Symmetry, locality and hyperintensionality

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Summary

Traditional notions of hyperintensionality

"A propositional context is hyperintensional if it does not license the substitution of logical equivalents salva veritate."

(Cresswell 1975, p. 25)

"A position in a sentence is hyperintensional if substitution of necessarily equivalents is not guaranteed to preserve truth value."

(Nolan 2014, p. 151)

A context is hyperintensional if it does not guarantee substitutivity with respect to classical logical equivalents.

(Leitgeb 2019, p. 307)

Hyperintensionality, intuitively

What is then a hyperintensional context, intuitively?

More fine-grained than intensionality: e.g. belief contexts, metaphysical grounding, essence, counterfactuals etc.

Julie is blonde.
Julie is blonde and Andy (Julie's brother) is a redhead or is not a redhead.

Now imagine Anna has just met Julie and she doesn't know Julie has a brother yet.

Anna knows that Julie is blonde.
Anna does not know that Julie is blonde and Andy is a redhead or is not a redhead.

Hyperintensional contexts vs hyperintensional logics

For classical logic the three definitions coincide, but they do not in a non-classical logic. Since the assessment of a logic's hyperintensionality concerns non-classical logics only, this is especially important.

Leitgeb's (2019) HYPE claims to be a hyperintensional logic. But what does it mean for a *logic*, as opposed to a context/operator to be hyperintensional?

Some formulae or connectives in the logic must act as hyperintensional operators. But this affects the logic's consequence relation.

Odintsov and Wansing (2021): we should judge whether a logic is hyperintensional based on its logical consequence relation alone.
According to this criterion, HYPE is not hyperintensional.

Odintsov and Wansing's argument

The notion of substitutivity of equivalents salva veritate is captured by the notion of self-extensionality, or congruentiality.

Definition

A logic L with language $\mathcal{L}$ is self-extensional if, given its consequence relation $\vdash_L$, for all formulae φ, ψ, θ of $\mathcal{L}$ and all propositional variables p the following holds:

$$\varphi \dashv\vdash_L \psi \text{ implies } \theta(\varphi/p) \dashv\vdash_L \theta(\psi/p),$$

where $\theta(\varphi/p)$ indicates the result of substituting the formula φ for the variable p in θ.

A logic is said to be hyperintensional iff it is not self extensional.

Odintsov and Wansing's argument

HYPE is self-extensional (Leitgeb 2019, Observation 13). Therefore, it is not hyperintensional.

Objectives for this talk

Odintsov and Wansing only formalise one of the traditional definitions of hyperintensionality, Cresswell's.

But we can also formalise the two other notions of hyperintensionality, obtaining different formal criteria.

The aim for the rest of the talk will be to formalise the remaining two criteria and to assess HYPE against them.

This will serve to highlight some peculiarities of HYPE and compare HYPE to other non-classical logics which could be said to be hyperintensional.

The Hilbert calculus $\mathbf{QN}^\circ$ (Fischer et al., 2021)

Propositional language $\mathcal{L}$ with logical symbols $\wedge, \vee \neg, \rightarrow, \bot$ and a countable set of propositional variables Prop. Closing Prop under logical connectives gives rise to the set Form.

$$A \rightarrow (B \rightarrow A) \qquad A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$A \wedge B \rightarrow A \qquad A \wedge B \rightarrow B$$

$$A \rightarrow A \vee B \qquad B \rightarrow A \vee B$$

$$A \rightarrow (B \rightarrow A \wedge B) \qquad (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$$

$$\neg\neg A \rightarrow A \qquad A \rightarrow \neg\neg A$$

$$\frac{A \qquad A \rightarrow B}{B} \qquad\qquad \frac{A \rightarrow B}{\neg B \rightarrow \neg A}$$

$\mathbf{QN}^\circ$ is equivalent to the Gentzen-style calculus $\mathbf{G1h_{cd}}$.

Semantics

Definition

An involutive Routley frame is a triple $\mathfrak{F} = \langle W, \leq, * \rangle$ such that:

- W is a non-empty set, which we assume to be countable;
- $\leq$ is a preordering on W;
- $* : W \longrightarrow W$ is an antimonotone function on W such that $w^{**} = w$.

