Mathematics (0580)

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Comprehensive Cheat Sheet

The Practice Book
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1 Algebra and graphs

KEY FORMULAS

Classifying stationary points (second-derivative test)

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum}, \qquad \frac{d^2y}{dx^2} < 0 \Rightarrow \text{maximum}$$

Use once a stationary point has been located, to decide whether it is a local maximum or a local minimum. Differentiate a second time and substitute the x-value of the stationary point.

The discriminant

$$\Delta = b^2 - 4ac$$

Use to determine the number of real roots of $ax^2 + bx + c = 0$ without solving the equation. A positive Δ gives two distinct real roots, $\Delta = 0$ gives one repeated root, and a negative Δ gives no real roots.

The power rule

$$y = ax^n \implies \frac{dy}{dx} = anx^{n-1}$$

Use to differentiate any term of the form ax^n term by term. The derivative of a constant is 0, and the derivative of a sum is the sum of the derivatives.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use to solve any quadratic equation $ax^2 + bx + c = 0$ when factorising does not work cleanly over integers. Substitute a, b and c directly from the equation written in that standard form.

The three algebraic identities

$$(a+b)^2 = a^2 + 2ab + b^2, \quad (a-b)^2 = a^2 - 2ab + b^2, \quad a^2 - b^2 = (a+b)(a-b)$$

Use to expand a squared bracket or a product of a sum and a difference on sight, without multiplying out term by term. Recognising the pattern also speeds up factorising in reverse.

Equation of a tangent

$$y - y_1 = \left.\frac{dy}{dx}\right|_{x=x_1} (x - x_1)$$

Use to find the equation of the tangent to a curve at a given point (x_1, y_1). Find y_1 by substituting x_1 into the curve's equation, find the gradient by differentiating and substituting x_1, then apply the point-gradient form.

Factorising a quadratic trinomial

$$x^2 + bx + c = (x+p)(x+q) \text{ where } p + q = b,\ pq = c$$

Use to factorise a monic trinomial by finding two numbers that sum to the coefficient of x and multiply to the constant term. For a non-monic trinomial $ax^2 + bx + c$, find two numbers summing to b and multiplying to ac, then split the middle term.

KEY CONCEPTS

- **Choosing a method to solve a quadratic**: Three methods solve any quadratic: factorising, completing the square, and the quadratic formula. Try factorising first, since it is fastest when the trinomial factorises over integers. Use the quadratic formula when factorising fails. Use completing the square only when a question explicitly asks for it, or to find the vertex of the corresponding graph.
- **Composite functions**: The composite function $fg(x)$ means apply g first and then apply f to the result, written $fg(x) = f(g(x))$. Work from the inside outward: substitute the input into g, then substitute that result into f.
- **Function notation**: A *function* f assigns exactly one output $f(x)$ to each input x. The set of allowed inputs is the *domain*; the set of resulting outputs is the *range*. Evaluate $f(a)$ by substituting a for x wherever x appears.
- **Like terms**: Two terms are *like terms* only when their letter parts are identical, meaning the same letters raised to the same powers. Only like terms can be added or subtracted; combine them by adding their coefficients and keeping the letter part unchanged.
- **Stationary points**: A *stationary point* is a value of x at which $\frac{dy}{dx} = 0$, so the curve is momentarily flat. Find stationary points by differentiating, setting the derivative equal to zero, and solving for x.
- **Elimination vs substitution for simultaneous equations**: Two linear equations in two unknowns share one solution pair (x, y). *Elimination* adds or subtracts multiples of the equations to remove one letter, and works fastest when both equations are already in the form $ax + by = c$. *Substitution* replaces one letter using an equation already solved for it, and works fastest when one equation gives $y = \ldots$ or $x = \ldots$ directly.
- **Inverse functions undo the original**: The *inverse* f^{-1} reverses the effect of f, so $f^{-1}(f(x)) = x$ for every x in the domain. Find f^{-1} by writing $y = f(x)$, swapping x and y, solving for y, and replacing y with $f^{-1}(x)$.
- **Simplifying algebraic fractions by cancelling factors**: Simplify an algebraic fraction by factorising the numerator and denominator completely and then cancelling any factor common to both. Never cancel individual terms from within a sum; only complete bracketed factors may be cancelled.
- **Restricting the domain to make an inverse exist**: A function only has an inverse over a region where it is one-to-one, meaning no two inputs give the same output. A quadratic such as $f(x) = (x-1)^2 + 4$ is two-to-one over all real x, so a domain restriction such as $x \geq 1$ is imposed to make it one-to-one before an inverse can be found; take the root matching the sign of that restriction.
- **Why the second-derivative test works**: The second derivative $\frac{d^2y}{dx^2}$ measures how the gradient itself is changing. At a maximum the gradient falls from positive to negative through the stationary point, so the gradient is decreasing and $\frac{d^2y}{dx^2} < 0$. At a minimum the gradient rises from negative to positive, so the gradient is increasing and $\frac{d^2y}{dx^2} > 0$.

EXAM TIPS

TIP In general $fg(x) \neq gf(x)$, so always apply the functions in the order the notation gives, working from the letter closest to the bracket outward. Checking which function is written first prevents composing them the wrong way round.

TIP When the term outside a bracket is negative, every sign inside the bracket flips once the bracket is expanded. Underline the multiplier before expanding so the sign change is not missed; this is the single most common algebra error at this level.

TIP Because swapping x and y is exactly a reflection in the line $y = x$, the graph of f^{-1} is always the mirror image of the graph of f in that line. Use this to sketch f^{-1} quickly once f is drawn.

