Cut-elimination for Intuitionistic Logic with Actuality resolved via hypersequents

Fabio De Martin Polo

Ruhr University Bochum
Department of Philosophy I

0. Background. S. Niki and H. Omori [13] – motivated by both, a philosophical project whose roots can be found in the work of M. Dummett and by some recent papers by M. De, e.g., [7] – took up the challenge of extending intuitionism from mathematical discourse to empirical discourse. In their joint contribution, Niki and Omori proposed a system of intuitionistic propositional logic expanded via the addition of a so-called actuality operator, denoted '$@A$' (for some formula A). Their logic, namely **IPC@**, is firstly introduced in terms of possible words semantics and axiomatic system. In addition, the authors introduced another proof system, namely a sequent calculus, called **LGJ**, by modifying the characteristic rules of the sequent calculus of Titani [14] and Aoyama [1] for the so-called global intuitionistic logic (**GIPC**). Unfortunately, as noticed in the final part of Omori and Niki's paper, **LGJ** is not Cut-free. Therefore, the authors proposed an open question, that is, whether there is a Cut-free *hypersequent* calculus for **IPC@** [13, p. 477]. This paper aims at solving this problem.

1. Framework & preliminary results. In the first section, we will propose an alternative proof-theoretic characterization of **IPC@** by employing the well-established framework of hypersequents, i.e., a simple and natural generalization of sequents to multisets of sequents (see, among others, [2, 3, 4], [12], [9, 209ff]). Formally, a hypersequent is a structure of the form: $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, where each $\Gamma_i \Rightarrow \Delta_i$ (for $i = 1, \ldots, n$) is an ordinary sequent. Γ_i, Δ_i are multisets of formulas and we call them *internal contexts*. If Δ_i contains at most one formula, then the sequent is said to be *single-succedent*. '$\mathcal{G}$','$\mathcal{H}$', $\ldots$, denote side hypersequents and we refer to them as *external contexts*. Finally, each hypersequent $\mathcal{G}$ is a multiset of ordinary sequents.[1]

[1] Note that the *interpretation* of a sequent, denoted $\mathcal{I}(\Gamma \Rightarrow \Delta)$, is defined as $\bigwedge \Gamma \rightarrow \bigvee \Delta$, where $\bigwedge \Gamma$ ($\bigvee \Delta$) stands for the conjunction (disjunction) of formulas belonging to Γ (Δ). The interpretation of a hypersequent $\mathcal{I}(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n)$ is defined as: $\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1 \vee \cdots \vee \bigwedge \Gamma_n \rightarrow \bigvee \Delta_n$. Finally, $\bigwedge[\,] = \top$ and $\bigvee[\,] = \bot$.

Once all preliminary notions are fixed, we will introduce our intended hypersequent system (called, **HIPC@**). Roughly, for the propositional connectives $(\wedge, \vee, \rightarrow)$ we just add hypersequent versions of Maehara's sequent rules for intuitionistic logic (see [11] and, e.g., [4]). For the actuality operator, instead, we formulate a new group of inference rules (see Figure 1) to formalize its main properties. In particular, to ensure completeness we add a rule specifically constructed to provide terminating derivations of two distinctive axiom schemas of **IPC@**, namely, $@(A \vee B) \rightarrow (@A \vee @B)$ and $@A \vee (@A \rightarrow B)$.[2] The results contained in this section include proofs of soundness and completeness for **HIPC@**, as well as a comparison to other related works. A particular attention will be paid in understanding, on the one hand, the relationship between **HIPC@** and **LGJ**, and on the other, the differences (and similarities) between **HIPC@** and a hypersequent-style formulation of **GIPC** (namely, **HGI**) due to A. Ciabattoni [5].

2. Cut-elimination. The purpose of the second section is to provide a systematic and uniform proof of Cut-elimination for **HIPC@**.
Hypsersequent calculi usually include, along with logical rules, internal and external structural rules. Intuitively, the former group of rule act on formulas, while the latter one act on whole sequents. Now, a crucial problem for Cut-elimination proofs arises when one encounters a derivation where either the external or the internal contraction rule was applied:

$$\frac{\mathcal{G} \mid A, A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta}\ \textsc{ic, l} \qquad\qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \textsc{ec}$$

Roughly, in order to avoid problems with the contraction rules we will use more general versions of Cut to perform multiple cuts in different sequents at once, e.g.:

$$\frac{\mathcal{G} \mid \Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n \qquad \mathcal{H} \mid \Sigma \Rightarrow A, \Pi}{\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}}\ \textsc{cut}_1$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \qquad \mathcal{H} \mid \Sigma_1 \Rightarrow [A]^{\lambda_1}, \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow [A]^{\lambda_n}, \Pi_n}{\mathcal{G} \mid \mathcal{H} \mid \Gamma^{\lambda_1}, \Sigma_1 \Rightarrow \Delta^{\lambda_1}, \Pi_1 \mid \cdots \mid \Gamma^{\lambda_n}, \Sigma_n \Rightarrow \Delta^{\lambda_n}, \Pi_n}\ \textsc{cut}_2$$

[2] An early single-succedent formulation of the hypersequent calculus for **IPC@** included L@, R@ and SPLIT@, plus:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow @A \mid \Gamma \Rightarrow @B}{\mathcal{G} \mid \Gamma \Rightarrow @(A \vee B)}\ \textsc{r@}\vee \qquad\qquad \frac{\mathcal{G} \mid @A, \Gamma \Rightarrow C \qquad \mathcal{G} \mid @B, \Gamma \Rightarrow C}{\mathcal{G} \mid @(A \vee B), \Gamma \Rightarrow C}\ \textsc{l@}\vee$$

Unfortunately, such a formulation faces serious troubles with Cut-elimination. Indeed, the formula $@(\top \wedge (A \vee B)) \Rightarrow @A \vee @B$ (where $\top$ is some tautology), provable by an application of CUT on the left with the sequent $@(\top \wedge (A \vee B)) \Rightarrow @(A \vee B)$, has no CUT-free proof. In other words, the elimination strategy fails when the premises of CUT are derived by L@$\vee$ and L@, and R@$\vee$ and R@, respectively.

