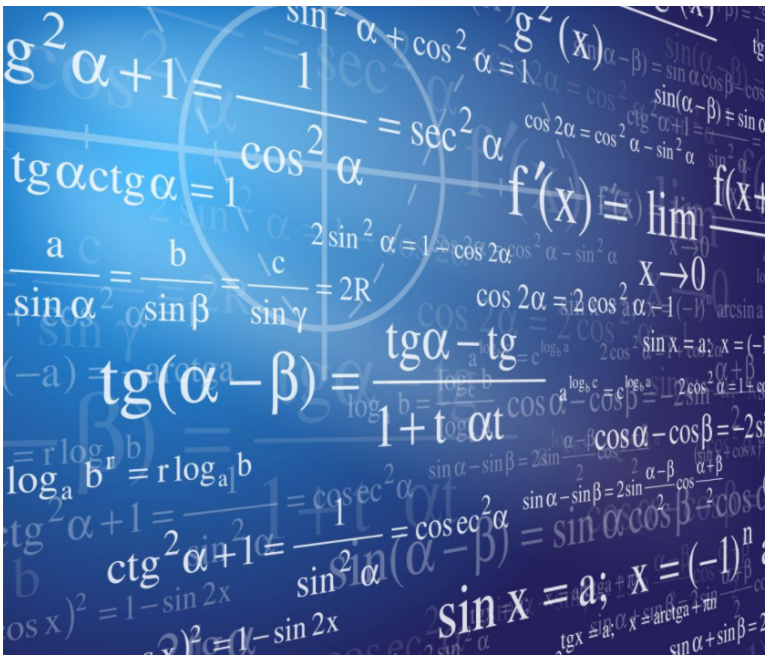
RTD EXAM PREP

Study Guide for the Math 30-1 Diploma Exam



Part 2 (pages 64 – 148)

- Radical and Rational Functions
- Trigonometry
- Permutations, Combinations, and Binomial Theorem

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Radical and Rational Functions

4

Course weighting: 8% Approx. Number of Exam Questions: 3 or 4

4.1 – Radical Functions **4.2** – The Graph of $y = \sqrt{f(x)}$ **4.3** – The Rational Function $y = \dfrac{f(x)}{g(x)}$

4.1 Radical Functions

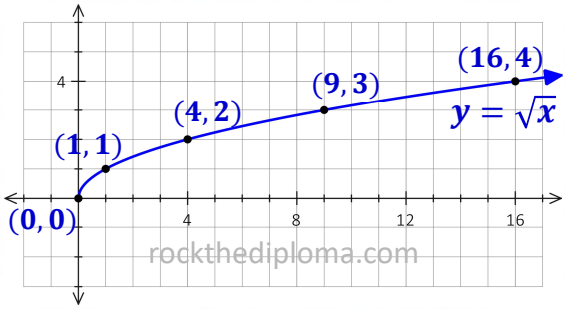
The basic radical function is $y = \sqrt{x}$.

Domain: $\{x \geq 0, x \in \mathbb{R}\}$
Can't square root negatives

Range: $\{y \geq 0, y \in \mathbb{R}\}$
Positive square roots only

For the **domain** of any radical function, we set whatever's under the radical sign as greater than / equal to zero.

For the **range**, consider transformations to the basic graph.



Worked Example Sketch the graph of $y = \sqrt{6 - 2x} - 1$, and state the domain, range, and coordinates of any intercepts. (nearest hundredth where applicable)

Solution:

- For **domain**, set whatever is under the radical sign greater than / equal to zero

$y = \sqrt{6 - 2x} - 1$

$6 - 2x \geq 0$

$6 - \geq 2x$

$3 \geq x$

Domain is $\{x \leq 3, x \in \mathbb{R}\}$

Or in *interval notation:* $(-\infty, 3]$

- For the ***y*-intercept**, substitute x with 0 and evaluate.

$y = \sqrt{6 - 2(0)} - 1$

$y = \sqrt{6} - 1$

$y \approx 1.45$ *approimate*

***y*-intercept is** $(0, 1.45)$

- For the ***x*-intercept**, substitute y with 0 and solve the resulting equation.

$0 = \sqrt{6 - 2x} - 1$

$1 = \sqrt{6 - 2x}$

$(1)^2 = \left(\sqrt{6 - 2x}\right)^2$ *Square both sides*

$1 = 6 - 2x$

$2x = 5$ ➡ $x = 5/2$ ***x*-intercept is** $(2.5, 0)$

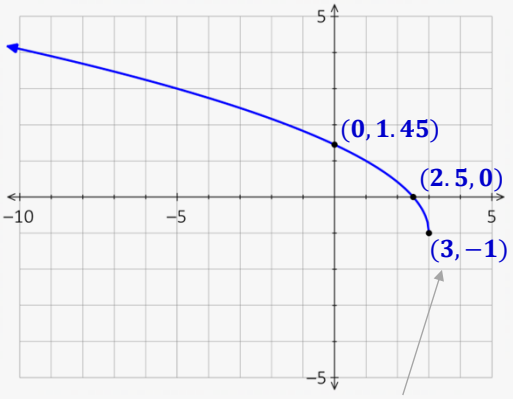
- For **range**, consider the transformations from the basic graph $y = \sqrt{x}$.

First re-write as: $y = \sqrt{-2(x - 3)} - 1$

The horiz. reflection, stretch by factor 0.5 and shift 3 right do NOT affect the range

*Vertical translation (affects range, think of basic graph shifted **1 unit down**)*

Range is $\{y \geq -1, y \in \mathbb{R}\}$ or $[-1, \infty)$



Start point on the basic graph is $(0,0)$
Here it's shifted 3 units right, 1 unit down

$y = \sqrt{-2(x - 3)} - 1$

Horizontal Reflection *(graph opens left instead of right)*

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Class Example 1 *Obtaining graph characteristics from a Radical equation*

Given a function $f(x) = -2\sqrt{3x-7}+5$ use an *algebraic process* to determine:

(a) The x and y intercepts of $f(x)$ *Exact values*

(b) The domain and range of $f(x)$

Class Example 2 *Radical Functions with Transformations*

You Try

Scan to check your work ➔

A function $f(x) = a\sqrt{3x+c}+d$ is transformed to $g(x)$ by applying a horizontal stretch by a factor of 2. Determine an expression representing:

(a) The domain of $g(x)$

(b) The y-intercept of $g(x)$

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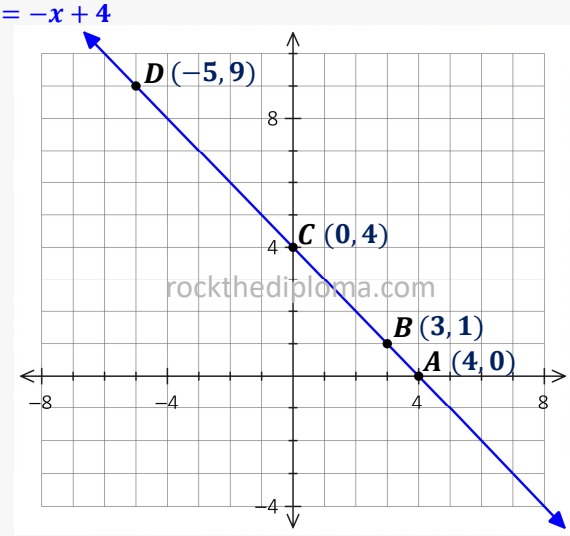
4.2 The Graph of $y = \sqrt{f(x)}$

Transforming $f(x)$ to $y = \sqrt{f(x)}$

Worked Example

Given the graph of $f(x) = -x + 4$, sketch the graph of $y = \sqrt{f(x)}$, and determine the domain and invariant points.



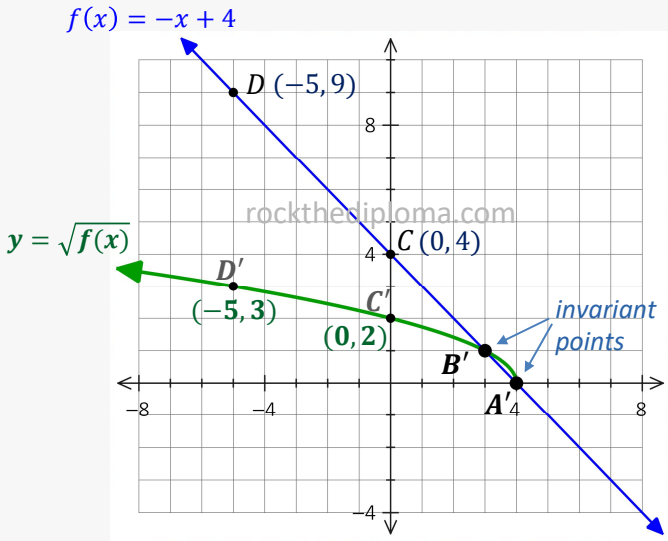
Solution:

Given the graph or equation of a function $y = f(x)$, we can obtain the graph of $y = \sqrt{f(x)}$ by transforming all points $(x, y) \to (x, \sqrt{y})$. ***See points D and C***

$\boldsymbol{D}$: $(-5, 9) \to (-5, 3)$ ($\sqrt{9}$)

$\boldsymbol{C}$: $(0, 4) \to (0, 2)$ ($\sqrt{4}$)

$\sqrt{\ }$ ***the y-coords***



The **domain** of $y = \sqrt{f(x)}$ can be found by considering the zeros / x-intercepts of $y = f(x)$. ***Point A***

Domain of $y = \sqrt{f(x)}$: $\{x \in \mathbb{R}, x \leq 4\}$ (x-intercept of $y = f(x)$)

Since we can't square root negatives, $y = \sqrt{f(x)}$ is defined wherever $f(x) \geq 0$, that is, wherever the graph is above the x-axis. ***To the left of point A***

The **invariant points** in the transformation from $y = f(x)$ to $y = \sqrt{f(x)}$ can be found by considering where the value of $f(x)$ is 0 or 1. ***See points A and B.***

↓ Obtaining characteristics of the graph of $y = \sqrt{f(x)}$ from the *equation* of $f(x)$ ↓

To determine the **domain** of the equation of $y = \sqrt{4 - x}$ ...

Whatever we are square rooting cannot be negative. (That is, it must be ≥ 0)

$$4 - x \geq 0$$
$$-x \geq -4$$
$$x \leq 4$$ ← **Domain**

*When dividing (or multiplying) both sides of an inequality by a negative, **reverse the direction** of the inequality!

Invariant points occur wherever the value of $f(x)$ is 0 or 1. (Since $\sqrt{0} = 0$ and $\sqrt{1} = 1$)

- 1st invariant point is where $f(x) = 0$

$$4 - x = 0$$
$$-x = -4$$
$$x = 4$$

So, coordinates of invariant point is $(4, 0)$

- 2nd invariant point is where $f(x) = 1$...

$$4 - x = 1$$
$$-x = -3$$
$$x = 3$$

So, coordinates of other invariant point is $(3, 1)$

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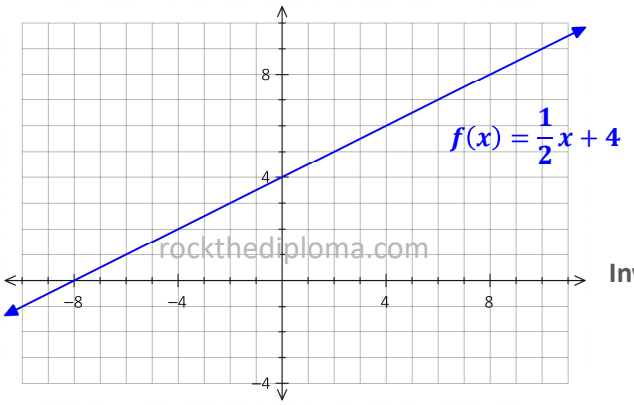
Class Example 3 *Obtaining the graph of $y = \sqrt{f(x)}$ given a graph of $f(x)$*

For the graph of $y = f(x)$ below, sketch the graph of $y = \sqrt{f(x)}$, and state its domain, range, and any invariant points.