TIP $\frac{x+3}{x+6}$ cannot be simplified to $\frac{1}{2}$, because $x+3$ is a sum, not a factor of the whole numerator. Only cancel something that multiplies the entire top and the entire bottom of the fraction.

2 Coordinate geometry

KEY FORMULAS

Equation of a straight line

$$y = mx + c$$

Use as the standard form of any non-vertical straight line, where m is the gradient and c is the y-intercept. Reading m and c straight from this form is the fastest way to sketch a line or compare two lines.

Gradient of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Use to find the gradient (steepness) of the line through two points (x_1, y_1) and (x_2, y_2). It is the change in y divided by the change in x, often remembered as rise over run.

Gradients of parallel lines

$$m_1 = m_2$$

Use to test for or construct parallel lines. Two lines are parallel exactly when their gradients are equal, so a line parallel to $y = mx + c$ keeps the same m and only c changes.

Gradients of perpendicular lines

$$m_1 \times m_2 = -1$$

Use to test for or construct perpendicular lines. When two lines meet at a right angle the product of their gradients is -1, so each gradient is the negative reciprocal of the other.

Length of a line segment

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Use to find the distance between two points (x_1, y_1) and (x_2, y_2). It is Pythagoras' theorem applied to the horizontal and vertical gaps between the points.

Midpoint of a line segment

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Use to find the point exactly halfway between two known endpoints (x_1, y_1) and (x_2, y_2). Average the two x-coordinates and average the two y-coordinates.

Finding the equation from the gradient and a point

$$c = y_1 - mx_1$$

Use once the gradient m and one point (x_1, y_1) on the line are known. Substitute the point and gradient into $y = mx + c$ to solve for the intercept c, then write the full equation.

KEY CONCEPTS

- **The coordinate plane**: A point is fixed by an ordered pair (x, y), where x is the horizontal distance from the origin and y is the vertical distance. The horizontal difference between two points is often written Δx and the vertical difference Δy; these two gaps drive every midpoint, length and gradient calculation.
- **The meaning of m and c**: In $y = mx + c$, the number m multiplying x is the *gradient* and the constant c is the y*-intercept*, the value of y where the line crosses the y-axis at $x = 0$. Two lines are identical only when both m and c match.
- **What the gradient measures**: The *gradient* measures how steeply a line rises or falls. A gradient of 3 means y increases by 3 for every 1 unit increase in x. Take the two coordinate differences in the *same order* for both x and y, or the sign of the gradient will be wrong.
- **Finding an endpoint from the midpoint**: The midpoint formula can be run in reverse. If M is the midpoint of AB and A is known, then $B = (2x_M - x_A,\ 2y_M - y_A)$, because each coordinate of M is the average of the matching coordinates of A and B. This reverses many "find the other end" questions in one step.
- **Positive, negative, zero and undefined gradients**: A *positive* gradient rises from left to right; a *negative* gradient falls from left to right. A horizontal line has gradient 0 (no rise), and a vertical line has an *undefined* gradient because the run is 0 and division by zero is not allowed.
- **The perpendicular bisector of a segment**: The *perpendicular bisector* of AB is the line that cuts AB at a right angle through its midpoint. Build it in two steps: find the midpoint of AB for a point on the line, and take the negative reciprocal of the gradient of AB for its gradient. Every point on it is equidistant from A and B.
- **Rearranging to read the gradient**: A line is often given in the form $ax + by = c$ rather than $y = mx + c$. Rearrange it to make y the subject before reading off the gradient. For example $2x + 3y = 6$ becomes $y = -\frac{2}{3}x + 2$, so its gradient is $-\frac{2}{3}$, not 2 or 3.

EXAM TIPS

TIP To get the gradient perpendicular to a line, turn the gradient upside down and change its sign. A gradient of $\frac{2}{3}$ becomes $-\frac{3}{2}$, and a gradient of 4 becomes $-\frac{1}{4}$. The one exception is a horizontal line, whose perpendicular is vertical and has an undefined gradient.

TIP When finding a gradient, subtract the coordinates in the *same order* on the top and the bottom of the fraction. If you start with $y_2 - y_1$ on top, you must use $x_2 - x_1$ underneath, not $x_1 - x_2$. Swapping the order on only one line flips the sign and turns an uphill line into a downhill one.

3 Geometry

KEY FORMULAS

Angle in a semicircle

$$90^\circ$$

Use whenever a triangle is inscribed in a circle with one side as the diameter; the angle at the vertex on the circle is always 90°.

Similar-shape scale factors

$$\text{length} : k, \qquad \text{area} : k^2, \qquad \text{volume} : k^3$$

Use once two shapes or solids are known to be similar with linear scale factor k; area scales by k^2 and volume by k^3.

Sum of exterior angles of a polygon

$$\text{Sum of exterior angles} = 360^\circ$$

Use for any convex polygon regardless of the number of sides; the exterior angles, one at each vertex, always total 360°.

Sum of interior angles of a polygon

$$\text{Sum of interior angles} = (n-2) \times 180^\circ$$

Use to find the sum of the interior angles of any n-sided polygon; substitute the number of sides directly.

Tangent perpendicular to radius

$$\angle(\text{tangent}, \text{radius}) = 90^\circ$$

Use whenever a tangent meets a circle; the radius drawn to that point always meets the tangent at a right angle, opening the way to Pythagoras or trigonometry.

Cyclic quadrilateral opposite angles

$$\angle A + \angle C = 180^\circ, \qquad \angle B + \angle D = 180^\circ$$

Use whenever all four vertices of a quadrilateral lie on one circle; each pair of opposite angles sums to 180°.

