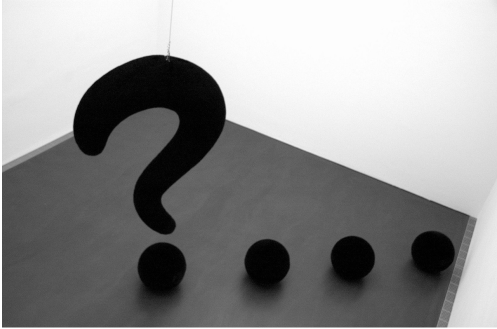
Questions in first-order logic

Ivano Ciardelli



PhDs in Logic XIII — Torino, September 1st 2022

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we deal with **statements** like (1) but not with **questions** like (2)

(1) Some EU countries are monarchies.

(2) Which EU countries are monarchies?

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Why?

A fundamental theoretical role is played by the notion of **truth**:

- logical operators are analyzed in terms of truth-conditional contribution;
- entailment is construed as necessary preservation of truth;
- ...

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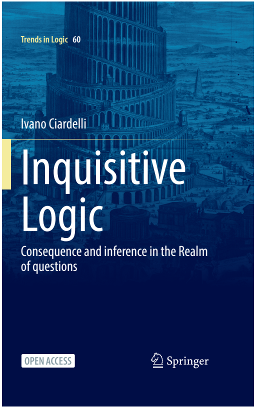
But truth is not applicable to questions. E.g., is (2) true or false?

Nevertheless, it is possible extend logic to questions in a natural way.

$\rightsquigarrow$ **Inquisitive Logic**



Inquisitive first-order logic, InqBQ

A conservative extension of classical first-order logic with questions.

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(3)	a.	Does every object satisfy P?	polar
	b.	What is one object satisfying P?	mention-some
	c.	What are the objects satisfying P?	mention-all
	d.	What is the object satisfying P?	unique-instance
	e.	…	

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We can study the interesting ways in which such questions are logically related and make inferences with questions.

Aim of this talk

- Introduce inquisitive first-order logic
- Survey some important open problems and recent advances

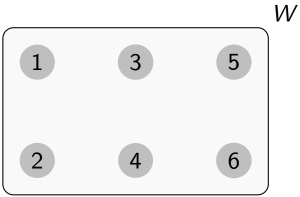
Part 1
Foundations of inquisitive logic

Intensional semantics

In intensional semantics, a model comes with a universe W of possible worlds, which represent different ways things could be.

Example

A die has been rolled. There are six "ways things could be" w.r.t. the outcome.



Standardly, semantics is given by a relation:

$$\begin{tikzcd} & w \arrow[dl] & \models \arrow[d] & A \arrow[dr] & \\ \text{world} & & \text{makes true} & & \text{statement} \end{tikzcd}$$

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Inquisitive semantics

Semantics given by a relation:

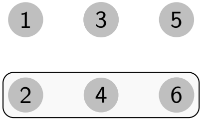
$$s \quad \models \quad A$$

information state supports sentence

Information states

An information state is modeled as a set $s \subseteq W$ of possible worlds:

- the worlds $w \in s$ are **compatible** with the information in s;
- the worlds $w \notin s$ are **ruled out** by the information in s.



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For convenience, I restrict to **consistent** information states, i.e., $s \neq \emptyset$.

Support for statements

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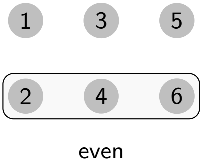

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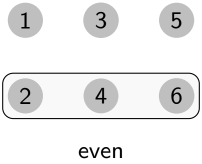
  $\phantom{s \models \text{even} \iff{}} \forall w \in s : w \models \text{even}$



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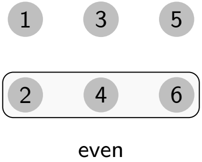

Truth-Support Bridge

$s \models A \iff \forall w \in s : w \models A$

Support for questions, 1

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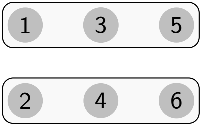
- parity := whether the outcome is even or odd
- $s \models \text{parity} \iff s \subseteq \{2, 4, 6\}$ or $s \subseteq \{1, 3, 5\}$



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Support for questions, 2

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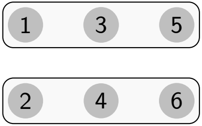

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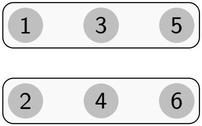

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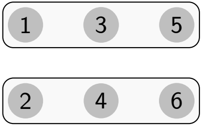
- $s \models \text{outcome} \iff s = \{1\}$ or $\ldots$ or $s = \{6\}$



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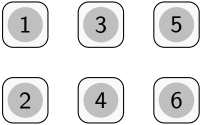

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Note

For statements, $\models$ coincides with $\models_{\text{truth}}$ (provided Truth-Support bridge holds).