Domain of $y = \sqrt{f(x)}$ _____

Range of $y = \sqrt{f(x)}$ _____

y-intercept of $y = \sqrt{f(x)}$ _____

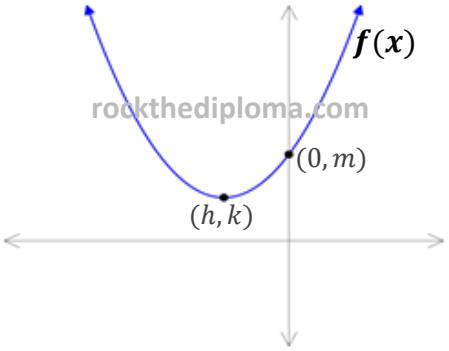
x-intercept of $y = f(x)$ _____

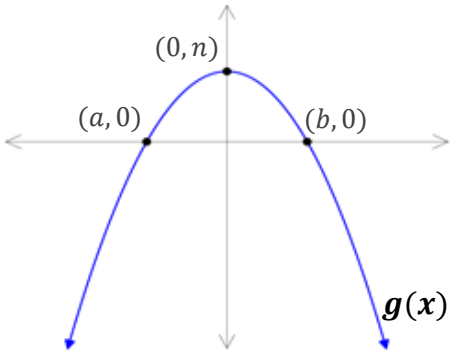
Invariant points: _____

x-intercept of $y = \sqrt{f(x)}$ _____

Class Example 4 *Determining domain, range, and invariant points*

Use the following graphs for each of the questions below





Determine an expression that represents:

(a) The **domain** _____ and **range** _____ of $y = \sqrt{f(x)}$

(b) The **domain** _____ and **range** _____ of $y = \sqrt{g(x)}$

(c) The y-intercept of $y = \sqrt{f(x)}$ _____

(d) The x-intercepts of $y = \sqrt{g(x)}$ _____

If the two functions have equations in the form $f(x) = a(x - h)^2 + k$ and $g(x) = ax^2 + k$, state the number of invariant points on:

(e) The graph of $y = \sqrt{f(x)}$, if $k = 1$ _____

(f) The graph of $y = \sqrt{f(x)}$, if $0 < k < 1$ _____

(g) The graph of $y = \sqrt{f(x)}$, if $k > 1$ _____

(h) The graph of $y = \sqrt{g(x)}$, if $n > 1$ _____

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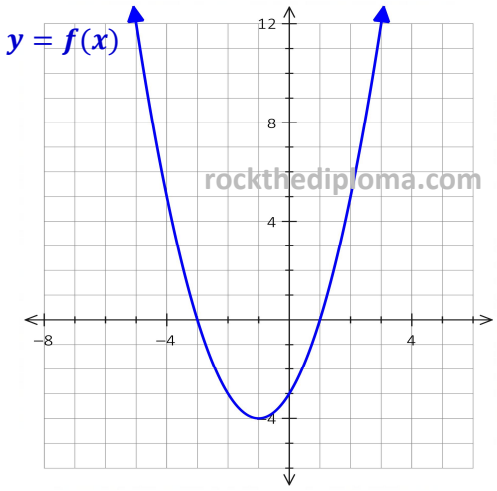
Class Example 5 *Obtaining the graph of $y = \sqrt{f(x)}$ when $f(x)$ is a quadratic*

Written Response *Use the following information to answer the first bullet*

The graph of a quadratic function $y = f(x)$ is shown.

The vertex and intercepts all have integer coordinates.



(a) On the same grid, provide a detailed sketch the graph of $y = \sqrt{f(x)}$. Indicate and label any invariant points, label the coordinates of any x or y intercepts, and state the domain and range of $y = \sqrt{f(x)}$. **[3 marks]**

Use the following information to answer the second bullet.

An equation for $f(x)$ can be written in the form $f(x) = (x - h)^2 + k$.

(b) Using the equation of $f(x)$. Determine the exact value of the coordinates of the invariant points on $y = \sqrt{f(x)}$. **[2 marks]**

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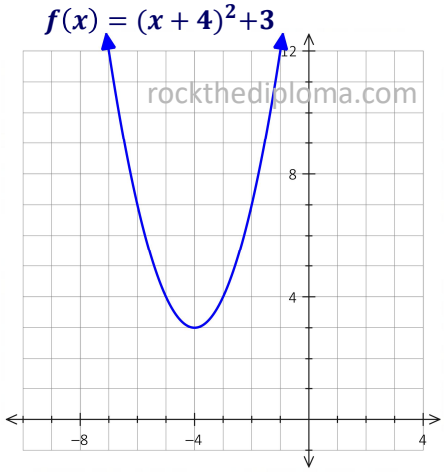
Radical and Rational Function Practice Questions *Part 1*

Radical and Rational Functions Practice Question 1

The graph of a quadratic function $y = f(x)$ is shown on the right.

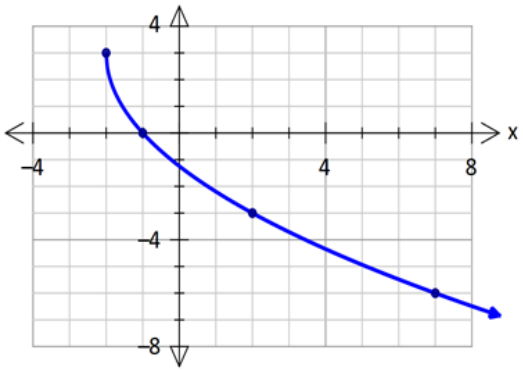
State the domain, range, and y-intercept of $y = \sqrt{f(x)}$. Sketch.



Radical and Rational Functions Practice Question 2

The following radical function has an equation of the form $y = a\sqrt{x-h}+k$. Determine the equation.

Need a hint? See bottom of the next page.



Radical and Rational Functions Practice Question 3

For a radical function $y = -2\sqrt{ax+9}+b$, determine an expression for each of the following:

Domain

Range

y-intercept

x-intercept
(challenge!)

