The Right Time to Enter

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Abstract

We analyze the dependence of future returns on past drawdowns of the monthly time series of the S&P500 index, from 1967 to 2012. From historical data, it appears that when the drawdown increases in absolute value, future returns increase, both in mean as well as in distributional values. We then run a Monte Carlo simulation on three models used in asset pricing, namely the Black-Scholes model, the CEV model and the exponential Ornstein-Uhlenbeck model, to assess whether these models reproduce this drawdown effect. It turns out that, while the first two do not exhibit a dependence of future returns on past drawdowns, in the exponential Ornstein-Uhlenbeck future returns increase when past drawdowns increase in absolute value, consistently with the empirical findings on the S&P500 index.

1 Introduction

The research of *the* right model in asset pricing is perhaps as utopistic as the research for the Holy Grail. Less ambitiously, much of the literature concentrates on presenting models which reproduce some stylized facts, like positivity of prices, independent increments, path continuity, or more subtle probabilistic properties.

This paper follows this approach in starting from a stylized fact: the dependence of future returns on past drawdowns. We then investigate whether three of the most used models in asset pricing, i.e. Geometric Brownian Motion (GBM), Constant Elasticity of Variance (CEV) and Exponential Ornstein-Uhlenbeck (EOU) succeed in reproducing this stylized fact with a reasonable choice of parameters.

The outline of the paper is the following. In Section 2, we present the empirical facts on the drawdown from where we started, i.e. the positive dependence of future returns on past drawdowns. In Section 3 we introduce a possible explanation of this *drawdown effect* that is the presence of mean reversion, and we compare it to the traditional Efficient Markets Hypothesis (EMH), done in financial markets and reflected in many asset pricing models, like in the Black-Scholes one (to cite the most famous). Section 4 briefly recalls the three well known asset pricing models we decided to test for the *drawdown effect* (GBM, CEV, EOU) and Section 5 presents the results of our test, obtained with a Monte Carlo simulation, to assess whether these models succeed in reproducing the *drawdown effect significantly*. Section 6 concludes. Finally, Appendix A presents some asset prices' distributional properties under the exponential Ornstein-Uhlenbeck model, while Appendix B shows how it is possible to make an exponential Ornstein-Uhlenbeck model co-exist with EMH.

2 Empirical facts on the drawdown

It is well known that returns and volatilities are typically uncorrelated. For an illustration, take the time series of the Standard&Poors 500 index, sampled monthly from 1967 to 2012, and plot the k-periods return over the time interval $[t-k,t]$, defined as

$$R_k(t) = \frac{P(t)}{P(t-k)} - 1$$

versus the annualized volatility on the same period $[t-k,t]$, defined as

$$Vol_k^2(t) = 12 \cdot Var[R(t-i)|i = 0, ..., k-l]$$

where Var indicates the sample variance and $P(t)$ are the price at time t of the S&P500. The result can be seen in Figure 1, where it is evident that returns and volatilities are uncorrelated.

Figure 1: S&P500 returns (on y axis) versus volatility (on x axis).

A less commonly investigated dependence is instead the one between returns and drawdowns, where the drawdown on the period $[t-k, t]$ is defined as

$$DD_k(t) = \min\{R_{u-v}(u) \mid t-k < v < u < t\} \tag{1}$$

i.e. as the maximum loss on returns on the period $[t-k, t]$. The result can now be seen in Figure 2, where it is evident that returns depend on drawdowns.

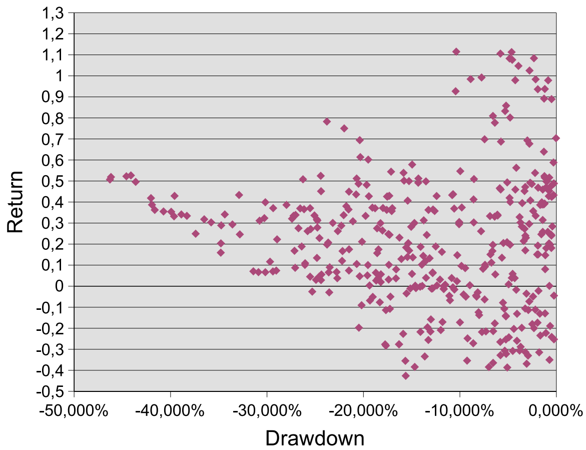

Figure 2: S&P500 returns versus drawdown.

