Continuum hypothesis in the ω-cofinality model

Inner model theory is a field of set theory that studies inner models, that is, transitive models of ZF that contain all the ordinals. The most extensively studied and historically the first inner model is Gödel's L, the class of constructible sets, which is also the smallest inner model. Scott's result showed in the 1960s that in the presence of large cardinals L cannot be all of V. Since then, inner model theorists have endeavoured to find inner models that have various large cardinals in them but that, on the other hand, have a canonical and well-understood structure similar to that of L.

In a recent novel approach Kennedy, Magidor and Väänänen [1] introduced several inner models obtained by replacing first-order logic by a stronger logic $\mathcal{L}^*$ in the definition of the constructible hierarchy. So for a logic $\mathcal{L}^*$, the corresponding hierarchy is defined as follows:

$$\begin{aligned} L'_0 &= \emptyset \\ L'_{\alpha+1} &= \mathrm{def}_{\mathcal{L}^*}(L'_\alpha) \\ L'_\lambda &= \bigcup_{\alpha<\lambda} L'_\alpha \text{ for limit } \lambda, \end{aligned}$$

where $\mathrm{def}_{\mathcal{L}^*}(L'_\alpha)$ means the set of subsets of L'_α that are $\mathcal{L}^*$-definable over L'_α. As in the case of L, the model of interest is $C(\mathcal{L}^*) =_{\mathrm{def}} \bigcup_{\alpha\in\mathrm{On}} L'_\alpha$, the union of the levels. Assuming the logic satisfies some undemanding conditions, the resulting model $C(\mathcal{L}^*)$ will be a model of ZFC. An example of this type of model studied previously is HOD, the class of hereditarily ordinal definable sets, which is obtained by using full second order logic in the construction of L. However, the approach of strong logics had not been studied systematically before.

An important class of models studied in [1] are C^*_κ, the models obtained by adding Shelah's [3] $\kappa-$cofinality quantifier $Q^{\mathrm{cf}}_\kappa xy$ to first order logic. The quantifier $Q^{\mathrm{cf}}_\kappa xy$ says that the pairs (x, y) of elements of the model satisfying the subformula form a linear order of cofinality κ. It can be proved that the resulting model C^*_κ is $L[\{\alpha \in \mathrm{On} : \mathrm{cf}(\alpha) = \kappa\}]$, the class of sets constructible relative to the class of ordinals of cofinality κ. Of special interest is the ω-cofinality model $C^* =_{\mathrm{def}} C^*_\omega$, which is the focus of our research.

Many properties of C^* are proved in [1]. It is known that C^* is closed under sharps, that is, $x^\sharp$ is in C^* for any $x \in C^*$ such that $x^\sharp$ exists. All Dodd-Jensen mice are in C^*, so the Dodd-Jensen core model K is contained in C^*.

On the other hand many central properties remain unsolved. It is not known if there is a measurable cardinal in C^*. Conversely, if there is a measurable cardinal in V, then V cannot be C^*. Importantly, it is not known if the continuum hypothesis holds in C^*. It can be proved that the continuum of C^* is at most $\aleph_2^V$ and that for any V-cardinal $\aleph_\alpha$

with $\alpha > 0$, $C^* \models 2^{\aleph_\alpha^V} \leq \aleph_{\alpha+1}^V$. But of course there may be C^*-cardinals between any $\aleph_\alpha^V$ and $\aleph_{\alpha+1}^V$. Indeed, if there is a Woodin cardinal, then $\aleph_1^V$ is a Mahlo cardinal in C^*, and if there is a proper class of Woodin cardinals, then regular $\aleph_\alpha^V$, $\alpha \geq 2$, are indiscernible in C^*.

Although the status of the continuum hypothesis in C^* is unknown, in [1] it is proved that assuming three Woodin cardinals and measurable cardinal above, there is a Turing cone $\{y \subset \omega : y \geq_T x\}$ of reals such that the continuum hypothesis holds in $C^*(y)$ for all y in the cone. $C^*(y)$ is like C^* but the construction is made relative to the real y as an additional predicate. The argument uses stationary tower forcing to show that the set $A =_{\text{def}} \{y \subset \omega : C^*(y) \models \text{CH}\}$ is a Σ^1_4 set of reals. The Woodins and the measurable cardinal guarantee Σ^1_4 determinacy, which by a result of Martin [2] implies that either A or its complement contain a cone, which is then used to derive the conclusion.

Our main research topic is the continuum hypothesis and the generalized continuum hypothesis in C^*. To study the possibility of GCH in C^*, we aim to extend the above result and find, assuming many enough Woodins and a measurable above, a Turing cone of reals y such that GCH holds in $C^*(y)$ also for uncountable cardinals. We can show using almost disjoint coding and arguments similar to the ones used in [1] that there are Turing cones of reals y such that GCH holds in $C^*(y)$ for small uncountable cardinals. Currently it is not known how far upwards this can be continued. A major difficulty is going beyond $\aleph_1^V$ since our arguments use the fact that a Woodin cardinal implies that $\aleph_1^V$ is inaccessible in $C^*(y)$ for any real y. If we are able to get a cone of reals y such that GCH holds in $C^*(y)$ up to $\aleph_2^V$, then, assuming a proper class of Woodins, we have full GCH in any such $C^*(y)$ by the indiscernibility of regular $\aleph_\alpha^V$, $\alpha \geq 2$.

References

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[3] Saharon Shelah. Generalized quantifiers and compact logic. *Trans. Amer. Math. Soc.*, 204:342–364, 1975.