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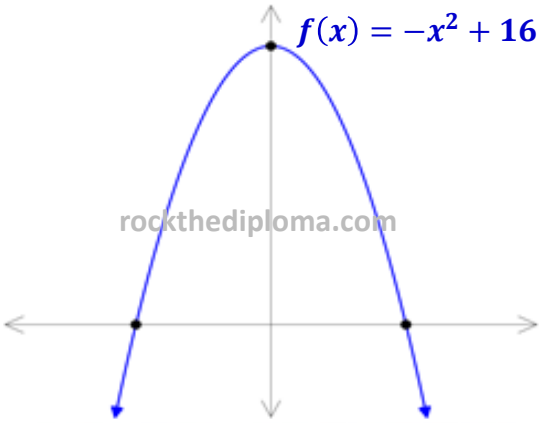
Radical and Rational Functions Practice Question 4



Given the function graph on the left,

(a) Label each axis (provide a scale) to sketch the graph of $y = \sqrt{f(x)}$, and state its domain and range.

(b) Use an algebraic process to determine the coordinates of any invariant points.

Radical and Rational Functions Practice Question 5

$f(x)$ is a quadratic function transformed to $y = \sqrt{f(x)}$. The graph of $f(x)$ is a parabola that opens upward, and the equation has the form $f(x) = ax^2 + k$. Provide a possible equation for $f(x)$ for each scenario:

(a) *There are no invariant points* **(b)** *There is exactly one invariant point* **(c)** *There are exactly two invariant points* **(d)** *There are exactly three invariant points* **(e)** *There are exactly four invariant points*

Radical and Rational Functions Practice Question 6

Given a function $f(x) = (x-1)^2 - 9$, sketch the graph of $y = \sqrt{f(x)}$ and state the following characteristics:

Domain

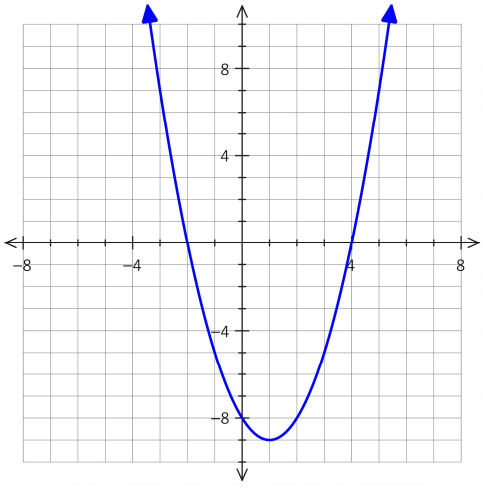
Range

y-intercept

Coord's of pt on graph of $y = \sqrt{f(x)}$ when $x = 6$.

x-intercept(s)

Coordiniates of Invariant Points



Hint for Practice question 2: This question relates to transformations. Compare to the basic graph of $y = \sqrt{x}$. (on page 49) The graph here "starts" at $(-2, 3)$, which are your h and k values in $y = a\sqrt{x-h} + k$. Then, use any other point on the graph, such as $(-1, 0)$ to solve for "a".

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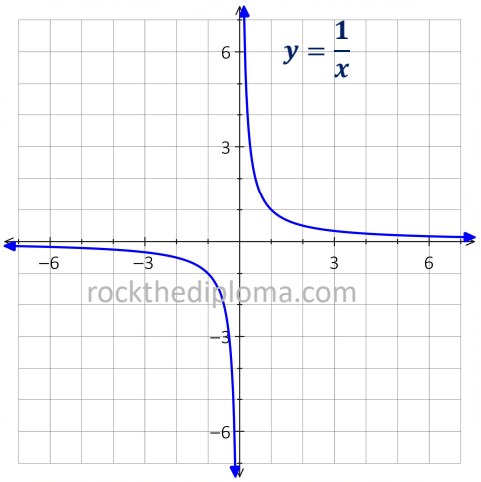
4.3 Rational Functions

A rational function has the form $y = \frac{f(x)}{g(x)}$, where f and g are polynomials. $g(x) \neq 0$

i – The graph of $y = \frac{1}{x}$



The Basic Rational Function

$\{x \in \mathbb{R} | \, x \neq 0\}$ — Domain

$\{y \in \mathbb{R} | \, y \neq 0\}$ — Range

n/a — x-intercept

n/a — y-intercept

Horiz. Asymptote **(H.A.) at $y = 0$**

Vert. Asymptote **(V.A.) at $x = 0$**

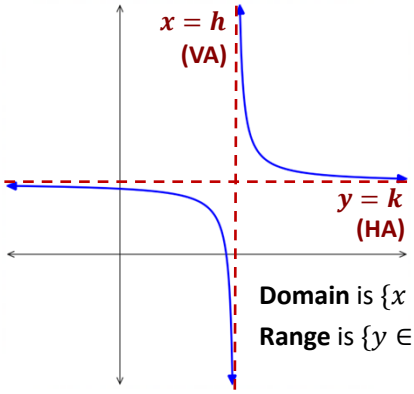
Asymptotes

ii – The graph of $y = \frac{a}{x-h} + k$

For a rational function in this form, consider the horizontal and vertical translations to the basic graph shown above, $f(x) = \frac{1}{x}$

$y = \frac{a}{x-h} + k$ ← H.A. shits k units up / down

V.A. shits h units left / right



Domain is $\{x \in \mathbb{R}, x \neq h\}$

Range is $\{y \in \mathbb{R}, y \neq k\}$

Worked Example Identify the key characteristics for the graph of $y = \frac{4}{x+2} + 3$

Solution: $y = \frac{4}{x+2} + 3$ ← H.A. shits 3 units up ➔ **H.A. at $y = 3$**

V.A. shits 2 units left ➔ **V.A. at $x = -2$**

For **domain**, consider non-permissible values (NPVs) …. *Can't divide by 0* ➔ **Dom. is** $\{x \in \mathbb{R}, x \neq -2\}$

For **range of a function in this form,** consider the H.A. … *Later we'll encounter rational function graphs that cross their H.A.s, but not here!* ➔ **Range is** $\{y \in \mathbb{R}, y \neq 3\}$

For x-int, set $y = 0$ & solve:

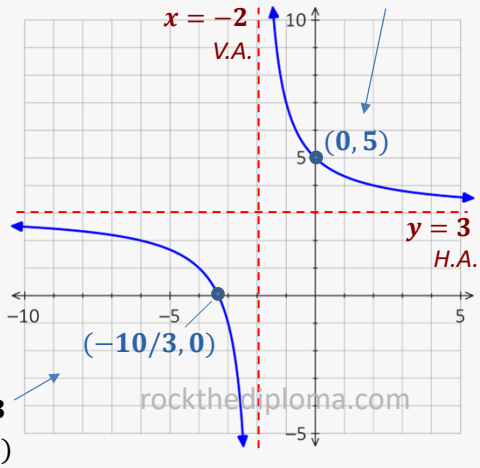
$0 = \frac{4}{x+2} + 3$ ➔ $\frac{4}{x+2} = -3$ ➔ $-3(x+2) = 4$ ➔ $-3x = 10$ ➔ $x = -10/3$

➔ **x-intercept at $(10/3, 0)$**

For y-int, set $x = 0$ and evaluate: $y = \frac{4}{(0)+2} + 3$

➔ **y-intercept at $(0, 5)$** $= 2 + 3$

$= 5$



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iii – The graph of $y = \frac{f(x)}{g(x)}$; f and g are linear functions

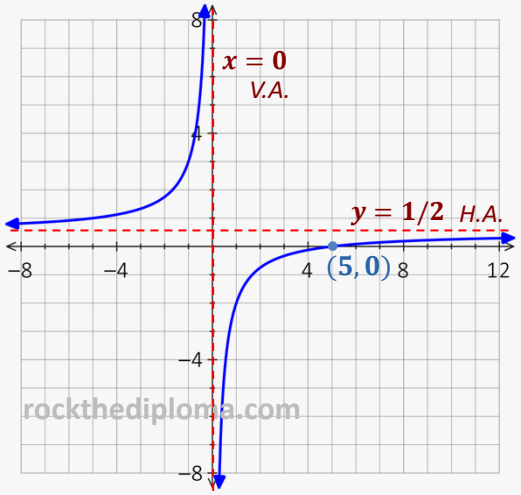
Worked Example Given $y = \frac{x-5}{2x}$ Sketch the graph, state the domain and range, sketch / state the equation of any asymptotes, and plot / state the coordinates of any x or y intercepts

Solution: Functions in this form can be thought of as transformations of the basic graph, $y = \frac{1}{x}$. (graph has the same shape)

$y = \frac{x-5}{2x}$ ← ➔ *x-int at* $x = 5$, *that is at* $(5, 0)$ *(Value of x that makes the "top" 0)*

➔ *V.A. at* $x = 0$ *(Value of x that makes the "bottom" 0)*

➔ *H.A. at* $y = \frac{1}{2}$ *(Same degree to & bottom, so H.A. is at* ***ratio of the lead coefficients****)* ➔ $y = \frac{1x-5}{2x}$



➔ **Domain** $\{x \in \mathbb{R},\ x \neq 0\}$ and ➔ **range** $\{y \in \mathbb{R},\ y \neq 1/2\}$

➔ No y**-int** (since V.A. at $x = 0$... this can be confirmed by showing there is no solution to $0 = \frac{x-5}{2x}$)

When the numerator / denominator of a rational function are both **linear**, then

- There is at most one discontinuity on the graph (Either a Vertical Asymptote, or Point of Discontinuity)
- There is at most one x-intercept
- The graph will never cross any Horizontal Asymptote, so if there is one, the range is $\{y \in \mathbb{R} | y \neq$ H.A. $\}$

Or, if applicable, the y-coord of the Point Discontinuity

Class Example 6 *Characteristics of a rational function from the equation – numerator / denom. are linear*

For the function $y = \frac{-2x-4}{x-3}$, state the indicated characteristics, and then sketch the graph.
Label all graph characteristics.

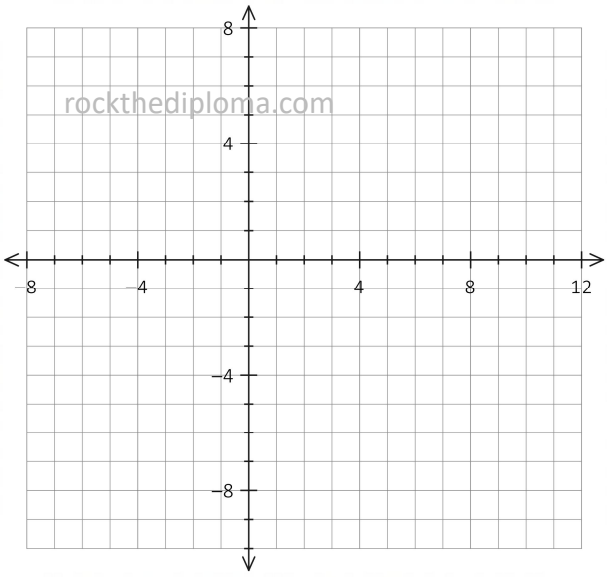
Asymptotes

Domain

Range

x-intercept

y-intercept



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iv – The graph of $y = \frac{f(x)}{g(x)}$; g is a quadratic function

Worked Example For $y = \frac{2}{x^2 - x - 6}$, graph and state domain, any asymptotes, and coordinates of any intercepts.

Solution:

Start by factoring the denominator, to determine any NPVs.
(Non-permissible values…. can't divide by zero)

$$y = \frac{2}{(x-3)(x+2)}$$

→ **No x-int** as there is no x-term in the numerator ("top" can never be 0)

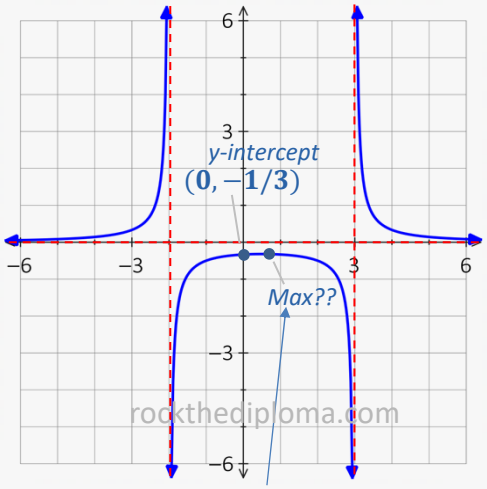
→ *V.A.s at* $x = 3$, and $x = -2$ → **Domain** is: $\{x \in \mathbb{R},\ x \neq -2,\ 3\}$
(Values of x that makes the "bottom" 0)