More in details, one can see that, when the drawdown increases in absolute value, future returns tend to be more and more positive, both in mean as well as in distributional values. A more quantitative description is given in the following table.

	% negative Returns	Mean Return	Max. Return	Min. Return	Volatility	How many Times?
DD $\leq$ 0	24.8	23.70%	120.00%	−43.40%	15.00%	500
DD $\leq$ 5%	23.8	18.70%	111.50%	−42.60%	14.70%	277
DD $\leq$ 10%	17.8	20.10%	111.50%	−42.60%	14.20%	213
DD $\leq$ 15%	12.3	22.60%	78.30%	−42.60%	13.70%	155
DD $\leq$ 20%	4.2	27.40%	78.30%	−19.70%	12.60%	96
DD $\leq$ 25%	1.9	28.90%	52.70%	−2.60%	12.10%	54
DD $\leq$ 30%	0	33.20%	52.70%	6.60%	12.20%	30

One possible explanation for this stylized fact could be that markets (or at least the S&P500 index) are mean-reverting, i.e. high and low prices are temporary and that an index price will tend to move to its *average* price over time. For this reason, after there has been a significant drawdown, there could be the moment when prices start to go up again. One practical use of this is that, if an investor waits to invest in the S&P500 until the drawdown reaches a threshold fixed in advance, then his/her return should better than the one obtained by investing at a generic time.

3 Mean reversion in financial markets versus efficient markets

Strictly speaking, mean reversion is not a mathematical concept, but rather a stylized fact, which can be reproduced by many models for stock returns. For example, in discrete time the easiest model to exhibit mean-reversion is an AR(1) process, defined as

$$X(n+1) = X(n) + a(b - X(n)) + W(n+1), \qquad n = 0, 1, \dots$$

with $(W(n))_{n\geq 0}$ i.i.d. random variables, and $a > 0$ is the speed of mean reversion. The corresponding equivalent in continuous time is the Ornstein-Uhlenbeck process, used for the first time in mathematical finance in the Vasicek model for interest rates (1977), and more recently also by CONSOB (2003) to detect market abuses. The process has a continuous-time dynamics given by:

$$dX(t) = -a(b - X(t))\ dt + \sigma\ dW(t), \qquad t > 0$$

where W is a Brownian motion.

Conversely, a classical assumption on financial markets is that of market efficiency, of which three definitions exist. The strongest version (Efficient Markets Hypothesis, or EMH, by Fama, Samuelson, 1960s) is equivalent to say that returns are random walks, with discrete time dynamics

$$X(n+1) = X(n) + b + W(n+1), \qquad n = 0, 1, \dots$$

with $(W(n))_{n\geq 0}$ i.i.d. random variables, the equivalent continuous time dynamics being now that of a Brownian motion with drift

$$dX(t) = b\ dt + \sigma\ dW(t), \qquad t > 0 \tag{2}$$

where W, as before, is a Brownian motion. The concept of market efficiency received much criticism, mainly in the '90s, by behavioural finance: this was partially solved by Jung and Schiller (2005), which concluded that efficient markets hypothesis is valid for stock prices, but not for the aggregate market (e.g. indexes).

It is quite clear that mean reversion and efficient markets are two phenomena which contradicts one another. In fact, in practical terms, with mean reverting markets the rule of thumb "buy low, sell high" should work. Conversely, with efficient markets there is not "high" or "low" in the markets, as stock prices are random walks.