Finding an unknown side using the linear scale factor

$$k = \frac{\text{corresponding side, enlarged}}{\text{corresponding side, original}}$$

Use once two shapes are known to be similar, to find k from one matching pair of sides, then multiply or divide any other side by k.

Interior angle of a regular polygon

$$\text{Interior angle} = \frac{(n-2) \times 180^\circ}{n}$$

Use once a polygon is known to be regular with n equal sides, to find each interior angle directly without first finding the total.

KEY CONCEPTS

- **Angle at the centre is twice the angle at the circumference**: When a chord subtends an angle at the centre and an angle at the circumference on the same arc, the centre angle is always double the circumference angle.
- **Angle facts at a point and on a line**: Angles on a straight line sum to 180°; angles at a point sum to 360°; vertically opposite angles, formed where two straight lines cross, are equal.
- **Angles in the same segment are equal**: Angles subtended by the same chord, from points on the *same* side of that chord, are equal to one another.
- **Parallel line angle rules**: When a transversal cuts two parallel lines, corresponding angles (an F shape) are equal, alternate angles (a Z shape) are equal, and co-interior angles (a C shape) sum to 180°.
- **Similar shapes have equal angles and proportional sides**: In similar figures, every pair of corresponding angles is equal and every pair of corresponding sides is in the same fixed ratio, the linear scale factor.
- **Tests for similar triangles**: Two triangles are similar if any one of these holds: two pairs of angles are equal (AAA), all three pairs of sides are in the same ratio (SSS), or two pairs of sides are in the same ratio with the included angle equal (SAS).
- **Alternate segment theorem**: The angle between a tangent and a chord drawn from the point of tangency equals the angle subtended by that chord in the alternate segment, the segment on the other side of the chord.
- **Exterior angle of a triangle**: The exterior angle of a triangle equals the sum of the two non-adjacent (opposite) interior angles. This follows directly from the interior angles summing to 180°.
- **Similar triangles formed by a line parallel to one side**: When a line is drawn parallel to one side of a triangle, it cuts the other two sides in the same ratio and creates a smaller triangle similar to the original, sharing the vertex angle.
- **Interior and exterior angle relationship at a vertex**: At each vertex of a polygon, the interior and exterior angles are supplementary, summing to 180°. Use this as a quick check once either angle has been calculated.
- **Two tangents from an external point**: If two tangents are drawn from the same external point, they are equal in length, and the line from the external point to the centre bisects both the angle between the tangents and the angle between the radii to the points of tangency. This follows from RHS congruence between the two triangles formed with the centre.
- **Why area scales by k^2 and volume by k^3**: Area is the product of two lengths, so each of its two factors picks up the linear scale factor once, giving $k \times k = k^2$. Volume is the product of three lengths, so it picks up the factor three times, giving k^3. The same reasoning extends to any measurement built from a product of lengths.

EXAM TIPS

TIP Stating a numerical answer alone earns no method mark. Quote the rule in brackets every time, for example "alternate angles, parallel lines" or "angle sum of a triangle".

TIP Write the theorem name exactly, for example "angles in the same segment" or "tangent-radius", not just the numerical result; the mark is for the reasoning.

TIP Given a ratio of areas, take the square root to reach the linear scale factor; given a ratio of volumes, take the cube root. Never divide the areas or volumes directly to get the linear ratio.

TIP "Angles in the same segment" only applies when both angles are on the *same* side of the chord. On opposite sides, the angles instead satisfy the cyclic-quadrilateral rule and sum to 180°.

4 Mensuration

KEY FORMULAS

Surface area of a cylinder and a sphere

$$A_{\text{cylinder}} = 2\pi rh + 2\pi r^2, \qquad A_{\text{sphere}} = 4\pi r^2$$

Use for a closed cylinder (curved side plus two circular ends) and for a sphere. For an open-top cylinder such as a tube or trough, drop one or both πr^2 end discs.

Volume of a cone, pyramid and sphere

$$V_{\text{cone}} = \tfrac{1}{3}\pi r^2 h, \qquad V_{\text{pyramid}} = \tfrac{1}{3} \times \text{base area} \times h, \qquad V_{\text{sphere}} = \tfrac{4}{3}\pi r^3$$

Use for the pointed and round solids. A cone or pyramid holds exactly one third of the prism or cylinder on the same base and height; h is the *perpendicular* height, not a slant edge.

Volume of a prism and a cylinder

$$V = A_{\text{cross-section}} \times l, \qquad V_{\text{cylinder}} = \pi r^2 h$$

Use for any solid with a uniform cross-section: multiply the area of the cross-section by the length. A cylinder is the special case where the cross-section is a circle of radius r.

Area and volume scale factors for similar solids

$$\frac{A_2}{A_1} = k^2, \qquad \frac{V_2}{V_1} = k^3 \quad \text{where } k = \frac{\text{length}_2}{\text{length}_1}$$

Use when two solids are mathematically similar and one measurement is known on each. A length ratio of k scales every area by k^2 and every volume by k^3; find k from a pair of matching lengths first.

Curved surface area of a cone

$$A_{\text{curved}} = \pi r\ell, \qquad A_{\text{total}} = \pi r\ell + \pi r^2, \qquad \ell = \sqrt{r^2 + h^2}$$

Use $\pi r\ell$ for the sloping surface of a cone, where ℓ is the slant height. Add the base disc πr^2 for a closed cone. When only the perpendicular height h is given, find the slant first with Pythagoras.

Rate of flow through a pipe

$$\text{Volume} = A \times v \times t$$

Use when liquid flows through a pipe or channel of cross-sectional area A at speed v for time t. The volume delivered equals cross-section times speed times time; rearrange to find a missing time, speed or area.

