Government of Karnataka

MATHEMATICS

9

Ninth Standard

Part-I

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

Karnataka Textbook Society (R.)
100 Feet Ring Road, Banashankari 3rd Stage, Bengaluru - 85

FOREWORD

The National Curriculum Framework (NCF), 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the national Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognize that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

This aims imply considerable change is school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather then a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

The National Council of Educational Research and Training (NCERT) appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in science and mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P. Sinclair of IGNOU, New Delhi for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organizations which have generously permitted us to draw upon their resources, material and personnel. We are especially grateful to the members of the National Monitoring Committee, appointed by the Department of Secondary and Higher Education, Ministry of Human Resource Development under the Chairpersonship of Professor Mrinal Miri and Professor G.P. Deshpande, for their valuable time and contribution. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
20 December 2005

Textbook Development Committee

Chairperson, Advisory Group in Science and Mathematics

J.V. Narlikar, *Emeritus Professor*, *Chairman,* Advisory Committee, Inter University Centre for Astronomy & Astrophysics (IUCAA), Ganeshkhind, Pune University, Pune

Chief Advisor

P. Sinclair, Director, NCERT and *Professor of Mathematics* , IGNOU, New Delhi

Chief Coordinator

Hukum Singh, *Professor* (Retd.), DESM, NCERT

Members

A.K. Wazalwar, *Professor and Head*, DESM, NCERT

Anjali Lal, *PGT*, DAV Public School, Sector-14, Gurgaon

Anju Nirula, *PGT*, DAV Public School, Pushpanjali Enclave, Pitampura, Delhi

G.P. Dikshit, *Professor* (Retd.), Department of Mathematics & Astronomy, Lucknow University, Lucknow

K.A.S.S.V. Kameswara Rao, *Associate Professor*, Regional Institute of Education, Bhubaneswar

Mahendra R. Gajare, *TGT,* Atul Vidyalya, Atul, Dist. Valsad

Mahendra Shanker, *Lecturer* (S.G.) (Retd.), NCERT

Rama Balaji, *TGT*, K.V., MEG & Centre, ST. John's Road, Bangalore

Sanjay Mudgal, *Lecturer*, CIET, NCERT

Shashidhar Jagadeeshan, *Teacher and Member,* Governing Council, Centre for Learning, Bangalore

S. Venkataraman, *Lecturer*, School of Sciences, IGNOU, New Delhi

Uaday Singh, *Lecturer*, DESM, NCERT

Ved Dudeja, *Vice-Principal* (Retd.), Govt. Girls Sec. School, Sainik Vihar, Delhi

Member-Coordinator

Ram Avtar, *Professor* (Retd.), DESM, NCERT (till December 2005)
R.P. Maurya, *Professor*, DESM, NCERT (Since January 2006)

ACKNOWLEDGEMENTS

The Council gratefully acknowledges the valuable contributions of the following participants of the Textbook Review Workshop: A.K. Saxena, *Professor* (Retd.), Lucknow University, Lucknow; Sunil Bajaj, *HOD*, SCERT, Gurgaon; K.L. Arya, *Professor* (Retd.), DESM, NCERT; Vandita Kalra, *Lecturer,* Sarvodaya Kanya Vidyalya, Vikas Puri, District Centre, New Delhi; Jagdish Singh, *PGT,* Sainik School, Kapurthala; P.K. Bagga, *TGT,* S.B.V. Subhash Nagar, New Delhi; R.C. Mahana, *TGT,* Kendriya Vidyalya, Sambalpur; D.R. Khandave, *TGT,* JNV, Dudhnoi, Goalpara; S.S. Chattopadhyay, *Assistant Master,* Bidhan Nagar Government High School, Kolkata; V.A. Sujatha, *TGT,* K.V. Vasco No. 1, Goa; Akila Sahadevan, *TGT,* K.V., Meenambakkam, Chennai; S.C. Rauto, *TGT,* Central School for Tibetans, Mussoorie; Sunil P. Xavier, *TGT,* JNV, Neriyamangalam, Ernakulam; Amit Bajaj, *TGT,* CRPF Public School, Rohini, Delhi; R.K. Pande, *TGT,* D.M. School, RIE, Bhopal; V. Madhavi, *TGT,* Sanskriti School, Chanakyapuri, New Delhi; G. Sri Hari Babu, *TGT,* JNV, Sirpur Kagaznagar, Adilabad; and R.K. Mishra, *TGT*, A.E.C. School, Narora.