The relevant question is now: which one of the two behaviors above does a model for asset prices have to satisfy, given its past observations? To answer to this question, we plot in Figure 3 the monthly returns of S&P500, from 1967 to 2012, to see whether it should be represented better by a random walk or by a mean reverting process.

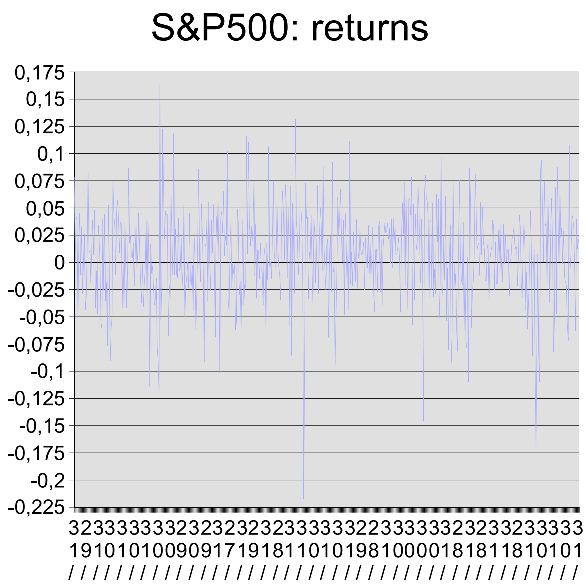

Figure 3: S&P500 historical returns, 1967–2012.

From Figure 3, it is quite clear that returns are not random walks but rather tend to concentrate around 0 (or at least a value near 0), behaviour typical of a mean-reverting process. An important consequence is that one should be able to implement the strategy of Section 2, i.e. to enter the market at the "right" time, in order to maximize returns, for example by waiting until a given drawdown is verified. Of course, to do this one must postulate that future returns depend on past quantities. This is, again, in contrast with EMH.

4 How do commonly used asset pricing models cope with this?

The relevant question is now whether commonly used asset pricing models succeed in reproducing the "drawdown phenomenon" described in Section 2, more in details in Figure 2 and in the subsequent table. To answer to this question, we select three commonly used models (Black- Scholes, Constant Elasticity of Variance and exponential Ornstein-Uhlenbeck) and run a Monte Carlo simulation on them, to see whether waiting for a given drawdown would increase the future returns as done in Section 2 with the S&P500 historical data.

We now present more in details the three tested models.

4.1 The Black-Scholes model

In the Black-Scholes model (1973), asset prices follow the dynamics

$$dS(t) = S(t)(\mu \, dt + \sigma \, dW(t)), \qquad t > 0 \tag{3}$$

i.e. returns are (continuous time) random walks. This model is consistent (and contemporary to) EMH: this is really not a case, as this model was really used firstly by McKean and Samuelson (1965), but became famous only after its use in Black and Scholes (1973) and Merton (1973).

4.2 The CEV model

In the Constant Elasticity of Variance (CEV) model, by Cox-Ross (1976), asset prices follow the dynamics

$$dS(t) = S(t)\mu \, dt + \sigma S(t)^{\gamma} dW(t), \qquad t > 0 \tag{4}$$

with $0 < \gamma \leq 1$: this model exhibits a stochastic volatility, which depends on the asset price, with a leverage effect: in fact, volatility is high when prices are low. Returns here are no more random walks, and this model is used for defaultable assets, as S can go to 0 with positive probability.

4.3 The exponential Ornstein-Uhlenbeck process

In the exponential Ornstein-Uhlenbeck (expOU) model, we define $S(t) = e^{X(t)}$, where

$$dX(t) = a(b - X(t)) \, dt + \sigma \, dW(t), \qquad t > 0 \tag{5}$$

This is a stationary, Gaussian, and Markovian process, which over time tends to drift towards its long-term mean b. This model is commonly used for commodity prices, but also by CONSOB (2003), as already mentioned in Section 2.