→ *H.A. at* $y = 0$ *(since higher degree is on the bottom)*

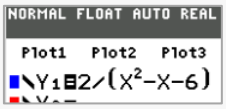
→ **y-int** is $(0, -1/3)$ ← $y = \frac{2}{(0-3)(0+2)}$

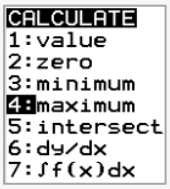
→ $= \frac{2}{-6}$

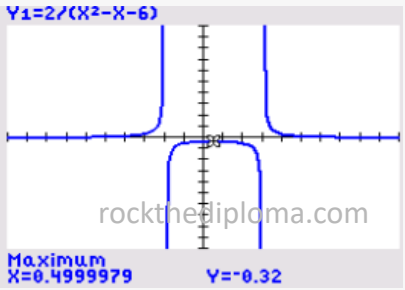
→ **Range** is $\{y | y \leq -0.32$ *or* $y > 0, y \in \mathbb{R}\}$
(y-coord of local max; Horiz. Asym)



Note: For Range, we need to use our graphing calc to determine the y-coord of the local max







Note: This is actually 0.5 (calc glitch!)

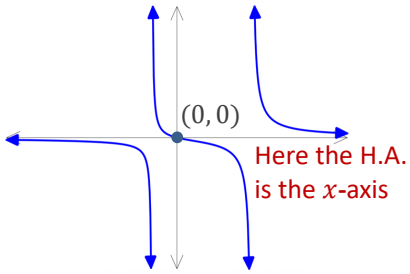
Horizontal Asymptotes

We've seen how Vertical Asymptotes (VAs) can occur at non-permissible values (NPVs) of x.

Horizontal Asymptotes (HAs) occur when the values of the function (the y-coords) approach some fixed value as the x-coords get larger and larger. (In either the pos. or neg. direction)

Case 1: Higher Degree - Denominator *"Higher Degree on bottom"*

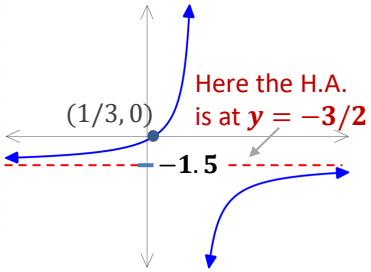
Example: $y = \frac{x}{(x+1)(x-3)}$



→ **H.A.** at $y = 0$

Case 2: Same Degree in Numerator / Denominator

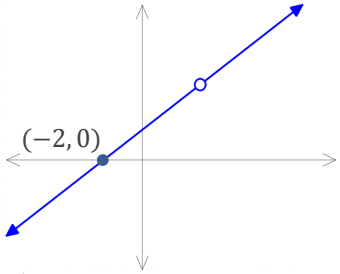
Example: $y = \frac{-3x+1}{2x-5}$



→ **H.A.** at $y =$ *Ratio of Lead Coefficients*

Case 3: Higher Degree - Numerator *"Lower degree on bottom"*

Example: $y = \frac{(x+2)(x-3)}{x-3}$



→ There is no **H.A.**

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Class Example 7 *Characteristics of a rational function from the equation –denom. are quadratic*

For the function $y = \dfrac{x-2}{x^2-2x-8}$, state the indicated characteristics.

Domain

V.A.

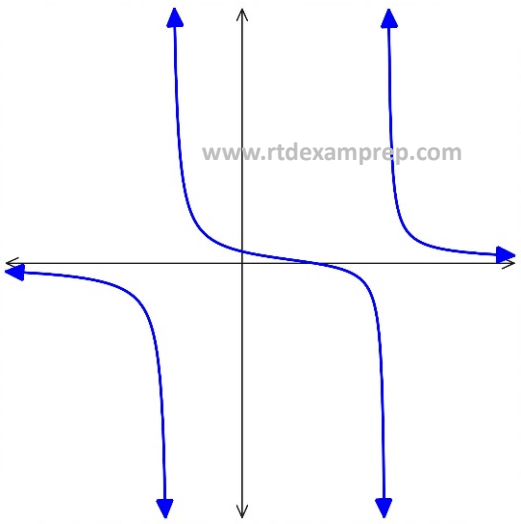
H.A.

Range

x-intercept

y-intercept

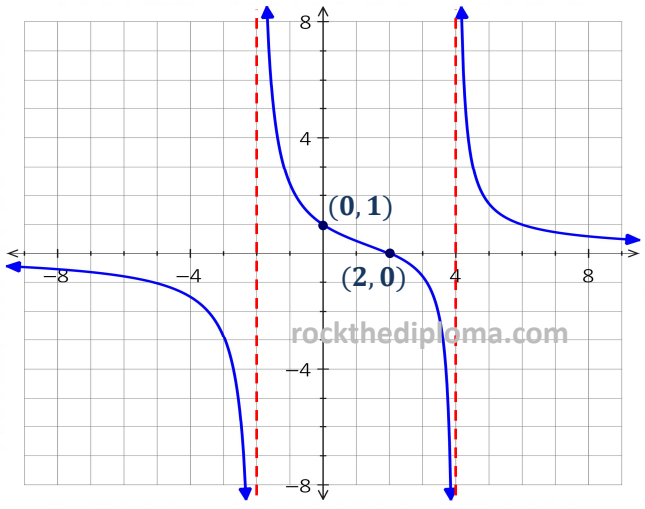
Sketch / label equation of any asymptotes, and plot / indicate coordinates of any intercepts.



Class Example 8 *Obtaining a Rational Function Equation from the Graph*

Diploma Example

The graph of the function below can be expressed in the form: $y = \dfrac{ax-b}{x^2-2x-c}$.



NR The values of a, b and c are, respectively, ____ , ____ , and ____.