HYPE is sound and complete with respect to the class of involutive Routley frames (Speranski 2021).

A propositional HYPE-model based on an involutive Routley frame is a tuple $\mathfrak{M} = \langle \langle W, \leq, * \rangle, v \rangle$, with $v\colon \texttt{Prop} \longrightarrow W$ a valuation function.

The characterization of the satisfiability of formulae in a HYPE-model $\mathfrak{M}$ is as follows (with $p \in \texttt{Prop}$, $\varphi, \psi \in \texttt{Form}$):

- $\mathfrak{M}, s \vDash p$ iff $s \in v(p)$,
- $\mathfrak{M}, s \vDash \varphi \wedge \psi$ iff $\mathfrak{M}, s \vDash \varphi$ and $\mathfrak{M}, s \vDash \psi$;
- $\mathfrak{M}, s \vDash \varphi \vee \psi$ iff $\mathfrak{M}, s \vDash \varphi$ or $\mathfrak{M}, s \vDash \psi$;
- $\mathfrak{M}, s \vDash \neg\varphi$ iff $\mathfrak{M}, s^* \nvDash \varphi$;
- $\mathfrak{M}, s \vDash \varphi \rightarrow \psi$ iff $\forall y \in W,$ s.th. $s \leq y,$ if $\mathfrak{M}, y \vDash \varphi$ then $\mathfrak{M}, y \vDash \psi$;
- $\mathfrak{M}, s \vDash \top$, $\mathfrak{M}, s \nvDash \bot$.

Consequence relation for HYPE

- The formula ψ follows from $\varphi_1, \varphi_2, \ldots \varphi_n$ in HYPE, i.e. $\varphi_1, \varphi_2, \ldots \varphi_n \vdash_{\mathrm{H}} \psi$ if for every HYPE model $\mathfrak{M}$ and state s $\mathfrak{M}, s \vDash \psi$ whenever $\mathfrak{M}, s \vDash \varphi_1, \ldots \mathfrak{M}, s \vDash \varphi_n$.

Odintsov-Wansing hyperintensionality

Recall from earlier: Odintsov and Wansing define hyperintensionality as the **failure of self-extensionality** in a logic L.

$$\varphi \dashv\vdash_L \psi \text{ does not imply } \theta(\varphi/p) \dashv\vdash_L \theta(\psi/p). \qquad (\mathsf{HYP}_L)$$

Obvious from before that HYPE does not satisfy HYP_L.

$$\varphi \dashv\vdash_H \psi \text{ implies } \theta(\varphi/p) \dashv\vdash_H \theta(\psi/p).$$

Claim

HYP_L is too strong, since it only tracks hyperintensional behaviour at the global level of the consequence relation. Although this is the most straightforward notion of hyperintensionality *for a logic*, it ends up just tracking negation-asymmetry. There are other, weaker notions of hyperintensionality.

Hyperintensionality and classical logic

"[m]easured by the standards of classical logic and semantics, HYPE's $\rightarrow$ and $\leftrightarrow$ are hyperintensional" whereas "relative to HYPE's own standards [...] they are intensional and hence non-extensional"

(Leitgeb 2019, p. 393)

Hyperintensionality and classical logic

Let $\vdash_{CL}$ be the classical consequence relation. For a logic L with consequence relation $\vdash_L$, we could simply define hyperintensionality with respect to classical logic as the failure of the following claim:

$$\varphi \dashv\vdash_{\mathrm{CL}} \psi \text{ implies } \theta(\varphi/p) \dashv\vdash_{\mathrm{L}} \theta(\psi/p).$$

But ...
even if the vocabulary of classical logic and of L are the same, the **meaning of the connectives** will be different, and the interpretation of classical connectives will not be directly transferrable to L.

