt{T_1 T_2}}$$

$$|G(j\omega)H(j\omega)|_{\omega=\frac{1}{\sqrt{T_1T_2}}} = \frac{KT_1T_2}{T_1+T_2}$$





The plot of $G(j\omega)\,H(j\omega)$ is shown in Fig. The infinite semicircular arc of the Nyquist contour maps into origin. As the point $(-1 + j0)$ is not encircled by the plot

∴ $N = 0$

$P = 0$

∴ $Z = 0$

Hence, the system is stable.

EXAMPLE - Using Nyquist criterion, determine the stability of the feedback system which has the following open loop transfer function.

$$G(s)\,H(s) = \frac{K}{s^2(1+sT)}$$ *(Type '2' system)*

Solution : Given that

$$G(s)\,H(s) = \frac{K}{s^2(1+sT)}$$

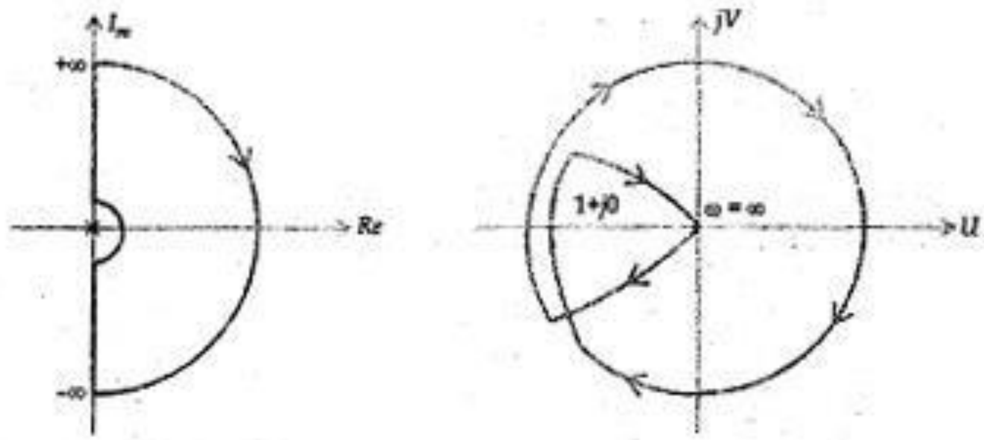
Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T)}$$

Rationalizing the equation (5.39) and separating the real and imaginary part

$$G(j\omega)\,H(j\omega) = \frac{K}{-\omega^2(1+\omega^2T^2)} + j\frac{K}{\omega(1+\omega^2T^2)}$$

The Nyquist diagram is shown in the Fig. Because of the double pole at $s = 0$, a small semicircular detour at the origin should be made.



The point $(-1 + j0)$ is encircled twice. Hence $N = 2$

$$P = 0$$

$$\therefore \quad Z = 2$$

Hence, the system is unstable.

EXAMPLE - Use Nyquist criterion, determine whether the closed loop system having following open loop transfer function is stable or not.

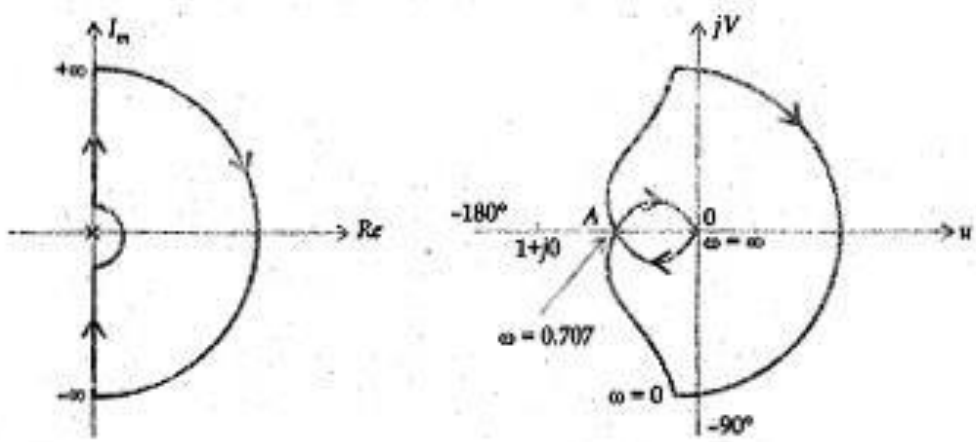
$$G(s)\,H(s) = \frac{1}{s\,(1+2s)\,(1+s)}$$

