Borel and Borel* Sets in Generalized Descriptive Set Theory

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Generalized descriptive set theory (GDST) aims at developing a higher analogue of classical descriptive set theory in which ω is replaced with an uncountable cardinal κ in all definitions and relevant notions. In the literature on GDST it is often required that $\kappa^{<\kappa} = \kappa$, a condition equivalent to κ regular and $2^{<\kappa} = \kappa$. In contrast, in this talk we use a more general approach and develop in a uniform way the basics of GDST for cardinals κ still satisfying $2^{<\kappa} = \kappa$ but independently of whether they are regular or singular. This allows us to retrieve as a special case the known results for regular κ, but it also uncovers their analogues when κ is singular. We also discuss some new phenomena specifically arising in the singular context (such as the existence of two distinct yet related Borel hierarchies), and obtain some results which are new also in the setup of regular cardinals, such as the existence of unfair Borel* codes for all Borel* sets. This is a joint work with Luca Motto Ros.

1. Preliminary notions in GDST

Let ν, λ be ordinals, $\nu \geq 2$. We consider the space ${}^{\nu}\lambda = \{f \mid f : \nu \to \lambda\}$ and we set ${}^{<\nu}\lambda = \bigcup_{\beta<\nu} {}^{\beta}\lambda$. We endow ${}^{\nu}\lambda$ with the bounded topology τ_b, i.e. the topology generated by the sets $N_s = \{x \in {}^{\nu}\lambda \mid s \subseteq x\}$, with $s \in {}^{<\nu}\lambda$. The weight of $({}^{\nu}\lambda, \tau_b)$ coincide with its density character and has value $\lambda^{<\nu} = \sup_{\beta<\nu} \lambda^{\beta} = |{}^{<\nu}\lambda|$. Moreover, ${}^{\nu}\lambda$ is completely metrizable if and only if $\mathrm{cof}(\nu) = \omega$. We refer the reader to [**FHK14**] and [**AMR19**] for a detailed exposition of the theory and its basics.

Throughout all the thesis, we work with κ infinite cardinal such that $2^{<\kappa} = \kappa$. We show the spaces ${}^{\kappa}2$ and ${}^{\mathrm{cof}(\kappa)}\kappa$ are homeomorphic, extending to all singular κ's what it is proved in [**DMR**] for $\mathrm{cof}(\kappa) = \omega$.

Theorem 1.1. *Let κ be an infinite cardinal such that* $\mathrm{cof}(\kappa) < \kappa$ *and assume* $2^{<\kappa} = \kappa$.
If $\langle \kappa_i \mid i < \mathrm{cof}(\kappa) \rangle$ *is any sequence cofinal in* κ*, then*[1]*:*

$$^{\kappa}2 \approx {}^{\mathrm{cof}(\kappa)}\kappa \approx \Pi_{i<\mathrm{cof}(\kappa)}\kappa_i.$$

The result is in line with Proposition 6.6 in [**AMR19**]. We also argue that for our purposes, the "right" generalization of the that Baire space is ${}^{\mathrm{cof}(\kappa)}\kappa$, rather than ${}^{\kappa}\kappa$. Notice this is still coherent with what it is done in the regular case, as if $\mathrm{cof}(\kappa) = \kappa$ we retrieve as a special case the known results for regular κ's.

2. Generalized Borel Sets

The collection of κ^+-Borel subsets of the generalized Cantor space it is the smallest κ^+-algebra generated by the τ_b-open sets of ${}^{\kappa}2$. When κ is singular, it can equivalently be described as the smallest κ-algebra containing all τ_b-open sets. Let $X \subseteq ({}^{\kappa}2, \tau_b)$. As in the classical (and regular) case, κ^+-Borel sets can be naturally stratified in a hierarchy whose classes are denoted by $\mathbf{\Sigma}^0_\alpha(X, \kappa^+)$, $\mathbf{\Pi}^0_\alpha(X, \kappa^+)$ and $\mathbf{\Delta}^0_\alpha(X, \kappa^+)$. Alternatively, a new hierarchy natural arises from the definition of κ^+-Borel sets which is available in the singular case. Define by recursion on $\alpha \geq 1$:

$$\begin{aligned} \mathbf{\Sigma}^0_1(X,\kappa) &= \text{ open subsets of X}, \\ \mathbf{\Sigma}^0_\alpha(X,\kappa) &= \left\{ \bigcup_{\gamma<\lambda} A_\gamma \mid \lambda < \kappa \text{ and } X \setminus A_\gamma \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta(X,\kappa) \right\}, \end{aligned}$$

together with $\mathbf{\Pi}^0_\alpha(X,\kappa) = \{X \setminus A_\gamma \mid A_\gamma \in \mathbf{\Sigma}^0_\alpha(X,\kappa)\}$ and $\mathbf{\Delta}^0_\alpha(X,\kappa) = \mathbf{\Pi}^0_\alpha(X,\kappa) \cap \mathbf{\Sigma}^0_\alpha(X,\kappa)$. The two hierarchies are related in the following way:

Theorem 2.1. *Let* $\alpha < \kappa^+$ *be an ordinal* [2]*. Then the followings hold:*

1. *If α is even, then* $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa) = \mathbf{\Sigma}^0_{1+\frac{\alpha}{2}}(X,\kappa^+)$*. Moreover:*
   (a) *if $\alpha = 0$, then* $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa) = \big(\mathbf{\Delta}^0_1(X,\kappa)\big)_{\mathrm{cof}(\kappa)}$*;*
   (b) *if $\alpha = \beta + 1$ is successor, then* $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa) = \big(\mathbf{\Pi}^0_{1+\beta}(X,\kappa)\big)_{\mathrm{cof}(\kappa)}$*;*
   (c) *if α is limit, then* $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa) = \Big(\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X,\kappa)\Big)_{\mathrm{cof}(\alpha)}$.
2. *If α is odd, then* $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa) = \mathbf{\Pi}^0_{1+\alpha}(X,\kappa) \subsetneq \mathbf{\Delta}^0_{1+\frac{\alpha+1}{2}}(X,\kappa^+)$.

[1] The symbol $\approx$ denotes homeomorphism

[2] If $0 \leq \alpha < \kappa^+$ is an ordinal, it can be written uniquely in the form $\alpha = \gamma + n$, with $\gamma = 0$ or $\gamma < \kappa^+$ limit and $n \in \mathbb{N}$. We say that α is even (odd) if n is even (odd). If α is even, then $\alpha = \gamma + 2m$. In this case, we set $\frac{\alpha}{2} = \gamma + m$.

The Theorem also allows to determine the closure properties (under unions and intersections) of the classes $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa)$, $\mathbf{\Pi}^0_{1+\alpha}(X,\kappa)$. Finally, we prove that if the ordinal $0 \leq \alpha < \kappa^+$ is odd, then $\mathbf{\Sigma}^0_{1+\alpha}({}^\kappa 2,\kappa)$ has not a complete element, hence neither a ${}^\kappa 2$-universal one. This should be considered with the fact that when $2^{<\kappa} = \kappa$ the classes $\mathbf{\Sigma}^0_{1+\alpha}(X,\kappa^+)$ and $\mathbf{\Pi}^0_{1+\alpha}(X,\kappa^+)$ have both universal and complete elements.

3. Generalized Analytic Sets

A subset of ${}^\kappa 2$ is (κ,μ)-analytic if it is either empty or continuous image of a closed subset of ${}^\mu\kappa$. When $\mu = \mathrm{cof}(\kappa)$, we simply write κ-analytic and we denote the class of all κ-analytic sets by $\mathbf{\Sigma}^1_1(\kappa)$. Definitions of the classes $\mathbf{\Pi}^1_1(\kappa)$ and $\mathbf{\Delta}^1_1(\kappa)$ naturally follow. It is shown in [**HK18**] that $\mathbf{\Delta}^1_1(\kappa) \subsetneq \mathbf{\Sigma}^1_1(\kappa)$ when κ is a regular cardinal and that there exists a $\mathbf{\Delta}^1_1(\kappa)$ subset of ${}^\kappa 2$ which is not κ^+-Borel. If κ is singular, the following theorems holds.

Theorem 3.1. *Le κ be any uncountable cardinal and assume $2^{<\kappa} = \kappa$. Then $\Delta^1_1(\kappa) \neq \Sigma^1_1(\kappa)$ and, if κ has uncountable cofinality, κ^+-Borel $\neq \Delta^1_1(\kappa)$.*

Combining this with [**DMR**], we completely determine when the natural generalization of Souslin's theorem holds at a given infinite cardinal κ.

Theorem 3.2. *Let κ be an infinite cardinal. Then $\mathbf{\Delta}^1_1(\kappa)$ sets and κ^+-Borel sets coincides if and only if $\mathrm{cof}(\kappa) = \omega$.*

If κ is an infinite cardinal, then every closed $C \subseteq {}^\kappa 2$ is a retract of the whole space if and only if $\mathrm{cof}(\kappa) = \omega$. Moreover, we generalize Theorem 1.5 from [**LS15**] to any κ with uncountable cofinality.

Theorem 3.3. *Let κ be a cardinal such that $\mathrm{cof}(\kappa) > \omega$. Then, there is a closed non-empty subset of ${}^{\mathrm{cof}(\kappa)}\kappa$ that is not the continuous image of ${}^{\mathrm{cof}(\kappa)}\kappa$.*

As a consequence of Theorem 3.3 and [**DMR**], the characterization of non-empty κ-analytic sets as continuous image of ${}^{\mathrm{cof}(\kappa)}\kappa$ holds if and only if $\mathrm{cof}(\kappa) = \omega$.