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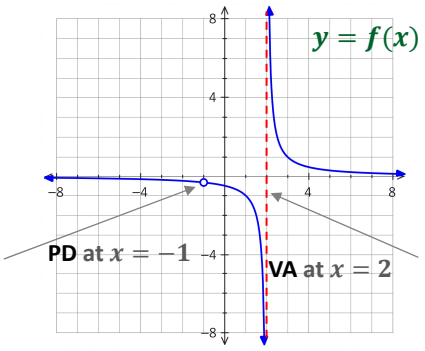
v – Types of Discontinuities

The graph of a **rational function** has two potential types of discontinuities at any non-permissible value.

(Values of x where the denominator is equal to 0)

Consider the graph of $f(x) = \dfrac{x+1}{(x+1)(x-2)}$



- If the factor that produces the discontinuity **cancels** with a term from the numerator, the graph will have a ***point of discontinuity.***
- If the factor does not cancel, the graph will have a ***vertical asymptote.***

Worked Example For $y = \dfrac{x+3}{x^2+x-6}$, determine the domain and describe the *nature of the discontinuities*.

Solution: *Factor the denominator to determine the NPVs (non-permissible values).*

$$y = \frac{x+3}{(x+3)(x-2)} \quad x \neq -3 \quad x \neq 2$$

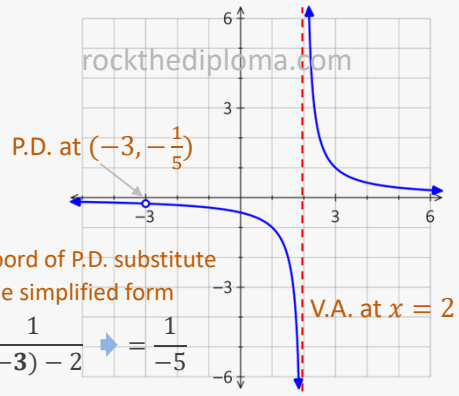
We can write this in simplified form…. $y = \dfrac{\cancel{x+3}}{\cancel{(x+3)}(x-2)} \Rightarrow y = \dfrac{1}{x-2};\ x \neq -3$

*Now, the NPV $x = -3$ comes from a factor that CANCELS OUT with a factor in the top …. So there is a **Point of Discontinuity at $x = -3$***

*And the NPV $x = 2$ comes from a factor that does NOT cancel out, So there is a **Vertical Asymptote at $x = 2$***

For domain, we are not concerned with the effect of the NPV on the graph… we simply state the restrictions:

Domain is $\{x \in \mathbb{R}, x \neq -3, 2\}$



NOTE: for y-coord of P.D. substitute $x = -3$ into the simplified form

$y = \dfrac{1}{(-3)-2} \Rightarrow = \dfrac{1}{-5}$

Class Example 9 *Characteristics of a rational function – types of discontinuities*

Sketch the following rational function, and describe the indicated characteristics.

$$y = \frac{x-6}{x^2-5x-6}$$

Domain

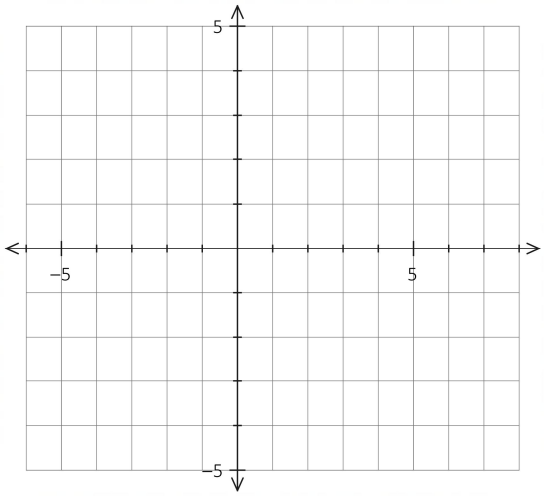
V.A.

H.A.

Range

x-intercept

y-intercept



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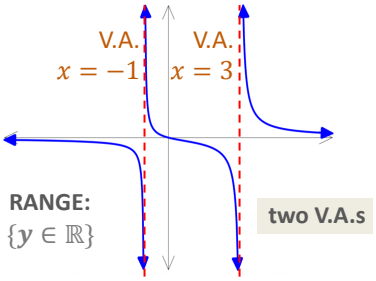
When the denominator of a rational function is **quadratic**, then:

- There can be up to two discontinuities on the graph (Each either a Vertical Asymptote or Point of Discontinuity)

EXAMPLE 1

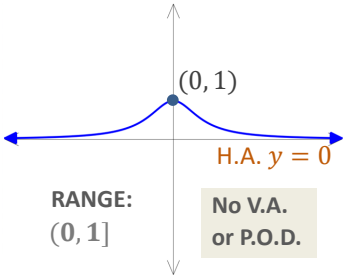
$y = \dfrac{x}{(x+1)(x-3)}$ — x-int at $x = 0$; V.A.s at $x = -1, 3$



EXAMPLE 2

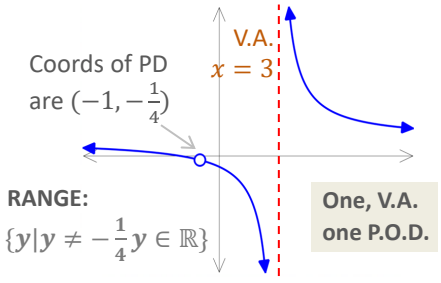
$y = \dfrac{1}{x^2+1}$ — No x-int this time (top can never be 0)

No discontinuity as denominator can never equal 0



EXAMPLE 3

$y = \dfrac{(x+2)(x+1)}{(x-3)(x+1)}$ — x-int at $x = -2$; PD at $x = -1$ (factors cancel); V.A. at $x = 3$



- Any value of x that makes the numerator (only) zero gives an **x-intercept**
- Any value of x that makes the denominator (only) zero gives a **V.A.** rockthediploma.com
- Any value x that makes both the numerator and denominator zero (factor cancels) gives a P.D.
- The graph can sometimes cross a Horizontal Asymptote, so if there is one, so the **range** takes careful analysis of the graph, and consideration of any H.A.s, y-coord's of P.D.s, and local max / mins.

Class Example 10 *Characteristics of Rational Functions from Equation*

Determine the listed characteristics for the following rational function:

$$y = \frac{x^2 - x - 6}{4x^2 - x - 18}$$

Domain

V.A.

