Krasner hyperfields and the model theory of valued fields

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The model theory of valued fields has been extensively studied after 1965, when the work of Ax-Kochen and Ershov had found remarkable applications in number theory [**1**].

Central for the problem of quantifier elimination for henselian valued fields (K, v), is the study of the multiplicative quotient group $K^\times/1 + \mathcal{M}_v$, where $\mathcal{M}_v$ is the maximal valuation ideal of v. The main point is to understand what kind of structure does the addition of the field K induce on this multiplicative group.

In 2011 Flenner defined a ternary relation to capture this structure introducing his RV-structures [**6**]. The success of RV-structures in the developments of valuation theory has been great. Not only they solve the problem of relative quantifier elimination for henselian valued fields of characteristic 0, but they also find a variety of other fruitful applications (see for instance [**2**, **7**]).

We noticed that the problem of understanding the additive structure of $K^\times/1 + \mathcal{M}_v$ had already been considered, not in relation to the model theory of valued fields, by Krasner in 1957 [**8**]. The approach of Krasner is a priori different from the one of Flenner: following the developments of Marty's work [**10**] who has proposed hypergroups as a generalization of the concept of group where the operation is allowed to be multivalued, Krasner observed that the additive structure of $K^\times/1+\mathcal{M}_v$ is that of a particular kind of hypergroup. The resulting structure with a multivalued addition and an ordinary multiplication is called by Krasner a hyperfield.

When dealing with the model theory of structures with a multivalued operation + one possibility is to encode it with the ternary relation $z \in x + y$. In the case of $K^\times/1 + \mathcal{M}_v$ we obtain in this way the same relation that Flenner defined 54 years after Krasner when introducing his RV-structures.

Many authors have considered Krasner hyperfields after him for different purposes [**3**, **4**, **9**, **11**, **5**]. Nowadays, the theory of structures with multivalued operations (including Krasner hyperfields) is experiencing a rapid development.

In fact, many mathematicians are active in the field of hypercompositional algebra and its applications. We believe that our observation can be a starting point to develop a close connection between two apparently distant areas of investigation.

Keywords. Model theory, valued field, hyperfield.

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