5 Monte Carlo simulation

We run a Monte Carlo simulation on the three models above to see whether waiting for a given drawdown would increase the future returns, in the same way that it would have done with the S&P500. Let us explain our Monte Carlo experiments in details. Using Euler-Maruyama method, for every model we simulate 10^4 price paths, starting from $S_0 = 50$ and simulating 500 time steps of length $\frac{1}{12}$ (500 monthly observations for every path, corresponding to more or less 40 years).

With a given (and fixed) set of price simulations for every model, we fix a DDR *time window*, i.e. the historical period on which we want to calculate the DDR index. For instance, if the window has length 12, then for every month the DDR index is calculated over the last 12 months. Then we fixed also a DDR threshold which is used to implement our investment strategy: for every month when the absolute DDR is above the threshold (remember that the DDR is negative), we do a 36-months investment and calculate the return of such investment. This is done for every simulated path.

Tables 1-9 below report the results of the experiments. For every DDR threshold, spanning from 0 to 50%, we report:

- the total number of investments done *on average* on every path
- the number of positive and negative (and the percentage of negative) investments done *on average* on every path
- the average return of all the investments done
- the min and max return of the investments done *on average*
- the standard deviation of the returns

We perform those analysis for different window lengths, with parameters consistent with market values. In the caption of every table the first number after the model name corresponds to the DRR window length used.

5.1 The Black-Scholes model

The model is the same of Section 4.1, with parameters $\mu = 0$, $\sigma = 0.3$.

From Tables 1, 2 and 3, it is quite evident that future returns are independent of past drawdown. More in details, the percentage of negative returns with our parameters is basically the same (around 60%) when we condition it to any given drawdown level. The mean return exibits a slightly positive dependence on the absolute value of the drawdown, but this trend is well into the confidence interval of the Monte Carlo simulation: in fact, the mean return ranges from 0.1% to about 4%, while its standard deviation is around 50%, so nothing can be concluded from this slight trend.

This non-dependence of the returns on the drawdown in the Black-Scholes model is not a surprise, as basically this model is the exponential of a drifted random walk.

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	452.00	180.97	271.03	60%	0.1%	203.5%	-73.5%	51.5%
5%	444.64	178.02	266.61	60%	0.2%	203.4%	-73.4%	51.6%
10%	408.98	163.77	245.21	60%	0.5%	202.5%	-73.0%	51.8%
15%	341.97	137.07	204.90	59%	0.9%	199.2%	-72.0%	52.0%
20%	264.40	106.13	158.27	59%	1.4%	192.1%	-70.6%	52.0%
25%	191.45	76.91	114.55	59%	1.9%	180.7%	-68.6%	51.7%
30%	130.20	52.29	77.92	58%	2.4%	165.2%	-65.9%	50.9%
35%	82.22	32.99	49.23	58%	2.8%	144.8%	-62.1%	49.4%
40%	47.26	18.95	28.31	58%	3.3%	118.5%	-56.5%	46.5%
45%	24.43	9.75	14.68	57%	3.5%	86.5%	-47.0%	40.4%
50%	11.81	4.73	7.08	57%	4.0%	54.3%	-32.5%	30.3%

Table 1: GBM 12-36

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	440.00	176.26	263.74	60%	0.1%	201.7%	-73.3%	51.4%
5%	439.93	176.23	263.69	60%	0.1%	201.7%	-73.3%	51.4%
10%	437.12	175.12	262.01	60%	0.2%	201.7%	-73.3%	51.4%
15%	420.94	168.60	252.34	60%	0.4%	201.2%	-73.0%	51.6%
20%	383.88	153.94	229.94	59%	0.9%	199.0%	-72.5%	51.8%
25%	330.90	132.77	198.13	59%	1.3%	194.2%	-71.6%	51.7%
30%	271.85	109.03	162.82	59%	1.8%	186.8%	-70.1%	51.5%
35%	213.61	85.61	128.00	58%	2.3%	175.3%	-68.2%	50.8%
40%	159.53	64.00	95.52	58%	2.8%	160.5%	-65.6%	49.7%
45%	112.32	45.01	67.31	57%	3.4%	142.8%	-61.9%	48.0%
50%	73.63	29.55	44.08	57%	3.9%	121.0%	-56.5%	44.7%

Table 2: GBM 24-36

5.2 CEV model

The model is the same as in Section 4.2, with parameters $\mu = 0$, $\sigma = 0.3$, $\gamma = 0.5$.