So this cannot be an adequate definition of hyperintensionality with respect to classical logic.(Odintsov and Wansing, 2021)

Hyperintensionality and classical logic

Assume we have some correspondence between the consequence relation of classical logic and that of L. Then we can define a syntactical translation criterion (STC) - something similar to a Gödel-Gentzen translation:

$$\Gamma \vdash_{CL} \varphi \text{ implies } \Gamma^L \vdash_L \varphi^L, \tag{STC}$$

If such STC is defined, we can define the failure of self-extensionality with respect to classical logic:

$$\varphi \dashv\vdash_{CL} \psi \text{ fails to imply } \theta(\varphi^L/p) \dashv\vdash_L \theta(\psi^L/p) \tag{N-Congr-CL}$$

Hyperintensionality with respect to classical logic, thus, is:

$$\text{If STC, then N-Congr-CL.} \tag{HYP$_{CL}$}$$

HYPE and classical logic

HYPE has several classical recapture properties (Fischer et al. 2021).

$$A \vdash_{\mathrm{CL}} B \text{ iff } A \cup \Phi \vdash_{\mathrm{H}B}$$

where
$\Phi = \{A \vee \neg A | A \in \texttt{Form}\} \cup \{D[A \to C] \leftrightarrow D[\neg A \vee C] | A, C \in \texttt{Form}\}$.

Also, HYPE has a nice relationship to intuitionsitic logic: for the propositional language with logical symbols $\{\wedge, \vee, \bot, \top, \to\}$ the following holds, with $\vdash_{\mathrm{INT}}$ being the intuitionistic consequence relation:

$$A_1, \ldots, A_n \vdash_{\mathrm{INT}} B \text{ iff } A_1, \ldots A_n \vdash_{\mathrm{H}} B$$

Since $\neg\varphi \dashv\vdash_{\mathrm{INT}} \varphi \to \bot$, but the equivalence does not hold in HYPE, we can define a version of the double negation translation for HYPE.

STC for HYPE

$$\varphi^H = (\varphi \to \bot) \to \bot \text{ for } \varphi \in \texttt{Prop}$$
$$(\neg\varphi)^H = \varphi^H \to \bot$$
$$(\varphi \wedge \psi)^H = \varphi^H \wedge \psi^H$$
$$(\varphi \vee \psi)^H = ((\varphi^H \to \bot) \wedge (\psi^H \to \bot)) \to \bot$$
$$(\varphi \to \psi)^H = \varphi^H \to \psi^H.$$

Lemma

$\Gamma \vdash_{CL} \varphi$ *implies* $\Gamma^H \vdash_H \varphi^H$.

Proof of Lemma

Proof.

Recalling the usual Gödel-Gentzen translation for intuitionistic logic (N-translation):

$$\begin{aligned} &\varphi^N = \neg\neg\varphi \text{ for } \varphi \in \texttt{Prop} \\ &(\neg\varphi)^N = \neg\varphi^N \\ &(\varphi \wedge \psi)^N = \varphi^N \wedge \psi^N \\ &(\varphi \vee \psi)^N = \neg(\neg\varphi^N \wedge \neg\psi^N) \\ &(\varphi \to \psi)^N = \varphi^N \to \psi^N. \end{aligned}$$

Since $\neg\varphi \dashv\vdash_{\mathrm{INT}} \varphi \to \bot$ the N-translation is equivalent to the H-translation in intuitionistic logic. Hence $\Gamma \vdash_{\mathrm{CL}} \varphi$ implies $\Gamma^H \vdash_{\mathrm{INT}} \varphi^H$. Since the H-translation is $\neg$-free, by Observation 41 in Leitgeb (2019), $\Gamma^H \vdash_{\mathrm{INT}} \varphi^H$ implies $\Gamma^H \vdash_{\mathrm{H}} \varphi^H$, hence $\Gamma \vdash_{\mathrm{CL}} \varphi$ implies $\Gamma^H \vdash_{\mathrm{H}} \varphi^H$. □

HYPE is not hyperintensional with respect to CL

Lemma

The converse also holds: $\Gamma^H \vdash_H \varphi^H$ *implies* $\Gamma \vdash_{CL} \varphi$, *since the axioms of HYPE are a subset of the axioms of classical logic.*

Corollary

If $A \dashv\vdash_{CL} B$ *then* $A^H \dashv\vdash_H B^H$.