H.A.

Range

x-intercept

y-intercept

Coords of any point of discontinuity

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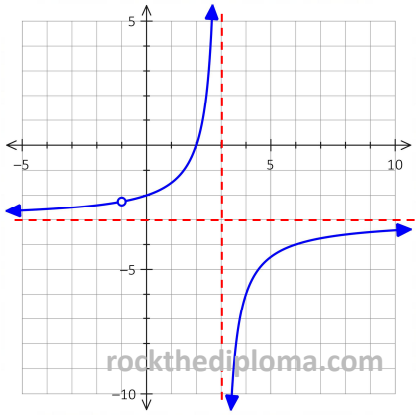
Class Example 11 *Obtaining an Equation from Rational Function Graph*

Diploma Example **Written Response**

The graph of $y = f(x)$ is shown on the right. The intercepts, point of discontinuity, and vertical asymptote all occur at integer values of x.

The horizontal asymptote occurs at a non-integer value of y.



(a) Determine an equation for $f(x)$. ***[3 marks]***

(b) State the domain and range of $f(x)$. ***[2 marks]***

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Radical and Rational Function Practice Questions *Part 2*

Radical and Rational Functions Practice Question 7

Determine the indicated characteristics for the following rational function:

$$y = \frac{2x - 1}{x^2 - 2x - 3}$$

V.A.

H.A.

x-intercept

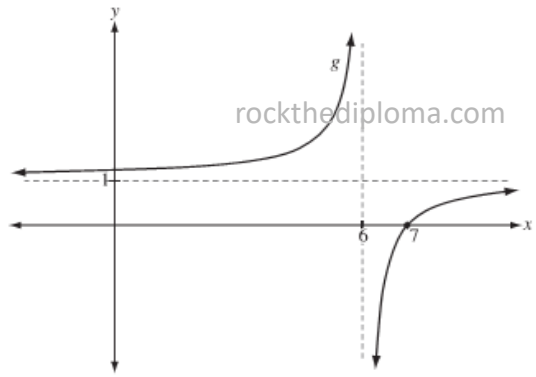
y-intercept

Domain

Range

Radical and Rational Functions Practice Question 8

The graph of $g(x)$ below can be represented in the form $g(x) = \frac{a}{x - b} + c$.



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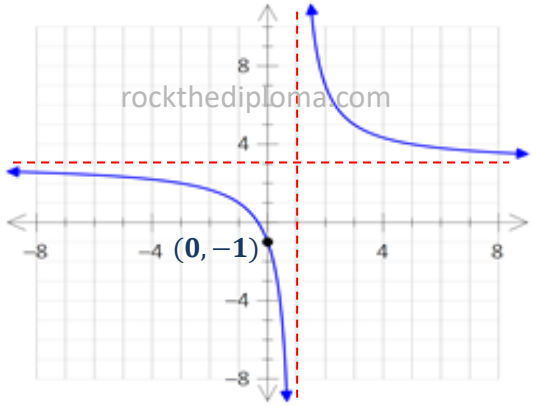
Diploma Example

If a function $f(x)$ is defined by the equation $f(x) = x^2 + 3x - 4$, the domain of the function $h(x) = (g \circ f)(x)$ will have the restriction:

A. $x \neq 6$ **B.** $x \neq 0$ **C.** $x \neq -4,\ 1$ **D.** $x \neq -5,\ 2$

Radical and Rational Functions Practice Question 9

(a) Determine an equation for the following function, in the form $y = \frac{a}{x - h} + k$ **and** $y = \frac{ax + b}{x - c}$.



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(b) Use the latter equation form to determine the x-intercept of the function.

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Radical and Rational Functions Practice Question 10

NR **Diploma Example**

A rational function $f(x) = \dfrac{x^2 - 8x + c}{x - b}$ has a point discontinuity at $(1, -6)$.

The value of c is _____.

Radical and Rational Functions Practice Question 11

A rational function is given by $f(x) = \dfrac{ax(x-b)}{x^2 - b^2}$; where $x \neq \pm b$

Use the indicated codes to answer the following:

Codes: **1** a **2** b **3** $-b$ **4** 0 **5** a/x

NR The graph of $f(x)$ has an x-intercept at $x =$_____, a horizontal asymptote at $y =$_____, and a point of discontinuity at $x =$_____.
first digit *second digit* *third digit*

Radical and Rational Functions Practice Question 12

For each of the following functions, factor and simplify to describe the graph characteristics. Sketch.

(a) $f(x) = \dfrac{x^2 - 6x + 8}{x - 2}$

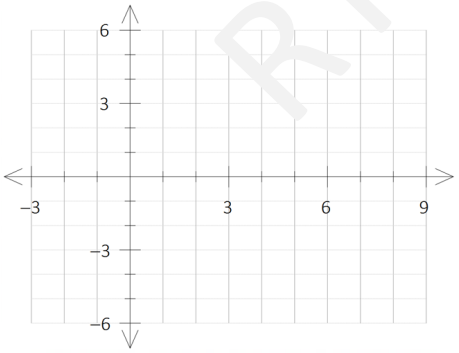
i - _____ Domain

ii - _____ Range

iii - _____ N.P.V.s (**VA**s / **PD**s)

iv - _____ x-intercept

v - _____ y-intercept



(b) $g(x) = \dfrac{4x^2 - 1}{2x^2 - 5x - 3}$

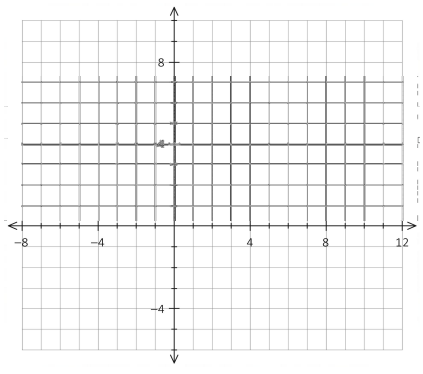
i - _____ Domain

ii - _____ Range

iii - _____ N.P.V.s (**VA**s / **PD**s)

iv - _____ x-intercept

v - _____ y-intercept



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