Surface area of a cuboid

$$A = 2(lw + lh + wh)$$

Use to add the areas of the three distinct pairs of rectangular faces of a cuboid. For a cube every edge is equal, so this reduces to $6l^2$.

Density, mass and volume

$$\rho = \frac{m}{V} \quad \Longrightarrow \quad m = \rho V$$

Use to link the mass of a solid to its volume through the density ρ of its material. Compute the volume from the shape formulae first, then multiply by the density; keep the mass and volume units consistent (for example g with cm^3).

Volume of a frustum

$$V = \tfrac{1}{3}\pi h\left(R^2 + Rr + r^2\right)$$

Use for a frustum, the solid left when the top of a cone is sliced off parallel to the base. Here R is the larger (base) radius, r the smaller (top) radius, and h the perpendicular depth of the frustum. Equivalently, subtract the small removed cone from the full cone.

KEY CONCEPTS

- **Compound solids: add or subtract volumes**: A compound solid is built from standard pieces, such as a hemisphere on a cylinder or a cone bored out of a block. Find the volume of each piece separately, then *add* the pieces that are present and *subtract* any piece that has been removed.
- **Perpendicular height, not slant height**: In every volume formula h means the *perpendicular* height, measured at right angles to the base. The slant edge ℓ of a cone or pyramid is longer than h. If a question gives only the slant, recover the perpendicular height with $h = \sqrt{\ell^2 - r^2}$ before substituting.
- **The prism rule**: A *prism* is a solid with the same cross-section along its whole length. Its volume is always the cross-sectional area multiplied by the length, so the method is identical whether the cross-section is a triangle, an L-shape or a circle. Find the area first, then multiply.
- **Converting between metric units of area and volume**: A length factor squares for area and cubes for volume. Because $1\ \mathrm{m} = 100\ \mathrm{cm}$, it follows that $1\ \mathrm{m}^2 = 100^2 = 10\,000\ \mathrm{cm}^2$ and $1\ \mathrm{m}^3 = 100^3 = 1\,000\,000\ \mathrm{cm}^3$. Convert *before* substituting so every length is in the same unit.
- **Surface area of a compound solid**: To find the outer surface of a compound solid, list every exposed face and curved surface, then add them. Include the base it rests on but exclude the hidden disc where two pieces meet. For a cylinder topped by a hemisphere this is the base circle plus the curved cylinder plus the curved hemisphere.
- **Nets of a solid**: A *net* is the flat, unfolded pattern of all the faces of a solid. The number and type of faces identify the solid: two congruent triangles with three rectangles fold into a triangular prism, and six squares fold into a cube. The total area of the net equals the surface area of the solid.
- **Working backwards from a volume or surface area**: Reverse problems give the finished measurement and ask for a missing length. Write the appropriate formula, substitute the known volume or area, and solve the resulting equation for the unknown. For a sphere of known volume, for instance, make r^3 the subject and take the cube root.

EXAM TIPS

TIP When two solids are joined, the circle or face where they meet is hidden inside and is not part of the outer surface. Add only the surfaces you could actually touch or paint; leaving out the internal join is the single biggest mark-loser in compound surface-area questions.

TIP The most common mensuration slip is writing $\frac{2}{3}\pi r^3$ (which is a hemisphere) or πr^3 for a whole sphere. Memorise the whole sphere as *four-thirds pi r cubed*; a hemisphere is exactly half of that, $\frac{2}{3}\pi r^3$.

TIP Carry π as a symbol through the working and only replace it with a decimal at the very end. This keeps cancellations tidy, avoids rounding error building up, and lets you give an exact answer instantly when a question says *in terms of* π.

TIP When solids are similar, a length ratio $a : b$ becomes an area ratio $a^2 : b^2$ and a volume ratio $a^3 : b^3$. Going the other way, take the square root of an area ratio or the cube root of a volume ratio to recover the length ratio before finding a missing side.

5 Number

KEY FORMULAS

Bounds interval for a rounded measurement

$$\left[m - \frac{u}{2},\, m + \frac{u}{2}\right)$$

Used to state the range of possible actual values for a measurement m rounded to a precision u (one whole unit at the rounded place). The lower bound is included and the upper bound excluded, since a value exactly on the upper bound would round up.

Percentage change

$$\text{Percentage change} = \frac{N - O}{O} \times 100\%$$

Used to find the percentage increase or decrease between an original value O and a new value N. A positive result is an increase; a negative result is a decrease.

Percentage multiplier

$$\text{new value} = \text{original} \times \left(1 \pm \frac{p}{100}\right)$$

Used to increase or decrease a quantity by $p\%$ in a single multiplication. Use $+$ for an increase and $-$ for a decrease.

Splitting and combining surds

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Used to split a surd into factors (to simplify it) or to combine two surds into one, for non-negative a and b with $b \neq 0$ in the second identity.

Bounds of a difference and a quotient

$$(a - b)_{\max} = a_{\max} - b_{\min}, \qquad (a \div b)_{\max} = a_{\max} \div b_{\min}$$

Used to find the upper bound of a difference or quotient of two positive rounded quantities: maximise the term that increases the result and minimise the term that decreases it. Swap every max for min (and vice versa) to find the lower bound instead.

Compound interest and depreciation

$$A = P(1 \pm r)^n$$

Used to find the value A after n periods of repeated percentage change on a principal P, where r is the per-period rate as a decimal. Use $+$ for compound interest or growth, $-$ for depreciation or decay.