Lemma

For any $\varphi, \psi \in \mathcal{L}_{CL}$, *if* $\varphi \dashv\vdash_{CL} \psi$ *then* $\theta(\varphi^H/p) \dashv\vdash_H \theta(\psi^H/p)$.

Proof.

This follows from the previous corollary and the fact that HYPE satisfies self-extensionality. □

Why do we need hyperintensionality with respect to classical logic for HYPE?

Leitgeb claims that HYPE is hyperintensional with respect to classical logic. This claim is criticized by Odintsov and Wansing as reported above, but Leitgeb could simply be meaning that HYPE is hyperintensional with respect to classical logic in the naïve way. Can this not be an acceptable claim?

Sure, but it's not a very meaningful one: even intuitionistic logic comes out to be hyperintensional with respect to classical logic on this criterion!

$$\neg A \vee A \dashv\vdash_{CL} A \rightarrow A \text{ does not imply } \theta(\neg A \vee A/p) \dashv\vdash_{INT} \theta(A \rightarrow A/p)$$

Self-extensionality and locality

A criterion for hyperintensionality with respect to necessary equivalence is harder to come up with.

- We cannot formulate it as the failure of

$$\Box\varphi \dashv\vdash_L \Box\psi \text{ implies } \theta(\varphi/p) \dashv\vdash_L \theta(\psi/v)$$

  because if in L $\Box\varphi \vdash \varphi$ we get back to Odintsov and Wansing's criterion.
- Therefore we need to define it using the conditional in L - and thus restrict our discussion to logics with a well-behaved conditional which satisfies the deduction theorem.
- Furthermore necessity is distinguished from theoremhood only if we consider semantical validity - to obtain a different criterion from OW's one, we need to consider formulae which are not provably equivalent. This notion of hyperintensionality is therefore not just based on a logic's consequence relation.

A local criterion for hyperintensionality with respect to necessary equivalents

$$\mathrm{HYP}_{\Box}\text{: There is a model } \mathcal{M} \text{ of } L \text{ such that } \mathcal{M} \vDash_L \Box(\varphi \leftrightarrow \psi) \text{ and } \mathcal{M} \nvDash_L \theta(\varphi/p) \leftrightarrow \theta(\psi/p).$$

This notion of hyperintensionality is very weak, so it would perhaps be more appropriate to assign to it the name of *weak hyperintensionality* or *strong intensionality*.

But should it be conflated with intensionality altogether? In classical intensional logics, although $A \leftrightarrow B$ does not entail $\theta(A/p) \leftrightarrow \theta(B/p)$ - because of the presence of non-local connectives such as the modal operators - $\Box(A \leftrightarrow B)$ licenses unrestricted substitution, as a consequence of the **K** axiom.

Neccessity and the HYPE conditional

Intuitively, a definition of necessity in HYPE would look like this:

$$\mathfrak{M}, s \vDash \Box A \text{ iff for all } s' \text{ such that } s \leq s', \mathfrak{M}, s' \vDash A.$$

However, HYPE is monotonic: if $\mathfrak{M}, s \vDash A$, then for all s' such that $s \leq s'$, $\mathfrak{M}, s' \vDash A$. This derives from the semantic clause for the HYPE conditional (and biconditional): $\mathfrak{M}, s \vDash A \rightarrow B$ iff for all s' such that $s \leq s'$, if $\mathfrak{M}, s' \vDash A$, then $\mathfrak{M}, s' \vDash B$.

Consequence

It suffices for a formula to be necessary in a HYPE model to hold at the least state s_l. If a formula holds at s_l then it has the same intension across states.

Therefore we can construct $\mathrm{HYP}_{\Box}$ in HYPE as failure of intersubstitutivity at the least state.

Weak hyperintensionality of HYPE

Lemma

We can construct a HYPE model $\mathfrak{M}$ in which in the least state s_l, $\mathfrak{M}, s_l \vDash A \leftrightarrow B$ but $\mathfrak{M}, s_l \nvDash \theta(A/p) \leftrightarrow \theta(B/p)$.