Special thanks are due to M. Chandra, *Professor* and *Head* (Retd.), DESM, NCERT for her support during the development of this book.

The Council acknowledges the efforts of *Computer Incharge,* Deepak Kapoor; *D.T.P. Operator,* Naresh Kumar; *Copy Editor,* Pragati Bhardwaj; and *Proof Reader,* Yogita Sharma.

Contribution of APC–Office, administration of DESM, Publication Department and Secretariat of NCERT is also duly acknowledged.

ಮುನ್ನುಡಿ

NCF–2005 ಆಧರಿಸಿ ರಚಿತವಾದ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯವಸ್ತುವಿನ ಆಧಾರದ ಮೇಲೆ ರಚಿಸಲಾದ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ 9ನೆಯ ತರಗತಿಯ ಆಂಗ್ಲ ಮಾಧ್ಯಮದ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ಕರ್ನಾಟಕದಲ್ಲಿ ಯಥಾವತ್ತಾಗಿ 2017–18ನೇ ಸಾಲಿನಿಂದ ಜಾರಿಗೊಳಿಸಲಾಗಿದೆ. ಈ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ಒಟ್ಟು 7 ಮಾಧ್ಯಮಗಳಲ್ಲಿ ಭಾಗ–1 ಮತ್ತು ಭಾಗ–2 ಎಂಬುದಾಗಿ ಎರಡು ಪುಸ್ತಕಗಳನ್ನು ಮುದ್ರಿಸಿ ಹೊರತರಲಾಗಿದೆ. NCF–2005ರ ಪಠ್ಯಕ್ರಮದ ಎಲ್ಲ ವೈಶಿಷ್ಟ್ಯಗಳನ್ನು ಈ ಪುಸ್ತಕ ಹೊಂದಿದೆ.

2005ರ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮವು ಈ ಕೆಳಗಿನ ವೈಶಿಷ್ಟ್ಯಗಳನ್ನು ಹೊಂದಿದೆ.

- ಕಲಿಕೆಯನ್ನು ಜೀವನದ ಅವಶ್ಯಕತೆಗಳೊಂದಿಗೆ ಜೋಡಿಸುವುದು.
- ಕಂಠಪಾಠ ವಿಧಾನದಿಂದ ಕಲಿಕೆಯನ್ನು ಮುಕ್ತಗೊಳಿಸುವುದು.
- ಪಠ್ಯಪುಸ್ತಕಗಳ ಹೊರತಾಗಿ ಪಠ್ಯಕ್ರಮವನ್ನು ಶ್ರೀಮಂತಗೊಳಿಸುವುದು.
- ಜ್ಞಾನದ ಅಭಿವೃದ್ಧಿಗೆ ಕಲಿಕಾ ಅನುಭವಗಳನ್ನು ಬಳಸುವುದು.
- ಭಾರತದ ಪ್ರಜಾಸತ್ತಾತ್ಮಕ ನೀತಿಯನ್ವಯ ಮಕ್ಕಳ ಅವಶ್ಯಕತೆಗಳಿಗೆ ತಕ್ಕಂತೆ ಸ್ಪಂದಿಸುವುದು.
- ಶಿಕ್ಷಣವನ್ನು ಇಂದಿನ ಹಾಗೂ ಭವಿಷ್ಯದ ಜೀವನಾವಶ್ಯಕತೆಗಳಿಗೆ ಹೊಂದುವಂತೆ ಮಾಡುವುದು.
- ವಿಷಯಗಳ ಮೇರೆಗಳನ್ನು ಮೀರಿ ಅವುಗಳಿಗೆ ಸಮಗ್ರ ದೃಷ್ಟಿಯ ಬೋಧನೆಯನ್ನು ಅಳವಡಿಸುವುದು.
- ಶಾಲೆಯ ಹೊರಗಿನ ಬದುಕಿಗೆ ಜ್ಞಾನ ಸಂಯೋಜನೆ.
- ಮಕ್ಕಳಿಂದಲೇ ಜ್ಞಾನವನ್ನು ಅಭಿವೃದ್ಧಿಪಡಿಸುವುದು.