Rationalising with a conjugate

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c$$

Used to remove a surd from a denominator of the form $a + b\sqrt{c}$: multiply numerator and denominator by the conjugate $a - b\sqrt{c}$. The product on the denominator is always rational.

KEY CONCEPTS

- **Estimating by rounding to 1 significant figure**: To estimate the result of a calculation, round every value in it to 1 significant figure first, then compute exactly with the rounded values. Comparing the estimate to a calculator answer catches misplaced decimal points and other large errors.
- **Recurring decimals are rational**: Every terminating decimal and every recurring decimal is a rational number, meaning it can be written as a fraction $\frac{a}{b}$ with a and b integers. The dot notation $0.\dot{2}\dot{4}$ marks the digits that repeat forever.
- **Reverse percentage**: A reverse percentage problem gives the value *after* a percentage change and asks for the value *before* it. The value after the change equals the original multiplied by the percentage multiplier, so the original is found by dividing, never by taking a percentage of the final value.
- **Upper and lower bound**: Every rounded measurement hides a small range of possible actual values. The *upper bound* is the largest value that would still round to the given figure; the *lower bound* is the smallest.
- **What a surd is**: A surd is a root, most commonly a square root, whose value is irrational, meaning it cannot be written as a fraction of integers and its decimal expansion never terminates or repeats. $\sqrt{4} = 2$ is not a surd because it is rational; $\sqrt{2}$ and $\sqrt{18}$ are surds.
- **Converting a recurring decimal to a fraction**: Let x equal the recurring decimal. Multiply x by a power of 10 that shifts the recurring block one full cycle to the left of the decimal point, then subtract the original x so the recurring parts cancel. Solving the resulting equation for x gives the fraction.
- **Expanding brackets containing surds**: Expand a product of two surd binomials with FOIL exactly as with algebraic brackets, using $\sqrt{a} \times \sqrt{a} = a$ to clear any squared surd term, then collect the rational terms and the surd terms separately.
- **Repeated percentage change is exponential, not linear**: A quantity that changes by the same percentage every period, such as annual depreciation on a car or repeated inflation, follows the exponential model $A = P(1 \pm r)^n$, not a straight-line reduction of $P \times r \times n$. Confusing the two is a common source of error whenever the change compounds over more than one period.
- **Why the max/min pairing works for a quotient**: To maximise $a \div b$ you want the numerator as large as possible and the denominator as small as possible, since dividing by a smaller positive number gives a larger result. The same logic in reverse, largest denominator and smallest numerator, minimises the quotient.

EXAM TIPS

TIP If a calculator answer and a 1-significant-figure estimate disagree by more than a small margin, a decimal point has likely been misplaced or a wrong operation entered. Recompute rather than trusting the display.

TIP To undo a 30% reduction that leaves a price of $\$84$, divide 84 by the multiplier 0.70 to get $\$120$. Subtracting 30% of $\$84$ instead gives a wrong answer, because that 30% applies to the *original* price, not the sale price.

TIP $\sqrt{a} + \sqrt{b}$ does not simplify to $\sqrt{a+b}$. Surds can only be combined by addition or subtraction once they have been simplified to the same number under the root, exactly like collecting like terms in algebra.

TIP A 20% increase followed by a 10% decrease is not a net 10% change. Multiply the individual multipliers together: $1.20 \times 0.90 = 1.08$, a net 8% increase overall.

TIP The upper bound of a rectangle's area is not (upper bound of length) $\times$ (given width); every rounded quantity in the expression must be replaced by its own bound before combining.

6 Probability

KEY FORMULAS

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Use to find the probability of A *given that* B has happened. The vertical bar reads "given"; the condition B becomes the new sample space.

Multiplication rule for independent events

$$P(A \text{ and } B) = P(A) \times P(B)$$

Use when two events are *independent*, meaning the outcome of one does not change the probability of the other, such as drawing with replacement or rolling two dice.

Multiplying along a tree diagram

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

Use for any multi-stage experiment: multiply the probabilities along a path from the start to the required outcome. For dependent events the second factor is the *conditional* probability of the second step given the first.

The complement

$$P(\text{not } A) = 1 - P(A)$$

Use to find the probability that an event does not happen, and as the fast route into any "at least one" question through $1 - P(\text{none})$.

Addition rule for mutually exclusive events

$$P(A \text{ or } B) = P(A) + P(B)$$

Use only when A and B are *mutually exclusive*, meaning they cannot both happen, such as scoring a 2 or a 5 on one roll of a die.

Conditional probability from counts

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Use when a Venn diagram or two-way table gives frequencies rather than probabilities. Divide the number of outcomes in both A and B by the number in B.

General addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Use for any two events to avoid double-counting the overlap. Subtracting $P(A \cap B)$ removes the outcomes that were counted once in $P(A)$ and again in $P(B)$.

Test for independence

$$P(A \cap B) = P(A) \times P(B)$$

Use to check whether two events are independent: they are independent exactly when this product rule holds, equivalently when $P(A \mid B) = P(A)$.