From Tables 4, 5 and 6, it appears that also here future returns are independent of past drawdown. More in details, the percentage of negative returns with our parameters is basically the same (around 45-55%) when we condition it to any given drawdown level, exibiting a slight nonlinear and non-monotone trend, which is however too small to be exploited in practice for investment purposes. An exception seems to be the last 1-2 rows of each table, which however contain a number of paths so small that it seems difficult to infer anything with some sense from them. The mean return does not exibits a defined dependence on the absolute value of the drawdown,

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	428.00	171.48	256.52	60%	0.1%	199.8%	-73.1%	51.3%
5%	428.00	171.48	256.52	60%	0.1%	199.8%	-73.1%	51.3%
10%	427.80	171.41	256.39	60%	0.1%	199.8%	-73.1%	51.3%
15%	424.69	170.14	254.55	60%	0.2%	199.7%	-73.0%	51.3%
20%	411.27	164.86	246.41	60%	0.5%	199.0%	-72.8%	51.5%
25%	382.55	153.39	229.16	59%	0.9%	196.9%	-72.4%	51.6%
30%	340.71	136.63	204.08	59%	1.4%	192.6%	-71.5%	51.5%
35%	291.82	116.91	174.91	59%	1.8%	185.2%	-70.3%	51.1%
40%	239.84	96.16	143.68	58%	2.4%	175.4%	-68.5%	50.4%
45%	189.13	75.78	113.35	58%	3.0%	162.8%	-66.1%	49.4%
50%	142.02	56.98	85.03	57%	3.7%	147.4%	-62.8%	47.8%

Table 3: GBM 36-36

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	452.00	221.94	230.06	51%	0.0%	18.9%	-16.2%	7.1%
5%	110.13	53.84	56.29	47%	0.6%	16.3%	-12.8%	6.9%
10%	9.08	4.39	4.69	46%	1.1%	6.4%	-4.0%	3.8%
15%	2.98	1.29	1.69	54%	-0.1%	1.5%	-1.9%	1.6%
20%	4.00	1.67	2.33	57%	-2.3%	1.3%	-6.3%	3.1%
25%								
30%								
35%								
40%								
45%								
50%								

Table 4: CEV 12-36

having also here a nonlinear and non-monotone dependence. Again, also here nothing can be concluded from this.

5.3 Exponential Ornstein Uhlenbeck

The model is the same as in Subsection 4.2, with parameters $\mu = \log(30)$, $\theta = 0.3$, $\sigma = 0.3$.

From Tables 7, 8 and 9, here finally a dependence on past drawdowns appears for future returns. More in details, the percentage of negative returns with our parameters is decreasing monotonically with the absolute value of the drawdown, ranging from 51% to about 35% with all the time horizons. Also the mean return exibits a positive dependence on the absolute value of the drawdown, ranging from about 8% to about 25-30%: this time these

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	440.00	215.94	224.06	51%	0.0%	18.7%	-16.1%	7.1%
5%	240.47	117.83	122.64	49%	0.4%	17.7%	-14.5%	7.0%
10%	48.64	23.77	24.87	45%	1.2%	11.9%	-8.6%	5.6%
15%	12.48	6.06	6.42	45%	1.2%	6.3%	-3.8%	3.3%
20%	6.46	2.90	3.56	50%	0.6%	3.6%	-2.4%	2.3%
25%	5.07	1.75	3.32	66%	-2.1%	0.7%	-4.8%	2.2%
30%	0.37	0.37	0.37	37%	37.1%	37.1%	37.1%	37.1%
35%								
40%								
45%								
50%								