4. Borel* Sets and Unfair codes

As D. Blackwell showed, in the classical setting Borel sets can be equivalently defined using games on well-founded trees. The generalization of those games leads to the class of (κ,μ)-Borel* subsets of ${}^\kappa 2$, i.e. the collection of subsets of ${}^\kappa 2$ that admits a (κ,μ)-Borel* code. When $\mu = \mathrm{cof}(\kappa)$, we simply write κ-Borel* sets (or codes). For every cardinal κ, the classes of κ^+-Borel and (κ^+,ω)-Borel* coincide. Moreover, we observe if $\mathrm{cof}(\kappa) = \omega$ then κ^+-Borel $= \mathbf{\Delta}^1_1(\kappa) = \kappa$-Borel*. In contrast:

Theorem 4.1. *Let κ be any cardinal of uncountable cofinality. Then:*

$$\kappa^+\text{-Borel} \subsetneq \mathbf{\Delta}^1_1(\kappa) \subseteq \kappa\text{-Borel}^* \subseteq \mathbf{\Sigma}^1_1(\kappa).$$

It's worth noticing $\mathbf{\Delta}^1_1(\kappa) \subseteq \kappa$-Borel* is a consequence of the following separation theorem which is the analogue of Lusins', but replacing Borel sets with κ-Borel* sets:

Theorem 4.2. *Let κ be an uncountable cardinal. Suppose A and B are disjoint $\mathbf{\Sigma}^1_1(\kappa)$ set. Then there are C_0 and C_1 κ-Borel* sets such that $A \subseteq C_0$, $B \subseteq C_1$ and C_0 and C_1 are duals.*

The class of κ-Borel* sets is closed under unions and intersections of size κ. A κ-Borel* code is said to be determined if each of its corresponding games are determined and a κ-Borel* set is determined if it admits such a code. Based on [**MV93**], we prove the following:

Theorem 4.3. *Let κ be any uncountable cardinal. Then $\mathbf{\Delta}^1_1(\kappa)$ sets are exactly the sets admitting a determined κ-Borel* code.*

We introduce in this context the notion κ-Borel* code *unfair* for player **I** (or **II**), that is a code in which player **I** (or **II**) does not have a winning strategy in the corresponding games. The next theorem shows that the existence of a κ-Borel* code which is not determined is equivalent to the existence of an undetermined Gale-Stewart clopen game on κ and of length $\mathrm{cof}(\kappa)$.

Theorem 4.4. *Let κ be an uncountable cardinal. The followings are equivalent:*

1. *There exists an undetermined κ-Borel* code.*
2. *There exists a Gale-Stewart game $G_{\mathrm{cof}(\kappa)}(A)$ on κ of length $\mathrm{cof}(\kappa)$ with clopen payoff set $A \subseteq {}^{\mathrm{cof}(\kappa)}\kappa$ which is not determined.*
3. *Every κ-Borel* set admits an unfair code.*

As a corollary we get:

Theorem 4.5. *Let κ be an infinite cardinal.*

- *If $\mathrm{cof}(\kappa) = \omega$, then all κ-Borel* codes are determined and κ^+-Borel $= \mathbf{\Delta}^1_1(\kappa) = \kappa$-Borel*.*
- *If $\mathrm{cof}(\kappa) > \omega$, then all κ-Borel* sets admits an unfair Borel* code and κ^+-Borel $\subsetneq \mathbf{\Delta}^1_1(\kappa) \subseteq \kappa$-Borel*.*

Moreover, sets in $\mathbf{\Delta}^1_1(\kappa)$ are exactly those κ-Borel which have both unfair and determined κ-Borel* codes.*

Descriptive set theory, Borel sets, analytic sets, Baire space, Cantor space, singular cardinals.

References

[AMR19] Andretta, A., Motto Ros, L., Souslin quasi-orders and bi-embeddability of uncountable structures. Accepted for publication on the Memoirs of the American Mathematical Society, (2019).

[DMR] Dimonte, V., Motto Ros, L., Generalized descriptive set theory at singular cardinals of countable cofinality. In preparation (2022).

[FHK14] Friedman, S.D., Hyttinen, T., Kulikov, V. Generalized descriptive set theory and classification theory. Memoirs of the American Mathematical Society, Vol. 230 No. 1081 (2014).

[HK18] Hyttinen, T., Kulikov, V. Borel* Sets in the Generalized Baire Space and Infinitary Languages. Jaakko Hintikka on Knowledge and Game Theoretical Semantics, Vol. 12. Outstanding Contributions to Logic. Springer (2018), 395–412.

[LS15] Lücke, P., Schlicht, P., Continuous images of closed sets in generalized Baire spaces. Memoirs of the American Mathematical Society, Vol. 230 No. 1081 (2014).

[MV93] Mekler, A., Väänänen, J., Trees and -subsets of ${}^{\omega_1}\omega_1$. Journal of Symbolic Logic, Vol. 58 No. 3 (1993).