KEY CONCEPTS

- **How a tree diagram works**: A tree diagram sets out the outcomes of a multi-stage experiment one stage at a time, with each branch carrying the probability of that step. *Multiply* the probabilities along a path to get the probability of that combined outcome, then *add* the results of every path that gives the required event.
- **Independent vs dependent events**: Events are *independent* when the first outcome leaves the second probability unchanged, and *dependent* when the first outcome changes it. Drawing an object and replacing it keeps the trials independent; drawing without replacing makes them dependent.
- **Reading a two-set Venn diagram**: A two-set Venn diagram splits the universal set into four regions: A only, B only, the overlap $A \cap B$, and the outside (neither). Once each region count is filled in, probabilities such as $P(A \mid B)$, $P(A \cup B)$ and $P(A')$ can be read straight off.
- **What conditional probability means**: $P(A \mid B)$ *restricts* the sample space to the outcomes in B and asks what fraction of that restricted space also lies in A. Once an event is known to have happened, only the outcomes consistent with it remain possible.
- **With replacement vs without replacement**: *With replacement*, the item is returned, so every stage uses the same total and identical branch probabilities. *Without replacement*, the item is kept, so the total falls by one at the next stage and the branch probabilities change.
- **Filling a Venn diagram from raw counts**: Work from the inside outwards. Enter the overlap $n(A \cap B)$ first, then find "A only" $= n(A) - n(A \cap B)$ and "B only" $= n(B) - n(A \cap B)$, and finally the outside $= n(S)$ minus the other three regions. Check that the four regions add up to $n(S)$.
- **Two-way tables**: A two-way table classifies each outcome by two properties at once, with row and column totals around the edge. Read a probability by dividing the relevant cell, or an edge total, by the overall total in the corner.
- **Mutually exclusive vs complementary events**: *Mutually exclusive* events cannot happen together, so $P(A \cap B) = 0$ and $P(A \text{ or } B) = P(A) + P(B)$. *Complementary* events are a special exclusive pair that also fills the whole sample space, so $P(A) + P(A') = 1$. Every complementary pair is mutually exclusive, but exclusive events need not be complementary.

EXAM TIPS

TIP When reading $P(A \mid B)$ from a Venn diagram or table, divide the count in $A \cap B$ by the count in B, never by the whole sample space. The condition shrinks the sample space to B alone.

TIP To find $P(\text{at least one})$, compute $1 - P(\text{none})$ instead of adding every separate case. Listing all the "one or more" outcomes wastes time and invites arithmetic slips.

TIP Reversing the bar asks a different question and usually gives a different value. $P(A \mid B)$ divides by $n(B)$ while $P(B \mid A)$ divides by $n(A)$, so stay clear about which event is the given condition.

TIP On a "without replacement" tree the second-stage total is one less than the first, so every second-stage fraction has the smaller denominator. Recount the objects remaining on each branch rather than reusing the first-stage denominator.

TIP At every branching point of a tree the probabilities of the branches leaving that node add up to 1. Use this to fill in a missing branch, and to check the two second-stage branches on each path before multiplying.

7 Statistics

KEY FORMULAS

Frequency density

$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}$$

Use to find the height of each bar in a histogram, especially when the class widths are unequal. Dividing by the class width makes the *area* of each bar, not its height, proportional to the frequency it represents.

Interquartile range

$$IQR = Q_3 - Q_1$$

Use to measure the spread of the middle half of the data, where Q_1 is the lower quartile and Q_3 is the upper quartile. A smaller IQR means the data is more consistent. It is less affected by extreme values than the range.

Mean from a frequency table

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Use to find the mean when each value x occurs with a frequency f. Multiply every value by its frequency, add these products, and divide by the total frequency $\sum f$. For grouped data, replace x with the midpoint of each class.

Median and quartile positions on a cumulative frequency curve

$$\text{median} \to \frac{n}{2}, \quad Q_1 \to \frac{n}{4}, \quad Q_3 \to \frac{3n}{4}$$

Use these cumulative frequency values to read estimates off a cumulative frequency curve. Read across from the value on the vertical axis to the curve, then down to the data axis. On a cumulative frequency graph the median uses $\frac{n}{2}$, not $\frac{n+1}{2}$.

Estimated mean of grouped data

$$\bar{x} \approx \frac{\sum fm}{\sum f}$$

Use when data is given in grouped classes and the exact values are unknown. Take m as the midpoint of each class, multiply by the frequency f, sum the products, and divide by the total frequency. The result is an *estimate* because the true values within each class are not known.

Frequency from a histogram bar

$$\text{frequency} = \text{frequency density} \times \text{class width}$$

Use to recover the frequency of a class from a histogram by rearranging the frequency density formula. Multiply the height of the bar (its frequency density) by the width of the class. This is how you read data back out of a completed histogram.

KEY CONCEPTS

- **Area represents frequency**: In a histogram the frequency of a class is shown by the *area* of its bar, not its height. The vertical axis is frequency density, so a wide class and a narrow class can represent the same frequency with bars of very different heights. This is what separates a histogram from an ordinary bar chart.

- **Measures of central tendency**: The mean, median and mode are single values that summarise where the centre of a data set lies. The *mean* is the sum of all values divided by how many there are; the *median* is the middle value once the data is ordered; the *mode* is the value that occurs most often.
- **What cumulative frequency means**: The cumulative frequency at a value is the running total of all frequencies up to and including that value, so it tells you how many data items are *less than or equal to* that value. Plotting cumulative frequency against the upper class boundaries gives the rising S-shaped cumulative frequency curve.
- **Choosing the most appropriate average**: The *mean* uses every value but is pulled towards extreme values. The *median* ignores extremes, so it is preferred when the data has outliers or is skewed. The *mode* is the only average that works for non-numerical data such as colours or shoe styles, and it is the value most likely to occur.
- **Histogram compared with a bar chart**: A *bar chart* uses equal-width bars whose *height* shows frequency, with gaps between the bars, and is used for discrete or categorical data. A *histogram* uses continuous data with bars that touch, and its *area* shows frequency through frequency density. Only a histogram copes correctly with unequal class widths.
- **Reading the median and quartiles from the curve**: To estimate the median, read across from $\frac{n}{2}$ on the cumulative frequency axis to the curve, then down to the data axis. Use $\frac{n}{4}$ for the lower quartile Q_1 and $\frac{3n}{4}$ for the upper quartile Q_3. The interquartile range $Q_3 - Q_1$ then measures the spread of the middle half.
- **Percentiles**: A *percentile* divides the data into hundredths. The k-th percentile is read from a cumulative frequency of $\frac{k}{100} \times n$, so the 25th percentile is the lower quartile and the 50th is the median. Percentiles are used to describe how a value ranks within a distribution, for example the mass below which the lightest 90 per cent of items fall.
- **The effect of outliers on the average**: An *outlier* is a value far from the rest of the data. Because the mean uses every value, one large outlier drags the mean towards it, sometimes making the mean unrepresentative of a typical value. The median and the interquartile range are *resistant* to outliers, since they depend on position rather than on the size of extreme values.