Table 5: CEV 24-36

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	428.00	210.00	218.00	51%	0.0%	18.6%	-16.1%	7.1%
5%	310.62	152.32	158.29	49%	0.3%	18.1%	-15.2%	7.1%
10%	97.26	47.60	49.66	45%	1.0%	14.0%	-10.7%	6.1%
15%	30.06	14.68	15.38	45%	1.3%	9.1%	-6.1%	4.4%
20%	14.10	6.82	7.28	47%	1.0%	6.1%	-4.1%	3.3%
25%	8.87	3.96	4.92	50%	0.8%	4.6%	-3.0%	2.7%
30%	6.23	2.64	3.59	60%	-1.0%	1.6%	-3.9%	2.2%
35%	7.00	4.00	3.00	25%	2.6%	5.6%	-1.8%	3.2%
40%	0.62	0.62	0.62	62%	62.2%	62.2%	62.2%	62.2%
45%								
50%								

Table 6: CEV 36-36

percentages have the same magnitude order of the standard deviation, thus the increase of the average return in dependence of the drawdown is this time significant.

6 Conclusions

We analyzed the dependence of future returns on past drawdowns of the monthly time series of the S&P500 index, from 1967 to 2012. From historical data, it appears that when the drawdown increases in absolute value, future returns increase, both in mean as well as in distributional values. On a theoretical basis, this stylized fact could be caused by the presence of mean reversion in the index prices. We then run a Monte Carlo simulation on three

DDR *thr.*	Number of investments *tot*	*pos*	*neg*	*% neg*	Return *avg*	*max*	*min*	*std*
0%	452.00	218.45	233.55	52%	7.5%	190.7%	-66.2%	46.9%
5%	444.74	215.91	228.83	51%	7.8%	190.6%	-66.0%	47.0%
10%	407.65	201.28	206.37	51%	8.7%	190.1%	-65.4%	47.3%
15%	335.92	170.68	165.24	49%	10.4%	188.1%	-63.9%	47.8%
20%	252.83	133.10	119.73	47%	12.5%	183.6%	-61.6%	48.3%
25%	176.46	96.36	80.10	45%	14.8%	175.9%	-58.7%	48.6%
30%	114.27	64.90	49.37	43%	17.3%	164.3%	-54.7%	48.6%
35%	67.75	39.91	27.84	41%	19.8%	147.8%	-49.0%	47.9%
40%	36.12	22.24	13.88	38%	22.8%	125.9%	-40.0%	45.4%
45%	17.19	11.00	6.19	36%	25.7%	97.3%	-24.8%	38.7%
50%	8.06	5.36	2.70	34%	29.3%	69.6%	-4.0%	27.3%

Table 7: expOU 12-36

DDR *thr.*	Number of investments *tot*	*pos*	*neg*	*% neg*	Return *avg*	*max*	*min*	*std*
0%	440.00	214.61	225.39	51%	8.0%	190.4%	-65.8%	47.0%
5%	439.96	214.60	225.36	51%	8.0%	190.4%	-65.8%	47.0%
10%	437.69	213.92	223.77	51%	8.1%	190.4%	-65.7%	47.1%
15%	421.48	208.33	213.15	51%	8.8%	190.2%	-65.4%	47.3%
20%	381.05	192.94	188.11	49%	10.2%	189.2%	-64.4%	47.6%
25%	321.58	168.09	153.49	48%	12.1%	186.7%	-62.8%	48.0%
30%	254.30	137.81	116.49	46%	14.3%	181.8%	-60.4%	48.3%
35%	188.97	106.44	82.53	43%	16.8%	174.6%	-57.1%	48.4%
40%	131.15	77.09	54.06	41%	19.5%	163.6%	-52.7%	48.0%
45%	84.15	51.51	32.64	39%	22.5%	149.0%	-46.5%	46.8%
50%	48.92	31.37	17.55	36%	25.9%	128.6%	-36.8%	43.5%

