On Tight Theories

Abstract

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The pervasive independence of set-theoretic assertions over ZFC has spawned a debate between pluralists and monists about set theory. Is there only one correct theory of sets? Monists think the answer is affirmative. They often appeal to the idea of a maximal and unique universe of sets, V, which complete theory we can hope to approximate in ever stronger extensions of ZFC. To them, independence is merely a shortcoming of our theories due to our expressive and epistemic limitations. Every set-theoretic assertion is either uniquely false or uniquely true. Pluralists answer in the negative. By assuming the consistency of ZFC, we have uncovered the consistency of a wide range of mutually incompatible extensions of the theory. Many, if not all, of these extensions seem to have interesting mathematical content in their own right. Pluralists argue that we should embrace all these incompatible theories as equally correct. Some set-theoretic assertions, on this view, are not uniquely true or uniquely false.

We argue in favor of pluralism about set theory. We do so by exploring the philosophical significance of the fact that ZF is a tight theory, where a theory T is tight if no two distinct theories extending T are bi-interpretable. Essentially, our argument is that the tightness of ZF limits the capacity of any extension of ZF to adequately interpret other meaningful mathematical theories; a capacity, we claim, any good foundational theory should have. Since there is plenty of evidence that various incompatible extensions of ZF are mathematically meaningful, we claim no single such extension can be an adequate foundation for mathematics. In conclusion, we point out how pluralism would seem to resolve this issue, by allowing that each such extension T is true of its own *sui generis* T-structure.

Visser (2006) proved that PA is a tight theory. Enayat (2016) proved that ZF also is tight.[1] This latter result might be somewhat surprising. Since Kurt Gödel and Paul Cohen showed us how to construct various models of theories extending ZF on the assumption that various other such extensions have models, set theorists have established a robust

[1]Enayat notes that Visser and Harvey Friedman proved this result independently in unpublished work.

hierarchy of interpretability between various set theories. Many such extensions, such as ZFC+CH and ZFC+¬CH, are mutually interpretable. What Enayat's theorem shows us is that any such mutual interpretability between any distinct extensions of ZF can never rise to the level of bi-interpretability.

We consider the claim that it is at least necessary for two distinct theories T_1 and T_2 to have the same mathematical content (or information, if you like) that they be bi-interpretable. It follows from this assumption and the tightness of ZF that no two distinct extensions of ZF express the same mathematical content. To some extent, this might seem a nice property to a set-theoretic monist. For whatever the complete theory of their unique V is, it is surely an extension of ZF, and by our remarks so far the *content* of that theory is unique. Hence, V is special in some sense, as no bi-interpretable structure can satisfy a distinct theory. Similar remarks could be made about the tightness of PA and the standard model $\mathbb{N}$.

However, if bi-interpretability really is a necessary condition on two theories having the same content, it seems tightness also limits the expressive power of any extension of a tight theory in important ways. In particular, it seems that, for no two distinct extensions of ZF or PA, can the one theory fully capture the content of the other theory. In the case of PA, this is arguably not a great issue, as there is no antecedent expectation that we should be able to adequately capture the content of all other mathematical theories with the use of PA. In the case of ZF and its extensions, we claim, this limitation is more severe. In addition to being a theory of transfinite hierarchies, like V, set theory is also meant to be a foundation of mathematics.

A foundation, among other things, should be able to adequately interpret the content of any meaningful mathematical theory. Arguably, even the monist should accept this. For even if they thought that, say, some completion of ZFC+CH is the only strictly speaking *true* theory of sets, it seems hardheaded to deny that ZFC+¬CH is at least mathematically *meaningful*. By what we have said so far, however, these two extensions cannot fully capture each other's content, so the monist, by adopting a unique theory of sets, will lose some mathematical content. In general, it seems that no single extension of ZF can play the role of an adequate foundation for mathematics.

We suggest that the more natural response to these observations would be to adopt a form of set-theoretic pluralism. For the pluralist embraces the existence of many distinct universes of sets, satisfying various distinct extensions of ZF. Each such universe has its own *sui generis* concept of set, and there is no expectation that they should be able to fully account for the concepts of set instantiated at other universes.

References

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