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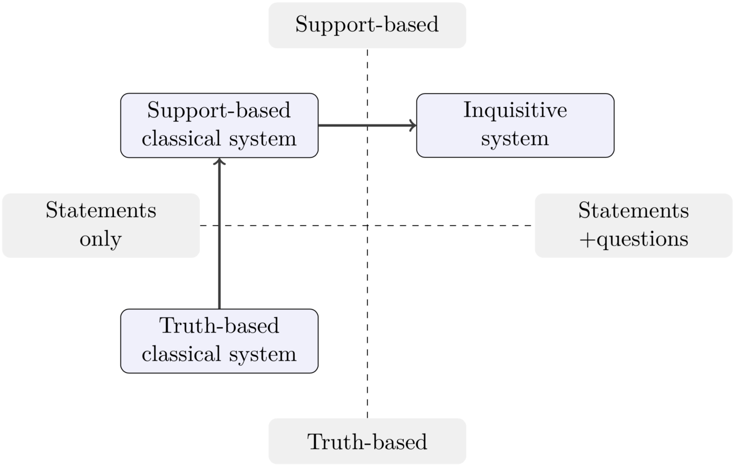
Note
For statements, $\models$ coincides with $\models_{\text{truth}}$ (provided Truth-Support bridge holds).
But now, **premises and conclusion may also be questions**!

Part 2

Inquisitive first-order logic, InqBQ


Support-based
Support-based classical system
Inquisitive system
Statements only
Statements +questions
Truth-based classical system
Truth-based

Part 2.1

Re-implement classical FOL based on support

Standard ingredients

- Signature $\mathcal{S}$: for simplicity, only predicate symbols.
- An infinite stock of first-order variables $(x, y, z, \ldots)$.

Classical formulas

$\alpha \;::=\; R(x_1, \ldots, x_n) \mid (x_1 = x_2) \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha \mid \forall x \alpha$

Defined operators

- $\neg\alpha := \alpha \to \bot$
- $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$
- $\exists x \alpha := \neg\forall x \neg\alpha$

This is essentially the standard language of first-order logic.

Models

- a non-emprty domain D of individuals;
- a non-empty universe W of possible worlds;
- for each world w and each predicate R, a suitable extension $R_w \subseteq D^n$.

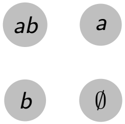
Remark

With a world w we can associate a relational structure $M_w = \langle D, \{R_w \mid R \in \mathcal{S}\}\rangle$.

Example

Signature: a unary predicate P.

- $D = \{a, b\}$
- $W = \{w_1, w_2, w_3, w_4\}$
- $P_{w_1} = \{a, b\},\ P_{w_2} = \{a\},\ P_{w_3} = \{b\},\ P_{w_4} = \emptyset$



Support for classical formulas

Given a model $M = \langle W, D, I \rangle$, a state $s \subseteq W$, and an assignment $g : \mathrm{Var} \to D$:

- $s \models_g R(x_1, \ldots, x_n) \iff \forall w \in s : \langle g(x_1), \ldots, g(x_n) \rangle \in R_w$
- $s \models_g x_1 = x_2 \iff g(x_1) = g(x_2)$ (simplification)
- $s \not\models_g \bot$
- $s \models_g \varphi \wedge \psi \iff s \models_g \varphi$ and $s \models_g \psi$
- $s \models_g \varphi \to \psi \iff \forall t \subseteq s :\ t \models_g \varphi$ implies $t \models_g \psi$
- $s \models_g \forall x \varphi \iff s \models_{g[x \mapsto d]} \varphi$ for all $d \in D$

Truth

$w \models_g \alpha \overset{def}{\iff} \{w\} \models_g \alpha$

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Proposition

$w \models_g \alpha \iff M_w \models_g \alpha$ in Tarskian semantics for FOL

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Proposition (Truth-support bridge)

For every classical formula α: $\quad s \models_g \alpha \iff \forall w \in s : \ w \models_g \alpha$.

