Distributed belief – combining uncertain information

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In *epistemic logic*, knowledge and belief are given precise formal definitions. Knowledge is defined indirectly, as lack of uncertainty. The standard way of giving semantics is through relational models [5]. The relations are interpreted as *indistinguishability*. Assumptions about the relations determine the notion of knowledge or belief.[1] This framework is a simple yet powerful tool with which to study higher-order reasoning, group notions of knowledge, and information dynamics. An important group notion is *distributed knowledge* [4, 3]. It is the "implicit" knowledge of a group: that which would be known if all the information of the individuals was pooled together. In relational semantics it has a straightforward, intuitive definition. It is defined using the intersection of the relations used to define individual knowledge of group members.

Weaker notions of *group belief* are also studied (e.g. in [1]). However, the move from knowledge to belief seems somewhat problematic for *distributed belief* (D). Distributed knowledge combines the knowledge of the individuals in the group, but for distributed belief, the information on which the individual beliefs are based is no longer certain and could be inconsistent. Disagreement in the group can then lead to distributed belief in everything. Looking for similar ways of combining the individual information of members of a group, that avoid this inconsistency, we are currently exploring what we are calling *cautious* distributed belief ($D^{\forall}$). We look at *maximal consistent* subgroups, and require that something is true given the combined information in all such groups.

The models considered are multi-agent Kripke models. For a finite non-empty set of agents A: $\mathcal{M} = \langle S, R, v \rangle$, where S is a set of possible worlds, $R = \{R_a \subseteq S \times S \mid a \in A\}$ assigns a relation to each agent, and v is a valuation function. For $a \in A$ we define the *conjecture set* relative to $s \in S$: $C_a(s) := \{s' \in S \mid sR_as'\}$. This is generalized in the combined conjecture set

[1] I take the central difference between knowledge and belief here to be that the former is factive (if something is known, then it must be true). This corresponds to reflexivity. Knowledge is usually taken to be captured by reflexivity, symmetry and transitivity, and belief by seriality, transitivity and Euclidicity.

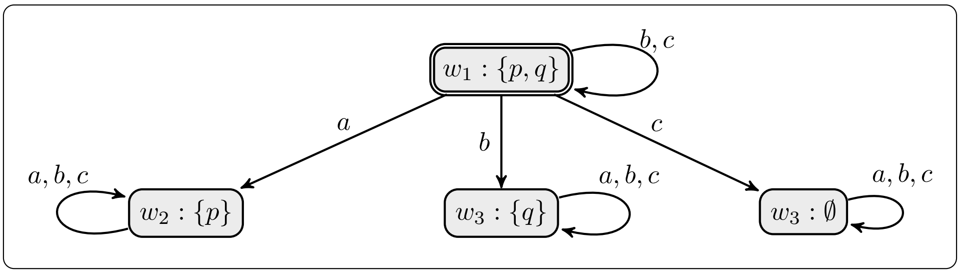
of a group $G \subseteq A$: $C_G(s) := \bigcap_{a \in G} C_a(s)$. Note that standard distributed belief is defined: $\mathcal{M}, s \vDash D_G\,\varphi$ iff $\forall s' \in C_G(s)$: $\mathcal{M}, s' \vDash \varphi$.

When we consider models that are not reflexive, the group can end up inconsistent ($C_G(s) = \emptyset$), even if all members are individually consistent. Consider a group of agents, who are all individually consistent, but disagree on some belief. Then take some belief that everyone in the group agrees on, e.g. that it is raining in Lisbon. The group has distributed belief that it is raining in Lisbon, but it also has distributed belief that it is not raining in Lisbon. This seems undesirable. We therefore look at ways of partially combining the information.

