Types in Weak Dependence Logic

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Introduction Dependence logic is an extension of first-order logic introduced by Jouko Väänänen [5]. By using the so-called team-semantics previously considered by Wilfrid Hodges [3], Väänänen gave an interpretation to suitable dependence atoms and showed that the expressive power of the logic thus obtained is equivalent to the existential fragment of second-order logic. Yang and Väänänen [6], have also introduced and studied a propositional version of dependence logic. Interestingly, it can be shown that, differently from classical or intuitionistic logic, the first-order version of dependence logic does not satisfy every validity of propositional dependence logic.

In our talk we overcome this mismatch between propositional and first-order dependence logic by defining a weak version of dependence logic, and we prove that it corresponds to propositional dependence logic in the sense of the De Jongh property. Most interestingly, we introduce a suitable notion of model-theoretic types and we apply the algebraic semantics of propositional dependence logic introduced in [4] to study its space of types. In particular, we show that over any model $\mathcal{M}$ its space of types is an Esakia space, whose subspace of maximal elements is the space of types of first-order logic, and which is dual to a suitable dependence algebra.

Propositional Dependence Logic We let the syntax of propositional dependence logic be $\mathcal{L}^{\otimes}_{\mathtt{InqB}} = \{\vee, \wedge, \otimes, \rightarrow, \bot\}$ and we define negation as $\neg\alpha := \alpha \rightarrow \bot$. A *propositional team* t over a set of atomic variables $\mathtt{AT}$ is a set of valuations $v : \mathtt{AT} \rightarrow 2$. The notion of a formula $\phi \in \mathcal{L}^{\otimes}_{\mathtt{InqB}}$ being *true in a team* $t \in \wp(2^{\mathtt{AT}})$ is defined as follows:

$$
\begin{array}{lll}
t \vDash p & \iff & \forall w \in t \; (w(p) = 1) \\
t \vDash \bot & \iff & t = \varnothing \\
t \vDash \psi \wedge \chi & \iff & t \vDash \psi \text{ and } t \vDash \chi \\
t \vDash \psi \otimes \chi & \iff & \exists s, r \subseteq t \text{ such that } s \cup r = t \text{ and } s \vDash \psi, r \vDash \chi \\
t \vDash \psi \vee \chi & \iff & t \vDash \psi \text{ or } t \vDash \chi \\
t \vDash \psi \rightarrow \chi & \iff & \forall s \; (\text{if } s \subseteq t \text{ and } s \vDash \psi \text{ then } s \vDash \chi).
\end{array}
$$

The propositional dependence atom $=\!\!(\vec{x}, y)$ is defined by letting $=\!\!(\vec{x}, y) := \bigwedge_{i\leq n}(x_i \vee \neg x_i) \to (y \vee \neg y)$. We define *propositional dependence logic* as the set $\mathtt{InqB}^{\otimes}$ of formulas valid in any team. We say that a formula $\alpha \in \mathcal{L}^{\otimes}_{\mathtt{InqB}}$ is *standard* if it does not contain the $\vee$ symbol. We recall the following normal form property.

Proposition 1 (Disjunctive Normal Form)**.** Let $\phi \in \mathcal{L}^{\otimes}_{\mathtt{InqB}}$, then there are standard formulas $\alpha_0, \ldots, \alpha_n \in \mathcal{L}^{\otimes}_{\mathtt{InqB}}$ such that $\phi \equiv_{\mathtt{InqB}^{\otimes}} \bigvee_{i\leq n} \alpha_i$.

First Order Weak Dependence Logic Formulas of *weak dependence logic* WDL over a signature $\mathcal{L}$ are defined as follows:

$$\phi ::= t = t' \mid R(t_0, \ldots, t_{n-1}) \mid \bot \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \exists x\phi \mid \forall x\phi;$$

where $t, t', t_0, \ldots, t_{n-1}$ are $\mathcal{L}$-terms and $R \in \mathcal{L}$. We let $\neg\phi := \phi \to \bot$. Let $\mathcal{M}$ be a $\mathcal{L}$-structure and V a set of variables, then a team X of $\mathcal{M}$ with domain V is a set of assignments $s : V \to \mathcal{M}$. If X is a team and $F : X \to \wp(\mathcal{M}) \setminus \{\varnothing\}$, we let $X[F/x] = \{s(a/x) : s \in X, a \in F(s)\}$ and $X[M/x] = \{s(a/x) : s \in X, a \in M\}$.