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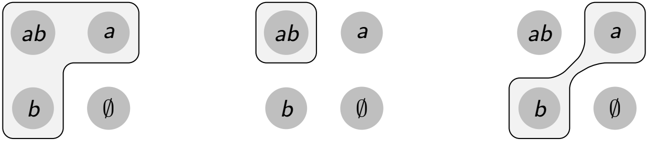
Illustration

Maximal supporting states for some classical formulas

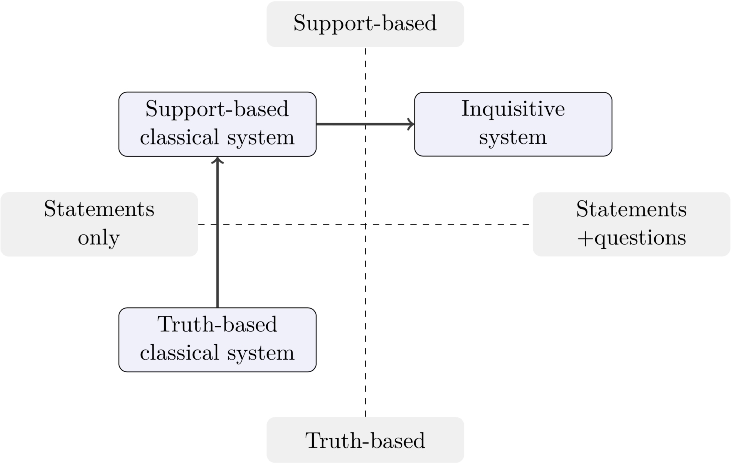


$\exists x P x$ $\forall x P x$ $\exists ! x P x$

Proposition: conservativity over classical FOL

For any set $\Gamma \cup \{\alpha\}$ of classical formulas:

$$\Gamma \models \alpha \iff \Gamma \models_{\mathsf{FOL}} \alpha$$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Part 2.2

Enrich FOL with questions

Full language of InqBQ

$$\varphi \ ::= \ R(x_1,\ldots,x_n) \mid (x_1 = x_2) \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \forall x \varphi \mid \exists\!\!\!\exists x \varphi$$

Think of $\mathbin{\vee\!\!\!\vee}$ and $\exists\!\!\!\exists$ as a question-forming operators:

- $\alpha \vee \neg\alpha \ \rightsquigarrow$ that α is true or false
- $\alpha \mathbin{\vee\!\!\!\vee} \neg\alpha \ \rightsquigarrow$ whether α is true or false ($?\alpha \ := \ \alpha \mathbin{\vee\!\!\!\vee} \neg\alpha$)
- $\exists x \alpha(x) \ \rightsquigarrow$ there is an x such that $\alpha(x)$
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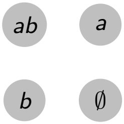
Support clauses

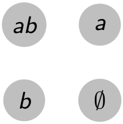
- $s \models_g \varphi \mathbin{\vee\!\!\!\vee} \psi \iff s \models_g \varphi$ or $s \models_g \psi$
- $s \models_g \exists\!\!\!\exists x \varphi \iff s \models_{g[x \mapsto d]} \varphi$ for some $d \in D$

Illustration

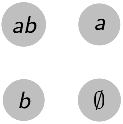
Let us look at three examples of first-order questions.



$?\forall x Px$



$\exists x Px$



$\forall x ? Px$

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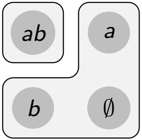

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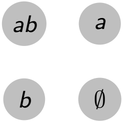

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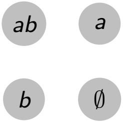

$\forall x ?Px$

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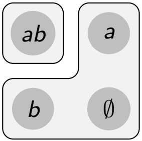

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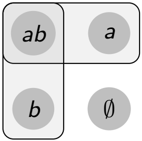

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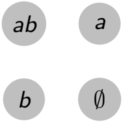

$\forall x ?Px$

$?\forall x Px$	$\rightsquigarrow$	Whether every object satisfies P
$\exists x Px$	$\rightsquigarrow$	What is one object satisfying P

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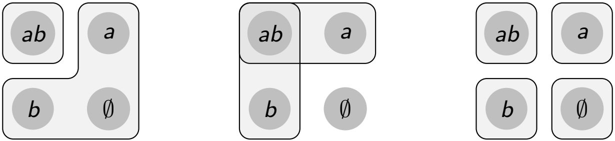

$?\forall x Px$ $\quad$ $\exists x Px$ $\quad$ $\forall x ? Px$

$$
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\exists x Px & \rightsquigarrow & \text{What is one object satisfying } P \\
\forall x ? Px & \rightsquigarrow & \text{Which objects satisfy } P
\end{array}
$$

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Dependency

$$\begin{array}{rcl} \forall x (Px \leftrightarrow \neg Qx), \forall x ? Px & \models & \forall x ? Qx \\ \forall x ? Px & \models & ? \forall x Px \\ \forall x ? Px & \not\models & \exists x Px \\ \exists x Px, \forall x ? Px & \models & \exists x Px \end{array}$$

Part 3
Open problems

Open problem 1
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Open problem 3

Is it possible to refute all non-entailments in a model with countable W and D?