Solution : Given that

$$G(s)\,H(s) = \frac{1}{s\,(1+2s)\,(1+s)}$$

Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{1}{j\omega(1+j2\omega)\,(1+j\omega)}$$



Rationalizing the equation and separate the real and imaginary part.

$$G(j\omega)\,H(j\omega) = \frac{-3}{(1+4\omega^2)(1+\omega^2)} - j\,\frac{1-2\omega^2}{\omega(1+4\omega^2)(1+\omega^2)}$$

SHORT QUESTIONS WITH ANSWER

Q1.What do you mean by Polar Plot ?

Ans: The Polar plot is a plot, which can be drawn between the magnitude and the phase angle of G (jω) H (jω) by varying ω from zero to infinite

Q2. What do you mean by Bode Plot ?

Ans: A Bode plot is a graph of the frequency response of a system. It is usually a combination of a Bode magnitude plot, expressing the magnitude (usually in decibels) of the frequency response, and a Bode phase plot, expressing the phase shift.

Q3. What do you mean by Nyquist stability criterion ?

Ans : Nyquist stability criterion states the number of encirclements about the critical point (1+j0) must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the 's' plane.

LONG QUESTIONS

Q1. State the rules for plotting a polar plot?

Q2. Write down the procedure to draw a bode plot ?

Q3. What is the need of a Nyquist plot , Explain with a suitable example

CHAPTER - 9

STATE VARIABLE ANALYSIS

9.1 STATE SPACE ANALYSIS:

State space analysis is an excellent method for the design and analysis of control systems. The conventional and old method for the design and analysis of control systems is the transfer function method. The transfer function method for design and analysis had many drawbacks.

Advantages of state variable analysis:

- It can be applied to non linear system.
- It can be applied to tile invariant systems.
- It can be applied to multiple input multiple output systems.
- Its gives idea about the internal state of the system.

9.1 CONCEPT OF STATE:

State:The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time $t=t_0$ (Initial condition), together with the knowledge of input for $\geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.

State vector:If n state variables are needed to completely describe the behaviour of a given system, then these n state variables can be considered the n components of a vector X. Such a vector is called a state vector.

***State space*:** The n-dimensional space whose co-ordinate axes consists of the $x_1$axis,x_2 axis,.... x_n axis, where x_1, x_2,...... $x_n$are state variables: is called a state space.

9.1.2 STATE MODEL:

Lets consider a multi input & multi output system is having

r inputs $u_1\ t$,$u_2\ t$,..... $u_r(t)$

m no of outputs $y_1\ t$,$y_2\ t$, ...$y_m(t)$

n no of state variables $x_1 t$,$x_2 t$,........ $x_n(t)$

Then the state model is given by state & output equation

X t =AX t +BU t .... state equation

Y t =CX t +DU t............ output equation

A is state matrix of size (n×n)

B is the input matrix of size (n×r)

C is the output matrix of size (m×n)

D is the direct transmission matrix of size (m×r)

X(t) is the state vector of size (n×1)

Y(t) is the output vector of size (m×1)