Definition 2 (Semantics)**.** Let $\mathcal{M}$ be a $\mathcal{L}$-structure, ϕ a $\mathcal{L}$-formula and X a team such that $FV(\phi) \subseteq dom(X)$, we define the *satisfaction* relation $\mathcal{M} \vDash_X \phi$ as follows:

$$\begin{array}{lcl}
\mathcal{M} \vDash_X t = t' & \iff & \forall s \in X, s(t) = s(t') \\
\mathcal{M} \vDash_X R(\vec{t}) & \iff & \forall s \in X, s(\vec{t}) \in R^{\mathcal{M}} \\
\mathcal{M} \vDash_X \bot & \iff & X = \varnothing \\
\mathcal{M} \vDash_X \psi \wedge \chi & \iff & \mathcal{M} \vDash_X \psi \text{ and } \mathcal{M} \vDash_X \chi \\
\mathcal{M} \vDash_X \psi \otimes \chi & \iff & \exists\, Y, Z \text{ such that } Y \cup Z = X \text{ and } \mathcal{M} \vDash_Y \psi \text{ and } \mathcal{M} \vDash_Z \chi \\
\mathcal{M} \vDash_X \psi \vee \chi & \iff & \mathcal{M} \vDash_X \psi \text{ or } \mathcal{M} \vDash_X \chi \\
\mathcal{M} \vDash_X \psi \to \chi & \iff & \forall Y \subseteq X, \mathcal{M} \vDash_Y \psi \text{ entails } \mathcal{M} \vDash_Y \chi \\
\mathcal{M} \vDash_X \exists x\psi & \iff & \exists F : X \to \wp(\mathcal{M}) \setminus \{\varnothing\} \text{ such that } \mathcal{M} \vDash_{X[F/x]} \psi \\
\mathcal{M} \vDash_X \forall x\psi & \iff & \mathcal{M} \vDash_{X[M/x]} \psi.
\end{array}$$

While this system does not contain the standard first-order dependence atom, we can define the restricted dependence $=\!\!(\vec{\phi}, \psi) := \bigwedge_{i\leq n}(\phi_i \vee \neg\phi_i) \to (\psi \vee \neg\psi)$, which expresses the idea of dependence among formulas. We let $\mathcal{L}^{\mathtt{WDL}}$ be the set of all formulas of weak dependence logic in the signature $\mathcal{L}$, $\mathcal{L}^{\mathtt{AT}}$ be the set of all atomic formulas, and $\mathcal{L}^{\mathtt{FO}}$ be the set of formulas of WDL that do not contain $\vee$. It can be shown (see e.g. [2]) that WDL-formulas have the following normal form.

Proposition 3. For any $\phi \in \mathcal{L}^{\mathtt{WDL}}$ we have $\phi \equiv \bigvee_{i<n} \alpha_i$ where $\alpha_i \in \mathcal{L}^{\mathtt{FO}}$ for all $i < n$.

Using the normal forms of InqB and WDL, we can show that weak dependence logic has the De Jongh property with respect to $\mathtt{InqB}^{\otimes}$, showing that WDL is the "right" first order version of propositional dependence logic. Given a map $\sigma : \mathtt{AT} \to \mathcal{L}^{\mathtt{AT}}$, namely a translation of propositional atoms into atomic WDL-formulas, we define ϕ^σ recursively: $p^\sigma = \sigma(p)$, $\bot^\sigma = \bot$ and $(\psi \odot \chi)^\sigma = \psi^\sigma \odot \chi^\sigma$ for $\odot \in \{\vee, \wedge, \otimes, \to\}$. The next result establishes that a propositional formula ϕ is a validity of $\mathtt{InqB}^\otimes$ if and only if every translation ϕ^σ is a validity of WDL.