- I.e., do countable models suffice to characterize the logic?

In short

Does InqBQ pattern with standard first-order logic or with second-order logic?

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Does InqBQ pattern with standard first-order logic or with second-order logic?

Key point

$\to$ introduces a second-order quantification on the state component:

$$s \models_g \varphi \to \psi \iff \forall t \subseteq s : t \models_g \varphi \text{ implies } t \models_g \psi$$

- Unclear whether a translation to FOL is possible.
- But also unclear how much second-order logic one can encode in InqBQ.

Part 4
Advances

Part 4.1
Well-behaved fragments

The classical antecedent (clant) fragment

We allow only classical formulas (i.e., $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, $\exists$-free) as antecedents.

A very broad fragment

All formulas that appeared in this talk are clant formulas:

- all classical formulas α
- polar questions $?\alpha$
- disjunctive questions $\alpha_1 \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n$
- mention-some questions $\exists \overline{x} \alpha(\overline{x})$
- mention-all questions $\forall \overline{x} ?\alpha(\overline{x})$
- ... and much more

Proposition

If α is classical then

$$s \models_g \alpha \to \psi \iff s \cap |\alpha|_M \models_g \psi$$

where $|\alpha|_M$ is the truth-set of α.

So, in the Clant fragment $\to$ does not really introduce 2nd-order quantification.

Clant completeness (Grilletti 2021)
The following natural deduction system is complete for the clant fragment.

Natural deduction system
Standard introduction/elimination rules for connectives, quantifiers, identity, plus following rules, where α is classical and $x \notin \mathrm{FV}(\psi)$:

Constant domains	$\mathbb{V}$-split	$\exists$-split	Class $\neg\neg$
$\dfrac{\forall x(\varphi(x) \mathbin{\mathbb{V}} \psi)}{\forall x\varphi(x) \mathbin{\mathbb{V}} \psi}$	$\dfrac{\alpha \to \psi \mathbin{\mathbb{V}} \chi}{(\alpha \to \psi) \mathbin{\mathbb{V}} (\alpha \to \chi)}$	$\dfrac{\alpha \to \exists x\varphi(x)}{\exists x(\alpha \to \varphi(x))}$	$\dfrac{\neg\neg\alpha}{\alpha}$

Example

$\forall x(Px \leftrightarrow \neg Qx),\ \forall x?Px \models \forall x?Qx$

Example

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$$
\cfrac{
\cfrac{\forall x ?Px}{?Py}\,(\forall\mathrm{e})
\qquad
\cfrac{\cfrac{[Py]_1 \quad \forall x(Px \leftrightarrow \neg Qx)}{\neg Qy}\,(\mathrm{class})}{?Qy}\,(\mathbb{V}\,\mathrm{i})
\qquad
\cfrac{\cfrac{[\neg Py]_1 \quad \forall x(Px \leftrightarrow \neg Qx)}{Qy}\,(\mathrm{class})}{?Qy}\,(\mathbb{V}\,\mathrm{i})
}{
\cfrac{?Qy}{\forall x ?Qx}\,(\forall\mathrm{i})
}\,(\mathbb{V}\,\mathrm{e}, 1)
$$

Example

$\forall x(Px \leftrightarrow \neg Qx),\ \forall x?Px \models \forall x?Qx$

$$
\dfrac{
\dfrac{
\dfrac{\forall x?Px}{?Py}\,(\forall\text{e})
\qquad
\dfrac{\dfrac{[Py]_1 \quad \forall x(Px \leftrightarrow \neg Qx)}{\neg Qy}\,(\text{class})}{?Qy}\,(\mathbb{V}\,\text{i})
\qquad
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}{?Qy}\,(\mathbb{V}\,\text{e}, 1)
}{\forall x?Qx}\,(\forall\text{i})
$$

Remark

- Wh-questions can be handled in inferences by pretty familiar rules.
- We built up these questions from simpler logical operators ($\forall$, $\mathbb{V}$, $\neg$); their logical properties follow from the properties of these operators.

Corollary
The set of Clant validities is recursively enumerable.