Table 8: expOU 24-36

models used in asset pricing, namely the Black-Scholes model, the CEV model and the exponential Ornstein-Uhlenbeck model, to assess whether these models reproduce this drawdown effect. From the Monte Carlo simulations it turned out that, while the first two does not exhibit a dependence of future returns on past drawdowns, in the exponential Ornstein-Uhlenbeck future returns increase when past drawdowns increase in absolute value, consistently with the empirical findings on the S&P500 index. In effects, of the three models this was the only one with mean reversion. Also, notice that we used parameters for the exponential Ornstein-Uhlenbeck process that, apart from the drift part, are exactly the same as those used in the Black-Scholes model, with dramatically different results.

The conclusion seems thus to be that, if we wish to use the *drawdown*

DDR	Number of investments				Return			
thr.	*tot*	*pos*	*neg*	*% neg*	*avg*	*max*	*min*	*std*
0%	428.00	210.15	217.86	51%	8.4%	189.9%	-65.5%	47.1%
5%	428.00	210.15	217.86	51%	8.4%	189.9%	-65.5%	47.1%
10%	427.92	210.13	217.79	51%	8.4%	189.9%	-65.5%	47.1%
15%	425.49	209.46	216.03	51%	8.5%	189.9%	-65.4%	47.2%
20%	412.29	205.20	207.09	50%	9.2%	189.7%	-65.0%	47.3%
25%	380.76	193.71	187.05	49%	10.4%	188.8%	-64.1%	47.6%
30%	331.61	173.94	157.67	47%	12.3%	186.6%	-62.7%	47.9%
35%	272.30	148.01	124.29	45%	14.5%	182.6%	-60.5%	48.2%
40%	210.58	119.17	91.41	43%	17.1%	176.3%	-57.4%	48.2%
45%	152.52	89.95	62.57	41%	19.9%	166.7%	-53.2%	47.8%
50%	102.01	63.05	38.96	38%	23.2%	153.1%	-47.0%	46.5%

Table 9: expOU 36-36

effect to invest, i.e. to wait for a given drawdown threshold in order to increase future returns, the mathematical model that we use should incorporate mean reversion in order to reproduce significantly this effect.

A More on Exponential Ornstein Uhlenbeck process

It is worth study some properties of the price process in Equation (5). Let us start with the solution of (5)

$$X_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)} dW_s$$

We know that, for all $t > 0$, X_t is Gaussian, with mean

$$\bar{X}_t := \mathbb{E}[X_t] = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$$

and variance

$$\bar{\sigma}_t^2 := \mathrm{Var}[X_t] = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$$

Using the properties of the log-normal process (5) we find:

$$\begin{aligned}
\mathbb{E}[S_t] &= \exp\left\{\bar{X}_t + \frac{1}{2}\overline{\sigma_t}^2\right\} = \\
&= \exp\left\{x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma^2}{4\theta}(1 - e^{-2\theta t})\right\} \\
\mathrm{Var}[S_t] &= \exp\{2\bar{X}_t + \overline{\sigma_t}^2\}(\exp\{\overline{\sigma_t}^2\} - 1) = \\
&= \exp\left\{2x_0 e^{-\theta t} + 2\mu(1 - e^{-\theta t}) + \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\right\}\left(\exp\left\{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\right\} - 1\right)
\end{aligned}$$

B Reconciling market efficiency with mean reversion

From what we described in this paper, a mathematical curiosity arises. The empirical evidence here is that, under the real world probability measure $\mathbb{P}$, a model which describes in a suitable way the *drawdown effect* seems to be the exponential Ornstein-Uhlenbeck model, where the log-price evolves as in Equation (5). However, the Black-Scholes model is widely used in option pricing, and under it the log-price evolves as in Equation (2), with $b = r - \frac{1}{2}\sigma^2$, with r being the risk-free rate, but under a different probability measure $\mathbb{Q}$, which should be equivalent to $\mathbb{P}$ to prevent arbitrages.