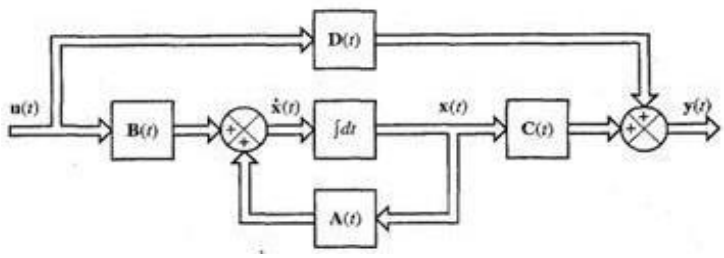
U(t) is the input vector of size (r×1)

BLOCK DIAGRAM OF THE LINEAR, CONTINUOUS TIME FUNCTIONS

The block diagram of the linear, continuous time functions is represented by following equations.

Xt =**AXt** +**ButY**

t =**CXt** +**Dut**



(Block diagram of the linear, continuous time functions)

It has multiple number of input, output and state variables, this has been represented by thick arrow as shown in above fig. Therefore n, parallel integrators must be present as there are n state variables where the output of each integrator is a separate variable.

OR

9.1. CONCEPTS OF STATE, STATE VARIABLE

State Variable

A **state variable** is one of the set of variables that are used to describe the mathematical "state" of a dynamical system. Intuitively, the state of a system describes enough about the system to determine its future behaviour in the absence of any external forces affecting the system. Models that consist of coupled first-order differential equations are said to be in state-variable form

- In mechanical systems, the position coordinates and velocities of mechanical parts are typical state variables; knowing these, it is possible to determine the future state of the objects in the system.
- In thermodynamics, a state variable is an independent variable of a state function like internal energy, enthalpy, and entropy. Examples include temperature, pressure, and volume. Heat and work are not state functions, but process functions.
- In electronic/electrical circuits, the voltages of the nodes and the currents through components in the circuit are usually the state variables. In any electrical circuit, the number of state variables are equal to the number of storage elements, which are inductors and capacitors. The state variable for an inductor is the current through the inductor, while that for a capacitor is the voltage across the capacitor.

The **state space model** of Linear Time-Invariant (LTI) system can be represented as,

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The first and the second equations are known as state equation and output equation respectively.

Where,

- X and $\dot{X}$ are the state vector and the differential state vector respectively.
- U and Y are input vector and output vector respectively.
- A is the system matrix.
- B and C are the input and the output matrices.
- D is the feed-forward matrix.

Basic Concepts of State Space Model

The following basic terminology involved in this chapter.

State

It is a group of variables, which summarizes the history of the system in order to predict the future values (outputs).

State Variable

The number of the state variables required is equal to the number of the storage elements present in the system.

Examples − current flowing through inductor, voltage across capacitor

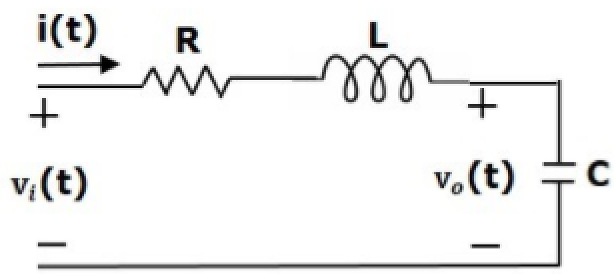
State Vector

It is a vector, which contains the state variables as elements.

In the earlier chapters, we have discussed two mathematical models of the control systems. Those are the differential equation model and the transfer function model. The state space model can be obtained from any one of these two mathematical models. Let us now discuss these two methods one by one.

9.2. STATE MODELS FOR LINEAR CONTINUOUS TIME FUNCTIONS(SIMPLE)

Consider the following series of the RLC circuit. It is having an input voltage, $v_i(t)$ and the current flowing through the circuit is $i(t)$.



There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two and these state variables are the current flowing through the inductor, $i(t)$ and the voltage across capacitor, $v_c(t)$.