We do this by looking at *maximal consistent* subgroups. A group G is consistent at a world s when $C_G(s) \neq \emptyset$. A subgroup $G' \subseteq G$ is maximally consistent relative to G (in symbols, $G' \subseteq_s^{max} G$), when it is consistent and there is no consistent $G' \subset H \subseteq G$.[2] Cautious distributed belief is defined:

$$\mathcal{M}, s \vDash D_G^{\forall}\,\varphi \qquad \text{iff} \qquad \forall G' \subseteq_s^{max} G, \forall s' \in C_{G'}(s)\text{: } \mathcal{M}, s' \vDash \varphi$$

We look at all the worlds in the conjecture sets of all maximal consistent subgroups. In the example above, we get cautious distributed belief that it is raining in Lisbon, but not in its negation. More interesting are cases where the proposition under consideration is not believed by everyone, and some combination of information is needed. Consider the model $\mathcal{M}$:



In the model, a believes q to be false while b believes it to be true. a believes p, while b is uncertain. c meanwhile, believes that $p \leftrightarrow q$. Consider first the group $G_1 = \{a, b\}$. At w_1, each agent in G_1 makes up a maximal consistent subgroup. Something then needs to be true in $\{w_1, w_2, w_3\}$ to be cautious distributed belief, thus $\mathcal{M}, w_1 \nvDash D_{G_1}^{\forall}\,p$. However, consider the group $G_2 = \{a, b, c\}$. b and c are consistent at w_1, so we end up with the maximal consistent subgroups $\{\{a\}, \{b, c\}\}$, and the worlds we quantify over are $\{w_1, w_2\}$. Thus $\mathcal{M}, w_1 \vDash D_{G_2}^{\forall}\,p$. Both groups have standard distributed belief in p, and in $\neg p$.

I will mention some results for $D^{\forall}$. First some basic results: whenever at least one agent is consistent, the cautious distributed belief of the group is consistent; when the individual relations are reflexive, standard and cautious

[2]Both the definition and the idea is similar to [2]. They use neighbourhood models, and maximal consistency is defined for sets that represent evidence, rather than for groups of agents.

distributed belief coincide; cautious distributed belief, unlike standard, is not coalition monotonic (beliefs can be lost upon adding agents). We have investigated whether some relational properties, associated with knowledge and belief, are inherited from individual relations. Seriality and reflexivity are each inherited without additional assumptions. Symmetry and Euclidicity both require reflexivity to be inherited, while transitivity requires either reflexivity or symmetry. This can be compared with the results for standard distributed belief, found in [1]. With the exception of reflexive models, the situation is the opposite: standard distributed belief inherits transitivity, symmetry and Euclidicity, while seriality requires reflexivity.

Finally, we have some results about the relative expressivity of the language $\mathcal{L}_{D^\forall}$ – the propositional language extended with the cautious distributed belief operators. First, $D^\forall_G$ is definable in terms of D_G:

$$\vDash D^\forall_G \varphi \leftrightarrow \bigwedge_{G' \subseteq G} \Big(\big(\neg D_{G'} \bot \wedge \bigwedge_{G' \subset H \subseteq G} D_H \bot \big) \to D_{G'} \varphi \Big).$$

Thus the language of standard distributed belief, $\mathcal{L}_D$, is at least as expressive as $\mathcal{L}_{D^\forall}$. $\mathcal{L}_{D^\forall}$ is however not as expressive as $\mathcal{L}_D$ (and thus $\mathcal{L}_{D^\forall}$ is strictly less expressive). To show this we define a notion of bisimilarity between models that preserves modal equivalence for $\mathcal{L}_{D^\forall}$ but not for $\mathcal{L}_D$. Finally, we have shown what could be added to $\mathcal{L}_{D^\forall}$ to make it equivalent to $\mathcal{L}_D$. Adding to $\mathcal{L}_{D^\forall}$ the operator $\asymp_G$ for $\emptyset \neq G \subseteq A$, expressing that G is inconsistent, makes the languages equally expressive.

I conclude with some future directions. We are interested in studying the complexity profile for $\mathcal{L}_{D^\forall}$, and in axiomatizing it. There are some challenges with the latter. For example, it seems difficult to find something to replace the coalition monotonicity axiom usually used for $\mathcal{L}_D$. We are also interested in studying a "bold" variant of distributed belief, that quantifies existentially rather than universally over the conjecture sets of maximally consistent subgroups: $\mathcal{M}, s \vDash D^\exists_G \varphi$ iff $\exists G' \subseteq^{max}_s G$, $\forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \vDash \varphi$. One thing worth mentioning is that while the dual use of universal quantification for $D^\forall_G$ means that it is a normal modal operator, $D^\exists_G$ is not a normal modal operator.

cautious distributed belief, distributed knowledge, epistemic logic, expressivity

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