Proof.

Let $\mathfrak{M}_1 = (\langle W_1, \leq_1, *_1 \rangle, v_1)$ and $\mathfrak{M}_2 = (\langle W_2, \leq_2, *_2 \rangle, v_2)$ be two HYPE models. Let all states s_1 in W_1 and all states s_2 in W_2 be such that $\mathfrak{M}_1, s_1 \vDash A \leftrightarrow B$ and $\mathfrak{M}_2, s_2 \vDash A \leftrightarrow B$. Let all states s_1 in $\mathfrak{M}_1$ be such that $\mathfrak{M}_1, s_1 \vDash \theta(A/p) \leftrightarrow \theta(B/p)$, but let the least state s in $\mathfrak{M}_2$ be such that $\mathfrak{M}_2, s \nvDash \theta(A/p) \leftrightarrow \theta(B/p)$, e.g. for a failure of contraposition. Then we can construct a model $\mathfrak{M}^+ = (\langle W = (W_1 \cup W_2 \cup \{s_l, s_g\}), \leq_+, *_+ \rangle, v_+)$ such that: for all $s_1 \in W_1$, $s_1 \leq_+ s_1'$ iff $s_1 \leq_1 s_1'$, for all $s_2 \in W_2$, $s_2 \leq_+ s_2'$ iff $s_2 \leq_2 s_2'$, and if $s_1 \in W_1$, $s_2 \in W_2$, $s_1 \nleq_+ s_2$ and $s_2 \nleq_+ s_1$; for all $s \in W$, $s_l \leq_+ s$ and $s \leq_+ s_g$; for all $s, s' \in W_1, W_2$, $*_+ = *$; for all $s \in W_1, W_2$, $v_+^{-1}(s) = v_{1/2}^{-1}(s)$, $v_+^{-1}(s_l) = \varnothing$ and $s_g = s_l^*$. Since all states $s \geq_+ s_l$ satisfy $A \leftrightarrow B$ and $\mathfrak{M}^+, s_l \nvDash A$, $\mathfrak{M}^+, s_l \nvDash B$, $\mathfrak{M}^+, s_l \vDash A \leftrightarrow B$. Therefore, according to our definition of necessity, $\mathfrak{M}^+, s_l \vDash \Box(A \leftrightarrow B)$. However, $\mathfrak{M}^+, s_l \nvDash \theta(A/p) \leftrightarrow \theta(B/p)$, by monotonicity. Therefore, formulae which are necessarily equivalent in a model of HYPE cannot be intersubstitutable salva veritate. □

Consequences of weak hyperintensionality

"Weak hyperintensionality" stems from non-locality and monotonicity of the HYPE conditional.

Leitgeb 2019, p. 307: two sentences have the same intension if they have the same truth value at each state. Sentences with the same intension are not substitutable salva veritate in HYPE.

But, since HYPE does not satisfy HYP_L, **theorems** of HYPE are always substitutable salva veritate.

Recap: the HYPE consequence relation is not hyperintensional, but the HYPE conditional can generate hyperintensional contexts within models, because of intensional behaviour + monotonicity.
Hyperintensionality notwithstanding, we shall see that results defined at all states of any model are more well-behaved than local results: this is because **gaps and gluts cancel each other out** in HYPE-models. Much of this "cancelling" is due to interesting interactions between the HYPE conditional and negation.

Concluding remarks

- HYPE is only hyperintensional in a very weak sense, because of the monotonicity of its conditional.
- However, the weak hyperintensionality of HYPE allows us to model hyperintensional operators at the semantic level. None of this hyperintensional behaviour emerges at the level of HYPE's consequence relation.
- HYPE is especially useful because its semantics is very flexible and its consequence relation relatively well-behaved.
- Although Leitgeb claims so, HYPE is *not* hyperintensional with respect to classical logic.
- Odintsov and Wansing's criterion of hyperintensionality is only fine-grained enough to capture a very obvious notion of hyperintensionality.

Thank you

Thank you!

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