Theorem 4 (De Jongh Property)**.** For every $\phi \in \mathcal{L}^{\otimes}_{\mathtt{InqB}}$ we have that $\mathtt{InqB}^\otimes \vDash \phi$ if and only if $\mathtt{WDL} \vDash \phi^\sigma$ for all $\sigma : \mathtt{AT} \to \mathcal{L}^{\mathtt{AT}}$.

Types in Weak Dependence Logic In the context of standard first-order logic, types are sets of formulas (possibly) realized by tuples of elements. Logics over team semantics enjoy a distinctive second-order flavour, which motivates different notions of type and realisation. If $B \subseteq \mathcal{M}^n$ we write $\mathcal{M} \vDash_B \phi(x_0, \ldots, x_{n-1})$ to mean $\mathcal{M} \vDash_X \phi(x_0, \ldots, x_{n-1})$ where X is the team induced by the relation B, i.e $X = \{s : (x_i)_{i<n} \to \mathcal{M} : (s(x_0), \ldots, s(x_{n-1})) \in B)\}$. An *$n$-type* p over A is a set of formulas $\phi(x, a)$, where $x = (x_0, \ldots, x_n)$ and $a \in A^m$, $m < \omega$. An n-type p is *realized* in $\mathcal{M}$ if there is a nonempty n-ary relation $B \subseteq \mathcal{M}^n$ such that $\mathcal{M} \vDash_B \phi(x, a)$ if $\phi(x, a) \in p$. An n-type p is *consistent* over $\mathcal{M}$ if there is an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ such that p is realized in $\mathcal{N}$. An n-type p is *complete* if it is consistent and for all n-ary formulas $\phi(x, a)$ either $\phi(x, a) \in p$ or $\neg\phi(x, a) \in p$. An n-type p is *deductively closed* if $p \vDash \psi$ entails $\psi \in p$ for all ψ with variables in $x_0, \ldots, x_{n-1}$. An n-type p is *prime* it is consistent, deductively closed and $\phi \vee \psi \in p$ entails $\phi \in p$ or $\psi \in p$. We let $S_n(\mathcal{M}, A)$ be the set of all complete n-types over $\mathcal{M}$ with parameters from A and $E_n(\mathcal{M}, A)$ be the set of all prime n-types over $\mathcal{M}$ with parameters from A. We let $S(\mathcal{M}, A) := \bigcup_{n<\omega} S_n(\mathcal{M}, A)$ and $E(\mathcal{M}, A) := \bigcup_{n<\omega} E_n(\mathcal{M}, A)$.

By showing that elementary embeddings preserve the satisfaction of WDL-formulas, we give a criterion for the consistency of types in $E_n(\mathcal{M}, A)$ analogous to that of first-order logic.

Proposition 5. Let $A \subseteq \mathcal{M}$ and $p \in E(\mathcal{M}, A)$, the following are equivalent:

(i) p is consistent;
(ii) For all $\phi_0, \ldots, \phi_n \in p$, $\mathcal{M} \vDash \exists x(\bigwedge_{i\leq n} \phi_n)$;
(iii) $Th(\mathcal{M}, A) \cup \{\phi(c^\eta, a) : \phi(x, a) \in q_\eta \text{ and } p \subseteq q_\eta \in S(\mathcal{M}, A)\}$ is satisfiable, where $c^\eta = (c^\eta_0, \ldots, c^\eta_n)$ are fresh constant symbols.

For any formula $\phi \in \mathcal{L}^{\mathtt{WDL}}$ we let $[\![\phi]\!] := \{p \in E(\mathcal{M}, A) : \phi \in p\}$. The next theorem provides a characterisation of the space of deductively closed types.

Theorem 6. The set of prime types $E(\mathcal{M}, A)$ forms an Esakia space with ordering $\subseteq$ and basis $\mathcal{B} = \{[\![\phi]\!] : \phi \in \mathcal{L}\} \cup \{[\![\phi]\!]^c : \phi \in \mathcal{L}\}$.

As maximal elements of Esakia spaces form a Stone space under the relative topology [1], we also obtain that the Stone space of maximal points in $E_n(\mathcal{M}, A)$ is exactly the Stone space of first-order types.