From the circuit, the output voltage, $v_0(t)$ is equal to the voltage across capacitor, $v_c(t)$.

$$v_0(t) = v_c(t)$$

Apply KVL around the loop.

$$v_i(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + v_c(t)$$

$$\Rightarrow \frac{\mathrm{d}i(t)}{\mathrm{d}t} = -\frac{Ri(t)}{L} - \frac{v_c(t)}{L} + \frac{v_i(t)}{L}$$

The voltage across the capacitor is -

$$v_c(t) = \frac{1}{C}\int i(t)dt$$

Differentiate the above equation with respect to time.

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \frac{i(t)}{C}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

Differential state vector, $\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix}$

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, C = [0 \quad 1] \; and \; D = [0]$$

OR

STATE SPACE REPRESENTATION OF NTH ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATION

Consider following nth order LTI system relating the output y(t) to the input u(t).

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y^1 + a_n y = u$$

Phase variables: The phase variables are defined as those particular state variables which are obtained from one of the system variables & its (n-1) derivatives. Often the variables used is the system output & the remaining state variables are then derivatives of the output. Let us define the state variables as

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \frac{dx}{dt}$$

$$x_3 = \frac{d\dot{y}}{dt} = \frac{dx_2}{dt}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = y^{n-1} = \frac{dx_{n-1}}{dt}$$

From the above equations we can write

$$\dot{x_1} = x_2$$

$$\dot{x_2} = x_3$$

$$\vdots \quad \vdots$$

$$\dot{x_{n-1}} = x_n$$

$$\dot{x_n} = -a_n x_1 - a_{n-1} x_2 - \cdots \ldots \ldots \ldots - a_1 x_n + u$$

Writing the above state equation in vector matrix form

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$\text{Where } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad A = \begin{bmatrix} 0 & 1 & 0 \ldots \ldots & 0 \\ 0 & 0 & 1 \ldots \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} \ldots \ldots . & -a_1 \end{bmatrix}_{n\times n}$$

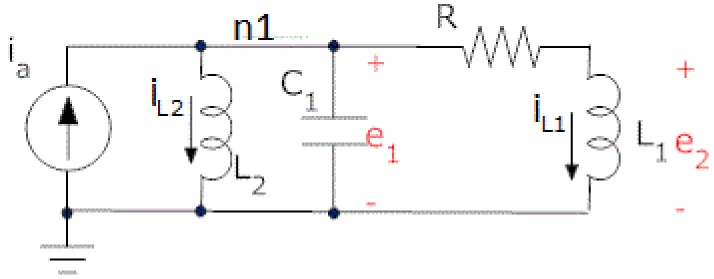
Output equation can be written as

$$Y(t) = CX(t)$$

$$C = [1 \quad 0 \ldots \ldots . \quad 0]_{1\times n}$$

DIRECT DERIVATION OF STATE SPACE MODEL (ELECTRICAL):

Derive a state space model for the system shown. The input is $i_a$and the output ise_1.



There are three energy storage elements, so we expect three state equations. Try choosingi_1, $i_2$and e_1 as state variables. Now we want equations for their derivatives. The voltage across the inductor L_2 is e_1(which is one of our state variables)

$$L_2 \frac{di_{L2}}{dt} = e_1$$

so our first state variable equation is

$$\frac{di_{L2}}{dt} = \frac{1}{L_2} e_1$$

If we sum currents into the node labeled n_1 we get

$$i_a - i_{L2} - i_{C1} - i_{L1} = 0$$

This equation has our input (ia) and two state variable (iL_2 and iL_1) and the current through the capacitor. So from this we can get our second state equation

$$i_{C1} = C_1 \frac{de_1}{dt} = i_a - i_{L2} - i_{L1}$$

$$\frac{de_1}{dt} = \frac{1}{C_1}\left(i_a - i_{L2} - i_{L1}\right)$$

Our third, and final, state equation we get by writing an equation for the voltage across L_1 (which is e_2) in terms of our other state variables

$$e_2 = L_1 \frac{di_{L1}}{dt} = e_1 - Ri_{L1}$$

$$\frac{di_{L1}}{dt} = \frac{1}{L_1}\left(e_1 - Ri_{L1}\right)$$

We also need an output equation:

$$e_2 = e_1 - Ri_{L1}$$

So our state space representation becomes

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} i_{L2} \\ e_2 \\ i_{L1} \end{bmatrix}$$