Special logical rules:

$$\frac{\mathcal{G} \mid @\Gamma \Rightarrow A}{\mathcal{G} \mid @\Gamma \Rightarrow @A}\ \text{R@}^{*} \qquad \frac{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid @A, \Gamma \Rightarrow \Delta}\ \text{L@}$$

Special structural rule:

$$\frac{\mathcal{G} \mid @\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{G} \mid @\Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}\ \text{SPLIT@}$$

*R@ is single-succedent.

Figure 1: $\mathbf{HIPC}^{@}$

Note that $[A]^{\boldsymbol{\lambda}_i}$ denotes a multiset with $\boldsymbol{\lambda}_i$ occurrences of the formula A, while the notation $\Sigma^{\boldsymbol{\lambda}_i}$ ($\Gamma^{\boldsymbol{\lambda}_i}$, $\Delta^{\boldsymbol{\lambda}_i}$, $\Pi^{\boldsymbol{\lambda}_i}$) indicates the result of deleting $\boldsymbol{\lambda}_i$ occurrences of A from Σ (Γ, Δ, Π).
Beside these considerations, we will introduce some auxiliary concepts, such as the notions of *marked rule instance* and *substitutive rule*, and present the proof of the Cut-elimination theorem by relying on two *reduction* lemmas. Intuitively, the proof strategy amounts to shift applications of Cut upwards in derivations in order to reduce either the cut-rank (i.e., the maximal complexity of the cut-formula) or the number of applications of Cut on formulas with a specific cut-rank.[3]

3. Conclusion & further research. In conclusion, we will suggest how the framework proposed throughout the talk can be accommodated to prove results on logics in the vicinity of **IPC**@ by giving some concrete examples and consider some methodological questions connected to hypersequent calculi.

Keywords. Intuitionistic logic, Actuality, Sequent calculus, Hypersequent, Cut elimination

References

[1] H. Aoyama. "The semantic completeness of a global intuitionistic logic". In: *Mathematical Logic Quarterly* **44**.2 (1998), pp. 167–175.

[2] A. Avron. "Hypersequents, logical consequence and intermediate logics for concurrency". In: *Annals of Mathematics and Artificial Intelligence* **4**.3 (1991), pp. 225–248.

[3] The proof strategy we have in mind was proposed in [12]. However, other applications of such a methodology can be found, for instance, in [6],[10], [8], [9, pp. 224–227].

[3] A. Avron. "The Method of Hypersequents in the Proof Theory of Propositional Non-classical Logics". In: *Logic: from Foundations to Applications, European Logic Colloquium*. Ed. by W. Hodges et al. Clarendon Press, 1996, pp. 1–32.

[4] A. Avron and O. Lahav. "A multiple-conclusion calculus for first-order Gödel logic". In: *Computer Science – Theory and Applications. CSR 2011*. Ed. by A. Kulikov and N. Vereshchagin. Vol. 6651. Lecture Notes in Computer Science. Springer, 2011, pp. 456–469.

[5] A. Ciabattoni. "A proof-theoretical investigation of global intuitionistic (fuzzy) logic". In: *Archive for Mathematical Logic* **44**.4 (2005), pp. 435–457.

[6] A. Ciabattoni, G. Metcalfe, and F. Montagna. "Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions". In: *Fuzzy sets and systems* 161.3 (2010), pp. 369–389.

[7] M. De. "Empirical negation". In: *Acta Analytica* **28**.1 (2013), pp. 49–69.

[8] A. Indrzejczak. "Eliminability of cut in hypersequent calculi for some modal logics of linear frames". In: *Information Processing Letters* **115**.2 (2015), pp. 75–81.

[9] A. Indrzejczak. *Sequents and Trees. An Introduction to the Theory and Applications of Propositional Sequent Calculi*. Springer, Cham, 2021.

[10] H. Kurokawa. "Hypersequent calculi for modal logics extending S4". In: *New Frontiers in Artificial Intelligence. JSAI-isAI 2013*. Ed. by Y. Nakano, K. Satoh, and D. Bekki. Vol. 8417. Lecture Notes in Computer Science. Springer, Cham, 2014, pp. 51–68.

[11] S. Maehara. "Eine Darstellung der intuitionistischen Logik in der klassischen". In: *Nagoya mathematical journal* 7 (1954), pp. 45–64.

[12] G. Metcalfe, N. Olivetti, and D. M. Gabbay. *Proof Theory for Fuzzy Logics*. Vol. 36. Springer, 2008.

[13] S. Niki and H. Omori. "Actuality in Intuitionistic Logic". In: *Advances in Modal Logic*. Vol. 13. 2020, pp. 459–479.

[14] S. Titani. "Completeness of global intuitionistic set theory". In: *The Journal of Symbolic Logic* **62**.2 (1997), pp. 506–528.