9ನೇ ತರಗತಿಯ ಗಣಿತ ಪುಸ್ತಕದಲ್ಲಿ ಅಂತರ್ಗತ ವಿಧಾನ (Integrated Approach), ರಚನಾತ್ಮಕ ವಿಧಾನ (Constructive Approach) ಹಾಗೂ ಸುರುಳಿಯಾಕಾರದ ವಿಧಾನ (Spiral Approach) ಗಳನ್ನು ಅಳವಡಿಸಲಾಗಿದೆ.

ಪಠ್ಯಪುಸ್ತಕಗಳ ವಿಷಯ ಹಾಗೂ ಅಭ್ಯಾಸಗಳು ವಿದ್ಯಾರ್ಥಿಗಳನ್ನು ಯೋಚನೆ ಮಾಡುವಂತೆ ಮಾಡಿ, ಚಟುವಟಿಕೆಗಳ ಮೂಲಕ ಜ್ಞಾನ ಹಾಗೂ ಸಾಮರ್ಥ್ಯಗಳನ್ನು ಪಡೆಯುವಂತೆ ಮಾಡುವ ಪ್ರಯತ್ನ ಮಾಡಲಾಗಿದೆ. ಪಠ್ಯವಸ್ತುಗಳೊಂದಿಗೆ ಅತ್ಯಂತ ಅವಶ್ಯಕ ಜೀವನ ಮೌಲ್ಯಗಳನ್ನು ಅಂತರ್ಗತವಾಗಿ ಬಳಸಲಾಗಿದೆ. ಈ ನೂತನ ಪಠ್ಯಪುಸ್ತಕಗಳು ಪರೀಕ್ಷಾ ದೃಷ್ಟಿಯಿಂದ ರಚಿತವಾಗಿಲ್ಲ. ಬದಲಾಗಿ ಅವುಗಳು ವಿದ್ಯಾರ್ಥಿಗಳ ಸರ್ವಾಂಗೀಣ ವ್ಯಕ್ತಿತ್ವ ವಿಕಸನಕ್ಕೆ ಪೂರಕವಾಗಿವೆ. ತನ್ಮೂಲಕ ಅವರನ್ನು ಸ್ವತಂತ್ರ ಭಾರತದ ಸ್ವಸ್ಥಸಮಾಜದ ಉತ್ತಮ ಪ್ರಜೆಗಳನ್ನಾಗಿ ಮಾಡುವ ಪ್ರಯತ್ನ ನಡೆದಿದೆ.

ನಿತ್ಯ ಜೀವನದಲ್ಲಿ ಗಣಿತವು ಎಲ್ಲಾ ಹಂತಗಳಲ್ಲೂ ಯಶಸ್ಸಿಗೆ ಅತ್ಯವಶ್ಯಕವಾಗಿದೆ. ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮ–2005ರಂತೆ ಗಣಿತವು ಕೆಲವು ಪರಿಕಲ್ಪನೆಗಳನ್ನು ಅರ್ಥಮಾಡಿಕೊಂಡು ಸಿದ್ಧಾಂತಗಳನ್ನು ನಿರೂಪಿಸಿ, ಪ್ರಮೇಯಗಳನ್ನು ಸಾಧಿಸಿ, ಲೆಕ್ಕಗಳನ್ನು ಮಾಡಿ ಪರೀಕ್ಷೆಯಲ್ಲಿ ಉತ್ತಮ ಅಂಕಗಳನ್ನು ಪಡೆಯುವುದರ ಜೊತೆಗೆ ಗಣಿತವನ್ನು ಜೀವನದ ಸಕಲ ಕ್ಷೇತ್ರಗಳಲ್ಲೂ ಬಳಸುವ ಸಾಮರ್ಥ್ಯವನ್ನು ಬೆಳೆಸಿಕೊಂಡು ಜೀವನದಲ್ಲಿ ಯಶಸ್ಸನ್ನು ಗಳಿಸುವಂತೆ ಮಾಡಬೇಕು. ಅದು ಸಹಕಾರಿ ಕಲಿಕೆಗೂ ಪೂರಕವಾಗಿರಬೇಕು.

