Categoricity for Strong Kleene Logics

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Broadly construed, logical inferentialists defend the view that notions in one's logical theory should be introduced by means of inferential relations between formulas.[1] For instance, they argue that the notion of logical consequence should be introduced by means of the derivability relation obtained from a particular proof system as opposed to being introduced via semantic notions such as truth-preservation. Even though the strong relationship between two different ways of introducing logical consequence can be shown by soundness and completeness proofs, whether we can *define* the semantic notions in one's logical theory by means of the inferences in a proof system is not at all trivial.[2]

For instance, an inferentialist classical logician may want to define the Boolean truth-conditions of connectives in terms of the provable inferences in a classical proof system. However, it is well-known in the literature, such as Carnap [4], Belnap and Massey [1], Gabbay [12], Humberstone [18], Garson [14], Smiley [27], Bonnay and Westerstahl [2], we cannot define the Boolean truth-functions for some connectives based on the provable inferences of a classical proof system in the logical framework `SET-FMLA` or `FMLA`.[3] This is because of the existence of non-standard models that satisfy the clasical derivability relation in `SET-FMLA`. In other words, proof systems of classical logic in `SET-FMLA` do not determine its intended semantics. In the literature, this is known as the *categoricity problem* or *Carnap's problem.*

Even though there is a growing literature on the categoricity problem of classical logic and its solutions, the notion of categoricity for non-classical logics has not been sufficiently investigated.[4] In this paper, we will investigate whether we can define the semantic notions of the logical theory of Strong

[1] See Gentzen [15], Dummett [8], Prawitz [21], and Brandom [3] for the pioneers of inferentialism.

[2] See Restall [23], Murzi and Steinberger [20] and Garson [14] for different inferentialist motivations to define the semantic notions in one's logical theory by means of inferences.

[3] See Humberstone [18] for logical frameworks.

[4] There are some notable exceptions such as Hjortland [17], Rumfitt [24], Johnson [19], Restall [23], Garson [13]. Additionally, some discussions in French and Ripley [11] and in Chemla and Egré [5] also deal with issues quite related to the notion of categoricity without taking it as their primary concern.

Kleene Logic (**K3**) and Logic of Paradox (**LP**) based on their inferences.[5] In particular, we will apply the strategies discussed in Belnap and Massey [1] and Bonnay and Westerstahl [2] to show their categoricity.

The structure of the paper is as follows: First, we will introduce some syntactic and semantic concepts to define categoricity precisely.[6] That is, we will say that a logic, conceived as a derivability relation, is categorical for its intended semantics with respect to a valuation space just in case the set of valuations induced by the derivability relation is precisely its intended set of valuations. Then, we discuss the categoricity problem with respect to Classical Logic and briefly discuss various different strategies of solving it. Similar to the way it is laid out in Bonnay and Westerstahl [2], we classify these different strategies into two broad categories, (1) the syntactic strategy, where focus is on the structural features of the consequence relation as in Carnap [4], Shoesmith and Smiley [26], Smiley [27], Rumfitt [25], and (2) the semantic strategy, where the focus is on the semantic restrictions on the valuation spaces as in Belnap and Massey [1], Bonnay and Westerstahl [2] and Church [7].

Then, we introduce the proof systems $\mathbb{LP}$ and $\mathbb{K}3$, and show that the consequence relations defined by them ($\vdash_{\mathbb{LP}}$ and $\vdash_{\mathbb{K}3}$) are not categorical for their Strong Kleene Semantics. We then show that, unlike Classical Logic, the semantic strategies discussed in Belnap and Massey [1], Bonnay and Westerstahl [2], and Church [7] do not work for these logics. That is, neither restricting the valuation space to be truth-functional and non-trivial, nor restricting it to interpret disjunction correctly are sufficient to show the categoricity of $\vdash_{\mathbb{LP}}$ and $\vdash_{\mathbb{K}3}$, because they have non-standard *truth-functions* determined by their inferences. However, we show that stronger restrictions, such as truth-functionality and interpreting the propositional constants correctly, or interpreting negation correctly, deliver the categoricity results. In other words, we can define the notions in their intended semantics from their inferences, but we need to appeal to stronger restrictions on the valuation spaces than the restrictions used for Classical Logic.

Last, we discuss philosophical consequences of these results. We first argue that if one thinks that the meaning of logical constants is determined by their inferential use, then the connectives in $\mathbb{LP}$ and $\mathbb{K}3$ do not have a determined meaning. For neither of the well-motivated solutions provided in Belnap and Massey [1] and Bonnay and Westerstahl [2] can be used in this setting, and the stronger restrictions mentioned above are not independently motivated. Hence, our results show that inferentialists who want to endorse these logics should use the syntactic strategy, i.e., should use more expressive proof systems. We then sketch a bilateralist proof system that can be used to show the categoricity of these logics.

Categoricity, Strong Kleene Logics, Inferentialism

[5] See Priest [22] for an introduction to these logics.

[6] We will appeal to the techniques used in Hardegree [16], Dunn and Hardegree [9], Garson [14] and Humberstone [18].

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