Proof-Theoretic Validity and Intuitionistic Logic

This work demonstrates the effect of recent work on proof-theoretic validity on the arguments Dummett (1991) put forward in the Logical Basis of Metaphysics. There Dummett argued that a proof-theoretic meaning theory could support intuitionism and certain antirealist metaphysical positions. It also aims to give some of the history leading up to these results.

In the Logical Basis of Metaphysics, Dummett argues that we can find answers to deep metaphysical questions about the world, the mind, and mathematical reality by constructing a satisfactory theory of meaning. Over half of the book constructs an inferentialist and proof-theoretic theory of meaning. And shows how this theory leads to anti-realist metaphysics. Dummett does not endorse this semantics as the correct one. But his sympathies lie with this proof-theoretic approach.

In the 3 decades since the publication of the book, proof-theoretic semantics has made great technical strides. This allows us to assess some of the claims Dummett made considering this new knowledge. These results appear to undermine Dummett's argument. But we will offer a response to this worry.

Word meanings in proof-theoretic semantics are proof rules. Dummett's proof-theoretic semantics has two interconnected approaches. The first requires that the set of proof rules for a connective be in harmony. This is often spelt-out as the requirement that introducing a term and then eliminating it proves nothing new. The second approach is proof-theoretic validity. Prawitz (1973) developed this semantics in the early seventies. Proof-theoretic validity separates potential proofs into those that are valid and those that are invalid. It does this based on a property like, but not identical to, normalisation. When we talk of the logic of proof-theoretic validity we mean the logic that includes all and only the valid proofs.

Our focus here will be on proof-theoretic validity, the more technical of the two notions. Because it is the one that turns out to have the most surprises in store. Prawitz conjectured, and Dummett hoped, that proof-theoretic validity would be intuitionistic. For Dummett, this is important because it is intuitionistic logic being the correct logic that leads to antirealist metaphysics. Initial results showed that, for subsets of the connectives, Prawitz's conjecture was correct. But it soon became clear that proof-theoretic validity had super intuitionistic features.

To show that Prawitz's conjecture was false took further work. Because Prawitz's initial definition had not clearly defined the meanings of the atomic formulas. Piecha and Schroeder-Heister (2019) showed that for any clarification of Prawitz's definition proof-theoretic validity was not intuitionistic. Rather it was superintuitionistic. In addition: Goldfarb (2016) showed that Dummett's definition was superclassical. Stafford (2021) showed that a clarification of Prawitz's system was inquisitive logic. Oliveira (2021) showed that if you focus on elimination rules it was intuitionistic. Sandqvist (2009) even showed that by fudging the treatment of 'or' you could get classical logic.

The topic considered here is what do all these results mean for Dummett's work. Let's

consider the two closest to the definition of proof-theoretic validity Dummett offered. These are Goldfarb's result for a modification of Dummett's definition and Stafford's result for a modification of Prawitz's definition. On the face of it, these results undermine the project. Goldfarb proved that Dummett's definition has superclassical validities. This removes Dummett's definition from serious consideration. Stafford's result proves that Prawitz's definition (spelt out) aligns with inquisitive logic. Inquisitive logic is a well-studied logic. It is a semantics for the joint treatment of questions and assertions. But it is not a semantics for assertions alone because it is not closed under substitution and its closure under substitution isn't decidable. Worse yet the rules of Inquisitive logic are not harmonious. This means that Dummett's two notions come apart. These results appear to demonstrate the failure of the project in the Logical Basis of Metaphysics.

The treatment of the atomic formulas causes these results. Every treatment takes a side on whether you can prove a disjunction without proving one of the disjuncts. It is this opinionatedness of the atomic formulas that lead to inquisitive logic.

Atomic formulas' meanings are always proof rules. But different approaches to proof-theoretic validity disagree on what the proof-theoretic analogue of a model is. Prawitz's treatment of atomic formulas has evolved since the seventies. His more recent view holds that the analogue to a model is a static language or a set of proof rules. But his earlier view was that the analogue to a model was a stage in a dynamic language. The language was dynamic in the sense that it was modelled growing as more rules were added. But why wasn't a view considered where the analogue of a model wasn't a stage in a dynamic language but the entire dynamic language?

It turns out that if one does this then the resulting logic is intuitionistic. If this treatment is justified, perhaps Dummett is saved from the technical results knocking at his metaphysical door. We will, however, end with a note of caution about Dummett's move from proof-theoretic semantics to intuitionistic logic via proof-theoretic validity.

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