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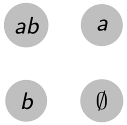
The Clant fragment seems like an attractive choice for a first-order inq logic:

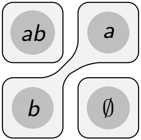
- it extends FOL in a simple way and has a rich repertoire of questions;
- retains key meta-theoretic properties of FOL;
- admits a simple proof system in which questions can be manipulated.

Part 4.2

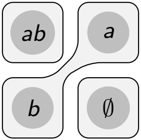
'How many' questions and inquisitive quantifiers



how many x are P

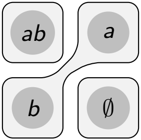


how many x are P



how many x are P

??



how many x are P

??

Question

Is there a formula χ of our logic such that

$$s \models \chi \iff \forall w, w' \in s : \#P_w = \#P_{w'}$$

Standard generalized quantifiers

Operators Q that combine with an predicate P to form a statement $Qx.Px$:

$$\left.\begin{array}{l}\text{At least three } x\\ \text{An even number of } x\\ \text{Infinitely many } x\end{array}\right\}\ \text{satisfy } Px$$

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Inquisitive generalized quantifiers

Operators Q which combine with a property P to form a question $Qx.Px$:

$$\left.\begin{array}{l}\text{Which } x\\ \text{How many } x\\ \text{Whether finitely or infinitely many } x\end{array}\right\}\text{ satisfy } Px$$

First step (Grilletti&C. 2021)
Take cardinality quantifiers (sensitive only to the cardinality of the extension): which such Q are definable in InqBQ ($Qx.Px$ is equiv to an InqBQ-formula)?

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Future work
Study InqBQ + 'how many' quantifier

Conclusion

Using inquisitive semantics, we can enrich classical FOL with questions, resulting in an **inquisitive first-order logic**.

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In this system we can regiment a broad variety of **first-order questions**, with an economical repertoire of primitives.

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Relations of **answerhood** and **dependency** involving such questions are captured as cases of entailment.

The key meta-logical questions about InqBQ are still **open problems**!
Challenges arise from $\to$ as a second-order quantifier on states.

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Much recent progress:

- Axiomatization of broad **fragments** of InqBQ which are in themselves rich inquisitive extensions of FOL.
- Game-theoretic characterization of **expressive power**.
- Characterization of definable **cardinality quantifiers**.

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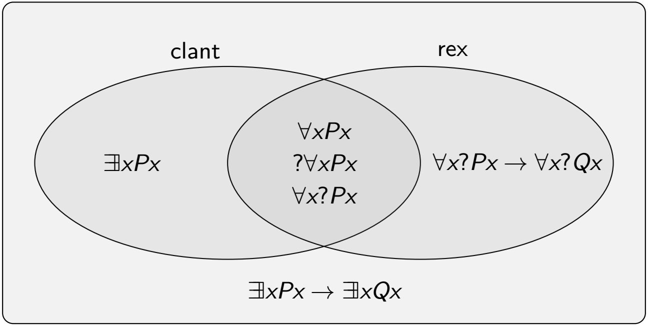
Much more to find out... $\rightsquigarrow$ Contributions are welcome!

The End

Restricted existential (rex) fragment

$\exists$ only allowed to occur in antecedents of implication.



Key property

For every rex formula φ there exists $n_\varphi \in \mathbb{N}$ such that for all states s:

$$s \not\models_g \varphi \implies \exists t \subseteq s \text{ with } \#t \leq n_\varphi : t \not\models_g \varphi$$

Theorem (C.&Grilletti 2022)

An extension of the above proof system is complete w.r.t. rex conclusions. That is, if $\Phi \models \psi$ and ψ is a rex formula then $\Phi \vdash^+ \psi$.

Example

$\forall x ? Px \models \exists x Px \veebar \forall x \neg Px$

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$\forall x ? Px \models \exists x Px \vee \forall x \neg Px$

$$
\cfrac{
\cfrac{\forall x ? P(x)}{? P(y)}\,(\forall e)
\qquad
\cfrac{\cfrac{[P(y)]_1}{\exists x P(x)}\,(\exists i)}{\exists x P(x) \vee \neg P(y)}\,(\vee i)
\qquad
\cfrac{[\neg P(y)]_1}{\exists x P(x) \vee \neg P(y)}\,(\vee i)
}{
\cfrac{
\cfrac{\exists x P(x) \vee \neg P(y)}{\forall x(\exists x P(x) \vee \neg P(x))}\,(\forall i)
}{\exists x P(x) \vee \forall x \neg P(x)}\,(\mathrm{CD})
}\,(\vee e, 1)
$$