EXAM TIPS

TIP Order the data first, then the median lies at position $\frac{n+1}{2}$. When n is odd this is a single middle value; when n is even it falls between two values, so take their mean. Forgetting to order the data first is the most common error.

TIP Each cumulative frequency point is plotted at the *upper* boundary of its class, never at the midpoint. This is because you only know the total number of items once the whole class has been counted. Plotting at midpoints is a frequent and costly slip.

TIP Histogram bars must touch because the data is continuous, so each bar spans the full width from one class boundary to the next. Leaving gaps, or drawing all bars the same width when the classes differ, are the two errors that lose the most marks in histogram questions.

TIP When reading any value from a cumulative frequency curve, draw the horizontal line from the correct cumulative frequency to the curve, then a vertical line down to the data axis. Reading in the wrong order, or reading from the class boundary instead of the curve, gives an answer that is out by a whole class width.

8 Transformations and vectors

KEY FORMULAS

Column vector between two points

$$\overrightarrow{AB} = \binom{x_2 - x_1}{y_2 - y_1}$$

Use to write the vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ as a column. Subtract the start coordinates from the end coordinates so the vector points from A towards B.

Magnitude of a vector

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

Use to find the length (magnitude) of the column vector $\binom{a}{b}$. It is Pythagoras' theorem applied to the horizontal and vertical components.

Translation by a column vector

$$(x, y) \mapsto (x + a, y + b)$$

Use to translate a point or shape by the column vector $\binom{a}{b}$: add a to every x-coordinate and b to every y-coordinate. Shape, size and orientation are all unchanged.

Vector in terms of position vectors

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

Use to write the vector joining A to B from their position vectors $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. Subtract the start position vector from the end one.

Adding and subtracting vectors

$$\binom{a_1}{b_1} \pm \binom{a_2}{b_2} = \binom{a_1 \pm a_2}{b_1 \pm b_2}$$

Use to add or subtract two vectors by combining matching components. Geometrically, addition places the second vector's tail at the first's head, and the resultant runs from the first tail to the last head.

Midpoint as a position vector

$$\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

Use to find the position vector of the midpoint M of AB from the position vectors $\mathbf{a}$ and $\mathbf{b}$. Average the two position vectors component by component, exactly like averaging coordinates.

Reflection in a vertical or horizontal mirror

$$(x, y) \mapsto (2k - x,\ y)$$

Use to reflect a point in the vertical line $x = k$: the y-coordinate stays fixed and the x-coordinate moves to the equal distance on the far side of the mirror. For a horizontal mirror $y = k$, reflect the other coordinate instead with $(x, y) \mapsto (x,\ 2k - y)$.

Scalar multiplication of a vector

$$k\binom{a}{b} = \binom{ka}{kb}$$

Use to scale a vector by a number k: multiply each component by k. The magnitude scales by $|k|$; a positive k keeps the direction and a negative k reverses it.

Section formula

$$\mathbf{p} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$$

Use to find the position vector of the point P that divides AB in the ratio $AP : PB = \lambda : \mu$. When $\lambda = \mu$ it reduces to the midpoint formula.

KEY CONCEPTS

- **Enlargement scale factor**: Under an enlargement with centre C and scale factor k, each image point P' satisfies $\overrightarrow{CP'} = k\,\overrightarrow{CP}$, so every length is multiplied by $|k|$. If $k > 1$ the figure grows and if $0 < k < 1$ it shrinks; object and image are always *similar*.
- **Position vector**: The *position vector* of a point P is the vector $\overrightarrow{OP}$ from the origin to P. If $P = (x, y)$ then its position vector is $\binom{x}{y}$, so position vectors and coordinates carry the same information written two ways.
- **The four transformations and their parameters**: Every transformation must be named *and* fully specified. A *translation* needs a column vector; a *reflection* needs its mirror line; a *rotation* needs a centre, an angle and a sense (clockwise or anticlockwise); an *enlargement* needs a centre and a scale factor.
- **What a vector is**: A *vector* has both magnitude (size) and direction, written as a column $\binom{a}{b}$ meaning "a across and b up", where a negative component means left or down. Vectors are printed in bold ($\mathbf{a}$) or underlined ($\underline{a}$) and drawn with an arrow.
- **Reflections in the axes and diagonals**: Reflection in the x-axis sends (x, y) to $(x, -y)$; in the y-axis to $(-x, y)$; in the line $y = x$ to (y, x); and in $y = -x$ to $(-y, -x)$. Learning these four sign-swaps turns most reflection questions into a one-line answer.
- **Rotation about the origin**: For an anticlockwise rotation about the origin, 90° sends (x, y) to $(-y, x)$, 180° to $(-x, -y)$ and 270° to $(y, -x)$. A 90° clockwise turn is the same as 270° anticlockwise. For a centre away from the origin, shift the centre to the origin, rotate, then shift back.
- **Collinear points**: Three points A, B, C are *collinear* (on one straight line) when $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$, that is $\overrightarrow{AC} = k\,\overrightarrow{AB}$. The scalar k also fixes the length ratio, so $|AC| = |k|\,|AB|$.
- **Negative scale factor enlargement**: An enlargement with a *negative* scale factor k places the image on the *opposite* side of the centre C from the object, at $|k|$ times the distance. It is equivalent to a 180° rotation about C followed by an enlargement of factor $|k|$, so the image is inverted as well as resized. This is an Extended-only idea and the main mark-loss trap of the chapter.
- **Parallel vectors**: Two non-zero vectors are *parallel* exactly when one is a scalar multiple of the other: $\mathbf{u} = k\mathbf{v}$ for some number k. If $k > 0$ they point the same way; if $k < 0$ they point in opposite directions. This scalar-multiple test is the foundation of most vector proofs.