The mathematical question is then: it is possible that the two models are consistent with each other? In other words: given the dynamics in Equation (5) for the log-price X under a probability $\mathbb{P}$, it is possible to find an equivalent probability $\mathbb{Q}$ such that the dynamics of X under $\mathbb{Q}$ are of the form in Equation (2), with of course now W being a $\mathbb{Q}$-Brownian motion driving Equation (2), different from the original $\mathbb{P}$-Brownian motion?

This question boils down to find a "double" dynamics for X of the form

$$\begin{aligned} dX_t &= \sigma\ dW_t && \text{(under } \mathbb{Q}) \\ &= -aX_t\ dt + \sigma\ d\tilde{W}_t && \text{(under } \mathbb{P}) \end{aligned}$$

In fact, it is well known that Girsanov transformations with constant kernel θ (which in these cases is b/σ or ab/σ) give rise to Girsanov densities which are proper martingales. The residual Girsanov transformation from $\mathbb{P}$ to $\mathbb{Q}$, or equivalently from $\mathbb{Q}$ to $\mathbb{P}$, is done with a density which is a proper martingale if, in the expression

$$d\tilde{W}_t = dW_t + \theta_t\ dt$$

the Girsanov kernel

$$\theta_t := \frac{a}{\sigma}X_t = aW_t + \frac{ax_0}{\sigma}$$

gives rise to a Girsanov density which is a proper martingale. In order to see that this is true, we use [1, Proposition 7.23(b)], which says that the Girsanov density $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is a proper martingale on an interval $[0, T]$ if there exist $C \in (0, +\infty)$, $\mu > 0$ such that

$$\mathbb{E}_{\mathbb{Q}}[e^{\mu\theta_t^2}] < C \qquad \text{for all } t \in [0, T] \tag{6}$$

Now we verify that Equation (6) holds for some constant C and μ. For $t \in [0, T]$ and $\mu > 0$, we have that

$$\mu\theta_t^2 = \mu\left(aW_t + \frac{ax_0}{\sigma}\right)^2 \leq 2\mu a^2 W_t^2 + 2\frac{\mu a^2 x_0^2}{\sigma^2}$$

so that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{\mu\theta_t^2}] &\leq \mathbb{E}_{\mathbb{Q}}\left[e^{2\mu a^2 W_t^2 + 2\frac{\mu a^2 x_0^2}{\sigma^2}}\right] = \\
&= e^{\frac{2\mu a^2 x_0^2}{\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{2\mu a^2 x^2 - \frac{x^2}{2t}}\, dx
\end{aligned}$$

which is finite if and only if $2\mu a^2 - \frac{1}{2t} < 0$, or $t < \frac{1}{4\mu a^2}$. By imposing this for all $t \in [0, T]$, the conclusion is that, if we choose $\mu < \frac{1}{4a^2 T}$, then $t < \frac{1}{4a^2\mu}$ for all $t \in [0, T]$, and we can conclude that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{\mu\theta_t^2}] &\leq e^{\frac{2\mu a^2 x_0^2}{\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{2\mu a^2 x^2 - \frac{x^2}{2t}}\, dx = \\
&= \sqrt{\frac{2\pi\left(\frac{1}{t} - 4\mu a^2\right)^{-1}}{2\pi t}} = \left(1 - 4\mu a^2 t\right)^{-1/2}
\end{aligned}$$

Now, if $\mu < \frac{1}{4a^2 T}$, then this quantity is less than $(1 - 4\mu a^2 T)^{-1/2} =: C$, which is finite. By virtue of Equation (6), we have that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, i.e. an exponential Ornstein-Uhlenbeck process under the real world probability measure $\mathbb{P}$ can have a Black-Scholes dynamics under an equivalent martingale measure $\mathbb{Q}$.

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