Connections to Algebraic Semantics The former study of types and the De Jongh property provide us with an interesting connection with the algebraic semantics of $\mathtt{InqB}^{\otimes}$. A *(classical) dependence algebra* $\mathcal{A}$ is a structure in the signature $\mathcal{L} = \{P, \wedge, \vee, \otimes, \rightarrow, \bot\}$ such that $\mathcal{A}\upharpoonright\{\wedge, \vee, \rightarrow, \bot\}$ is a Heyting algebra, $P^{\mathcal{A}}\upharpoonright\{\wedge, \otimes, \rightarrow, \bot\}$ is a Boolean algebra and for all $x, y, z \in \mathcal{A}$:

$$
\begin{aligned}
&(\mathsf{KP}) \quad \neg x \rightarrow (y \vee z) = (\neg x \rightarrow y) \vee (\neg x \rightarrow z);\\
&(\mathsf{Dist}) \quad x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z);\\
&(\mathsf{Mon}) \quad (x \rightarrow z) \rightarrow (y \rightarrow k) = (x \otimes y) \rightarrow (z \otimes k).
\end{aligned}
$$

It was shown in [4] that dependence algebras provide a complete algebraic semantics to $\mathtt{InqB}^{\otimes}$.

Given a structure $\mathcal{M}$, we write $\equiv^{\mathcal{M}}$ for the equivalence relation it induces over sets of formulas. We let $\mathcal{L}^{\mathtt{WDL}}_{\mathcal{M}}$ be the set of equivalence classes of formulas $\mathcal{L}^{\mathtt{WDL}}/\equiv^{\mathcal{M}}$ and $\mathcal{L}^{\mathtt{FO}}_{\mathcal{M}}$ be the set $\mathcal{L}^{\mathtt{FO}}_{\mathcal{M}} := \{[\alpha] \in \mathcal{L}^{\mathtt{WDL}}/\equiv^{\mathcal{M}} : \alpha \in \mathcal{L}^{\mathtt{FO}}\}$. The Lindenbaum-Tarski algebra of $\mathtt{WDL}$ over a structure $\mathcal{M}$ is the algebra:

$$\mathcal{T}(\mathcal{M}) := (\mathcal{L}^{\mathtt{WDL}}_{\mathcal{M}},\ \mathcal{L}^{\mathtt{FO}}_{\mathcal{M}},\ \vee,\ \wedge,\ \otimes,\ \rightarrow,\ \bot);$$

where each operator $\odot \in \{\vee, \wedge, \otimes, \rightarrow\}$ is interpreted by letting $[\phi] \odot [\psi] := [\phi \odot \psi]$ for all $\phi, \psi \in \mathcal{L}^{\mathtt{WDL}}$. The following theorem allows us to relate the space of types of weak dependence logic to the algebraic semantics of $\mathtt{InqB}^{\otimes}$.

Theorem 7. The Lindenbaum-Tarski algebra $\mathcal{T}(\mathcal{M})$ is a dependence algebra and, moreover, $\mathcal{T}(\mathcal{M})$ is the dual of $E(\mathcal{M})$ under Esakia duality.

References

[1] Leo Esakia. *Heyting Algebras: Duality Theory (Trends in Logic)*. Cham: Springer, 2019.

[2] Pietro Galliani. “Upwards closed dependencies in team semantics”. In: *Information and Computation* 245 (2015), pp. 124–135.

[3] W. Hodges. “Compositional Semantics for a Language of Imperfect Information”. In: *Logic Journal of the IGPL* 5.4 (1997), pp. 539–563.

[4] Davide Emilio Quadrellaro. “On intermediate inquisitive and dependence logics: An algebraic study”. In: *Annals of Pure and Applied Logic* (2022), p. 103143.

[5] Jouko Väänänen. *Dependence Logic: A New Approach to Independence Friendly Logic*. Cambridge University Press, 2007.

[6] Fan Yang and Jouko Väänänen. “Propositional Logics of Dependence”. In: *Annals of Pure and Applied Logic* 167.7 (2016), pp. 557–589. DOI: 10.1016/j.apal.2016.03.003.