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u \qquad \mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L_2} & 0 \\ -\frac{1}{C_1} & 0 & -\frac{1}{C_1} \\ 0 & \frac{1}{L_1} & -\frac{R}{L_1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{C_1} \\ 0 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{q} + Du \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & -R \end{bmatrix} \qquad D = 0$$

STATE SPACE TO TRANSFER FUNCTION

Consider the state space system:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{q}(t) + Du(t)$$

Now, take the Laplace Transform (with zero initial conditions since we are finding a transfer function):

$$s\mathbf{Q}(s) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{Q}(s) + DU(s)$$

We want to solve for the ratio of Y(s) to U(s), so we need so remove **Q**(s) from the output equation. We start by solving the state equation for **Q**(s)

$$s\mathbf{Q}(s) - A\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$(s\mathbf{I} - \mathbf{A})\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) = \mathbf{\Phi}(s)\mathbf{B}U(s); \quad \text{where } \mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

The matrix **Φ**(s) is called the state transition matrix. Now we put this into the output equation

$$Y(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B}U(s) + DU(s)$$
$$= (\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + D)U(s)$$

Now we can solve for the transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

EXAMPLE:

Find the transfer function of the system with state space representation

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{q} + Du = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} + 0 \cdot u$$

First find (s**I**-**A**) and the **Φ**=(s**I**-**A**)-1 (note: this calculation is not obvious. Details are here). Rules for inverting a 3x3 matrix are here.

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 4 & s+2 \end{bmatrix}$$

$$\mathbf{\Phi} = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{\begin{bmatrix} s^2+2s+4 & 2+s & 1 \\ -3 & s(2+s) & s \\ -3s & -3-4s & s^2 \end{bmatrix}}{s^3+2s^2+4s+3}$$

Now we can find the transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}\mathbf{\Phi}\mathbf{B} + D$$

$$= \frac{s+5}{s^3+2s^2+4s+3}$$

SHORT QUESTIONS WITH ANSWER

Q1. What do you mean by state variable?

Ans : A state variable is one of the set of variables that are used to describe the mathematical "state" of a dynamical system. Intuitively, the state of a system describes enough about the system to determine its future behaviour in the absence of any external forces affecting the system. Models that consist of coupled first-order differential equations are said to be in state-variable form.

Q2. How the state variable electrical circuit is represented?

Ans:In electronic/electrical circuits, the voltages of the nodes and the currents through components in the circuit are usually the state variables. In any electrical circuit, the number of state variables are equal to the number of storage elements, which are inductors and capacitors. The state variable for an inductor is the current through the inductor, while that for a capacitor is the voltage across the capacitor.

Q3. Write down the state space representation of a LTI system ?

Ans: The state space model of Linear Time-Invariant (LTI) system can be represented as,

$\dot{X}=AX(t)+BU(t)$

$Y=CX(t)+DU(t)$

LONG QUESTION

Q1. Derive the state space representation of a series RLC circuit.

LEARNING RESOURCES:

1. Conrtol Systems by Samarajit Ghosh-Pearson
2. Control Systems by Principles and Design by Gopal. M,-Tata McGraw-Hill
3. Automatic Control System by Kuo, B.C,-Prentice Hall
4. Modern Control Engineering byOgata, K -Prentice Hall
5. Modern Control Engineering by Nagrath& Gopal-New Age International, New Delhi
6. Control System Engg by P Ramesh Babu& R. Anandanatarajan -SCITECH