EXAM TIPS

TIP The vector $\overrightarrow{BA}$ is the exact reverse of $\overrightarrow{AB}$, so $\overrightarrow{BA} = -\overrightarrow{AB}$: negate *both* components. The two have the same length but opposite direction, and flipping the sign of only one component is a common slip.

TIP Always build a joining vector as *end minus start*: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, never $\mathbf{a} - \mathbf{b}$. Getting the order backwards reverses the vector and flips the sign of every result that depends on it.

TIP When a question says *describe fully*, state every parameter or lose marks: a rotation needs centre, angle *and* sense; an enlargement needs centre *and* scale factor (with its sign); a reflection needs the equation of its mirror line; a translation needs its column vector. Writing "rotation 90°" alone is incomplete.

9 Trigonometry

KEY FORMULAS

Area of a triangle using two sides and the included angle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Use to find the area of any triangle when two sides and the angle between them are known, without needing the perpendicular height.

Pythagoras' theorem

$$a^2 + b^2 = c^2$$

Use to find a missing side of a right-angled triangle when the other two sides are known, where c is the hypotenuse, the side opposite the right angle.

Space diagonal of a cuboid

$$d^2 = l^2 + w^2 + h^2$$

Use to find the length of the diagonal running from one corner of a cuboid to the opposite corner, given the length, width and height. It applies Pythagoras' theorem twice, once across the base and once up through the solid.

The cosine rule (finding a side)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Use to find the third side of a triangle when two sides and the angle between them, the included angle, are known.

The sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use in any triangle, not only a right-angled one, to find a missing side or angle when a complete side-angle pair is known alongside one more side or angle. Lower-case letters denote sides opposite the upper-case angles of the same letter.

The three trigonometric ratios (SOH CAH TOA)

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Use to find a missing side or angle in a right-angled triangle. Choose the ratio that links the side you already know with the side or angle you want, relative to the angle θ.

The cosine rule rearranged to find an angle

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Use to find an angle directly when all three sides of a triangle are known, by rearranging the cosine rule to make $\cos A$ the subject.

KEY CONCEPTS

- **Labelling the sides of a right-angled triangle**: Relative to the angle being used, the *hypotenuse* is always the longest side, opposite the right angle; the *opposite* side is across from the angle; the *adjacent* side is next to the angle and is not the hypotenuse. Relabel the opposite and adjacent sides if the angle used changes.
- **Setting up a right-angled triangle within a 3D solid**: Every 3D trigonometry problem reduces to a right-angled triangle drawn within the solid: identify the required length or angle, then find the plane containing both the right angle and the quantity needed. Redraw that triangle separately in 2D before applying Pythagoras or a trig ratio.
- **Angles of elevation and depression**: The *angle of elevation* is measured upward from the horizontal to a point above; the *angle of depression* is measured downward from the horizontal to a point below. Because the two horizontals are parallel, the angle of elevation from one point equals the angle of depression from the other, by alternate angles.
- **Choosing the sine rule or the cosine rule**: Use the *sine rule* when a complete side-angle pair is known alongside one more side or angle. Use the *cosine rule* when either all three sides are known, or two sides and the included angle between them are known, since no complete pair is available for the sine rule.
- **The angle between a line and a plane**: The angle between a line and a plane is measured between the line and its *projection* onto the plane, found by dropping a perpendicular from a point on the line to the plane. The resulting right-angled triangle has the line as its hypotenuse and the projection as the adjacent side.
- **Complementary angle relationships**: For any acute angle θ, $\sin\theta = \cos(90^\circ - \theta)$ and $\cos\theta = \sin(90^\circ - \theta)$, because the two non-right angles of a right-angled triangle sum to 90° and the side labelled opposite one angle is labelled adjacent to the other.

EXAM TIPS

TIP Sketch the single right-angled triangle relevant to the question as a flat 2D diagram, labelling only the lengths and angle needed. This avoids confusing a length in the solid with its projection onto the base, the most common error in 3D trigonometry.

TIP CAIE trigonometry questions expect answers in degrees unless stated otherwise; an answer that is wildly out of range for a simple ratio usually means the calculator is set to radians. Check the mode before starting a multi-step problem, not after finishing it.

TIP When the sine rule is used to find an angle from two sides and a non-included angle, two triangles can fit the given information, one with an acute angle and one with its obtuse supplement, 180° minus that angle. Check whether the diagram or the angle sum of the triangle rules out one of the two solutions.

TIP When the required triangle is not immediately visible, first find a diagonal within a face or the base using Pythagoras, then use that diagonal as a side of the second triangle that contains the required angle or length.