Löwenheim-Skolem-Tarski numbers for regularity quantifiers

Christopher Turner

University of Bristol

July 27, 2022

Let $P_0, \ldots, P_n$ be symbols of second order logic, with some canonical interpretation for every first order structure. The *Löwenheim-Skolem-Tarski* number $\mathrm{LST}(P_0, \ldots, P_n)$ is defined to be the least cardinal κ such that for every first order language $\mathcal{L}$ of cardinality less than κ, and for every $\mathcal{L}$ structure $\mathcal{A}$, there is some $\mathcal{L} \cup \{P_0, \ldots, P_n\}$ elementary substructure of $B \prec \mathcal{A}$ of cardinality $<\kappa$.

In [1], Magidor and Väänänen examine the LST numbers of two logical symbols I and Q^{ec}. I (known as the Härtig quantifier) takes two subsets of a structure as parameters, and returns true if they have the same cardinality in the background universe V. Q^{ec} takes in two ordered subsets of a structure as its parameters, and returns true if the orders have the same cofinality. They show that $\mathrm{LST}(I)$ is at least the first inaccessible, that $\mathrm{LST}(I, Q^{\mathrm{ec}})$ is at least the first Mahlo cardinal, and that modulo the consistency of a supercompact the LST numbers can exactly equal these lower bounds.

We expand this with a series of results for intermediate logics between these two extremes. We will begin by developing two new schemes of second order logical symbols R^ϵ and Q^ϵ (for ϵ an ordinal or $\epsilon = \infty$).

The idea behind R^ϵ is that it should take as its parameters a single subset S of a structure, plus some auxiliary information, and return true if the cardinality of S is regular and has Cantor-Bendixson rank $<\epsilon$ in the class of regular cardinals. Meanwhile, Q^ϵ should take two ordered sets (and some auxiliary information) and return true if the two orders have the same cofinality, and that cofinality has C-B rank $<\epsilon$ in the regulars. R^ϵ and Q^ϵ are intuitively similar to I and Q^{ec} respectively, but only apply on those elements of the class Reg of regular cardinals whose Cantor-Bendixson rank is $< \epsilon$.

A more formal definition of R^ϵ is as follows. It has two parameters: a set S and an auxiliary well-order X. It returns true if the order type of X is some ordinal $\delta < \epsilon$, and the cardinality of S has C-B rank precisely δ in the regular cardinals. (So for R^ϵ to be true about S, we it's not quite sufficient for S be regular and of rank below ϵ; we must also have "guessed" that rank using X. This useful restriction turns out not to be a major limitation to the strength of the symbol when working with LST numbers.) The formal definition of Q^ϵ is similar.

We will see that, with the exception of certain large (but set-size) choices of ϵ, the $\text{LST}(I, Q^\epsilon)$ and $\text{LST}(I, R^\epsilon)$ all fit in the interval between $\text{LST}(I)$ and $\text{LST}(I, Q^{\text{ec}})$, and order themselves neatly via a series of inequalities provable in ZFC. In particular, $\text{LST}(I, R^1) = \text{LST}(I, R^0) = \text{LST}(I)$ and $\text{LST}(I, Q^\infty) = \text{LST}(I, Q^{\text{ec}})$.

We will then establish an optimal lower bound for any $\text{LST}(I, Q^\epsilon)$ and $\text{LST}(I, R^\epsilon)$, given the consistency of a large enough supercompact. In all cases, that lower bound turns out to be the least regular cardinal which is not discussed by Q^ϵ and R^ϵ, i.e. the least regular cardinal whose Cantor-Bendixson rank in Reg is precisely ϵ.

Keywords. Löwenheim-Skolem-Tarski numbers, LST, regular cardinals, Härtig quantifier, set theory, model theory

References

[1] Menachem Magidor and Jouko Väänänen. "On Löwenheim–Skolem–Tarski numbers for extensions of first order logic". In: *Journal of Mathematical Logic* 11.01 (2011), pp. 87–113.