Why Ekman-reduction is no reduction (and therefore Ekman's paradox is no paradox)

Sara Ayhan

Ruhr University Bochum

This paper is concerned with questions about reductions in natural deduction and λ-calculus, and their relations to paradoxes both in natural deduction and in sequent calculus. The reductions for our usual connectives, corresponding to β-reductions in λ-calculus, are meant to eliminate unnecessary detours of the following form: There is a formula, called *maximal formula*, which is both the conclusion of an application of an introduction rule of a connective as well as the major premise of an applied elimination rule governing the same connective. It can and has been argued, however, that there are more reductions than the ones that are usually considered (see, e.g., Tennant (1995)). One of those, presented in (Ekman, 1994, 1998), is the following:

$$\dfrac{B \to A \qquad \dfrac{A \to B \qquad \begin{array}{c}\mathscr{D}\\ \vdots \\ A\end{array}}{B}\,{\to}E}{A}\,{\to}E \qquad \rightsquigarrow_{Ekman} \qquad \begin{array}{c}\mathscr{D}\\ \vdots\\ A\end{array}$$

The motivation to discuss this reduction is related to Tennant's (1982) proof-theoretic characterization of paradoxes. According to this, in the context of proof theory paradoxes are to be seen as non-normalizable derivations of $\perp$. They lead from certain paradoxical sentences to a proof of absurdity (or in Curry's paradox the proof of an arbitrary atomic formula), which is not normalizable. The proofs are in non-normal form and all attempts to apply reduction procedures eventually end up with the original proof. Thus, the normalization sequences for such proofs enter a loop, never ending with a proof in normal form.[1] What Ekman wanted to show with his reduction is that we can get a non-normalizable derivation of $\perp$, which is *not paradoxical* in nature, though, because it does not contain any "paradoxical sentences" like the Liar, for example. Thus, if we would accept this reduction, this would show that this feature of looping non-normalizability of a derivation of $\perp$ is not due to paradoxical sentences or connectives but can occur in a system without these, as well.

However, although Ekman-reduction is certainly a reduction in the sense that there is an elimination of what seems to be an unnecessary detour, I will argue that

[1] As Tennant (1995) later refined: it does not have to be a loop but can also be a non-terminating sequence, as in the case of Yablo's paradox.

it is not an acceptable reduction. There have been several different ways in the literature to respond to Ekman, e.g. (Schroeder-Heister/Tranchini, 2017) or responding to this respectively (Tennant, 2021). Schroeder-Heister and Tranchini argue that Ekman-reduction needs to be rejected because it would lead to a trivialization of identity of proofs in the sense that every derivation of the same conclusion would have to be identified (if reductions are taken to preserve identity of proofs). Tennant, on the other hand, points out that the peculiarity of this reduction resolves once we use what he calls *parallelized* elimination rules instead of the usual *serial* ones (as in the derivation above).[2] He claims that if we use the parallelized elimination rules in the construction of Ekman's paradox instead, then the derivation can actually be given in normal form, i.e., we do not get into a looping normalization sequence. Instead, I will give a criterion for the acceptability of reductions which will rule out Ekman-reduction from the beginning, i.e. whatever representation of rules we use, there is no looping normalization sequence. I will motivate this claim by two points. First of all, I will show that the redundancies we observe in a natural deduction representation of Ekman's paradox are not present if we transfer it into a derivation in sequent calculus, *although* it is indeed possible to transfer the general Ekman-redundancy and -reduction to sequent calculus. Secondly, I will argue that the question, which reductions we accept in our system, is not only important if we see them as generating a theory of proof identity but is also decisive for the more general question whether a proof has meaningful content.

My method will be to exploit the Curry-Howard-correspondence (see, e.g., Sørensen/ Urzyczyn (2006)) and look at proof systems annotated with λ-terms (in Curry-style). These make the structure of our derivations explicit and allow a much easier way to compare and transfer derivations of natural deduction and sequent calculus. Since both these points are important for my aim, I think it is advantageous to use term-annotated systems. This will allow us to show in a much simpler way what is wrong with potential reductions and why they should not be admitted in our system.

A comparison between Ekman-reduction and Ekman's paradox in natural deduction vs. sequent calculus then shows that there is something odd about it, which can itself be used as an argument against the proposed similarity to paradoxes. A reflection on reductions of terms gives us further means to show what is essentially wrong with this reduction: it allows to reduce a term of one type to the term of an *arbitrary* other. If we take reductions as inducing an identity relation then that would force us to identify proofs of different *arbitrary* formulas. But even if we reject this assumption (some people do not find this theory of proof identity very compelling), I will argue that allowing such reductions would render derivations in such a system *meaningless*. Therefore, I will propose a criterion for the acceptability of reductions that ensures the non-triviality of our system both with respect to identity of proofs as well as to their meaning.

[2]The parallelized elimination rules correspond to what elsewhere, e.g. in (Negri/von Plato, 2001), is called *general* elimination rules.

Proof-theoretic semantics, Reductions, Paradox, Natural deduction, Sequent calculus.

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