ಪ್ರಾಚೀನ ಭಾರತದ ಶ್ರೇಷ್ಠ ಗಣಿತ ಶಾಸ್ತ್ರಜ್ಞರ ಕೊಡುಗೆಗಳನ್ನು ಸೂಕ್ತ ಸಂದರ್ಭಗಳಲ್ಲಿ ಕೊಡಲಾಗಿದೆ. 9ನೇ ತರಗತಿಯ ಪಠ್ಯಪುಸ್ತಕಗಳು ಶೈಕ್ಷಣಿಕವಾಗಿ ವೈಶಿಷ್ಟ್ಯಪೂರ್ಣವಾಗಿವೆ. ಇತರ ಪಠ್ಯಪುಸ್ತಕಗಳಂತೆಯೇ ಈ ಪಠ್ಯಪುಸ್ತಕಗಳು ವಿದ್ಯಾರ್ಥಿ/ವಿದ್ಯಾರ್ಥಿನಿಯರಿಗೆ ಸಾಮರ್ಥ್ಯ ಹಾಗೂ ಕೌಶಲ್ಯಗಳನ್ನು ಬೆಳೆಸಿಕೊಳ್ಳಲು ಸಹಾಯ ಮಾಡುತ್ತವೆ.

ರಾಷ್ಟ್ರೀಯ ಮಟ್ಟದ ಶೈಕ್ಷಣಿಕ ಸುಧಾರಣೆ ಹಾಗೂ ಅಭಿವೃದ್ಧಿಗಳನ್ನು ಗಮನದಲ್ಲಿರಿಸಿ ಕರ್ನಾಟಕ ರಾಜ್ಯದ ವಿದ್ಯಾರ್ಥಿಗಳೂ ಸಹ ರಾಷ್ಟ್ರೀಯ ಮಟ್ಟದಲ್ಲಿ ಶೈಕ್ಷಣಿಕವಾಗಿ ಹಾಗೂ ಸ್ಪರ್ಧಾತ್ಮಕ ಪರೀಕ್ಷೆಗಳಲ್ಲಿ ಯಶಸ್ವಿಯಾಗಿ ತಮ್ಮ ಸಾಧನೆಯನ್ನು ಮಾಡಲು ಈ ಪಠ್ಯಪುಸ್ತಕ ಸಹಕಾರಿಯಾಗಲಿ ಎಂಬುವುದೇ ಸಾರ್ವಜನಿಕ ಶಿಕ್ಷಣ ಇಲಾಖೆಯ ಪ್ರಮುಖ ಆಶಯವಾಗಿದೆ.

ಈ ಪಠ್ಯಪುಸ್ತಕವು ವಿದ್ಯಾರ್ಥಿ ಸ್ನೇಹಿ ಹಾಗೂ ಶಿಕ್ಷಕ ಸ್ನೇಹಿಯಾಗಿದೆ. ಗಣಿತ ಕಲಿಕೆ ಸಂತೋಷದಾಯಕ ಹಾಗೂ ಅರ್ಥಪೂರ್ಣವಾಗುವಂತೆ ಮಾಡಲು ಈ ಪಠ್ಯಪುಸ್ತಕವು ಸೂಕ್ತವಾದ ದಾರಿಯನ್ನು ರೂಪಿಸುತ್ತದೆಯೆಂದು ನಾವು ಭಾವಿಸುತ್ತೇವೆ.

9ನೇ ತರಗತಿಗೆ NCERT ಯ ಗಣಿತ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ರಾಜ್ಯದಲ್ಲಿ ಪ್ರಥಮಬಾರಿಗೆ ಜಾರಿಗೆ ತರಲು ದಿಟ್ಟ ನಿರ್ಧಾರ ತೆಗೆದುಕೊಂಡು ಸಹಕರಿಸಿದ ಸರ್ಕಾರ ಮತ್ತು ಇಲಾಖೆಯ ಉನ್ನತ ಅಧಿಕಾರಿಗಳಿಗೆ ಆಭಾರಿಯಾಗಿದ್ದೇನೆ. ಈ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ನಮ್ಮ ರಾಜ್ಯದಲ್ಲಿ ಜಾರಿಗೆ ತರಲು ಅನುಮತಿ, ಸಹಕಾರ ಹಾಗೂ ಮಾರ್ಗದರ್ಶನವನ್ನು ನೀಡಿದ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಸಂಸ್ಥೆ ನವದೆಹಲಿ ಹಾಗೂ ಆ ಸಂಸ್ಥೆಯ ಅಧಿಕಾರಿಗಳಿಗೂ ಇಲಾಖೆ ತನ್ನ ಹೃತ್ಪೂರ್ವಕ ಕೃತಜ್ಞತೆಗಳನ್ನು ಅರ್ಪಿಸುತ್ತದೆ.

ಗೋಪಾಲಕೃಷ್ಣ ಹೆಚ್.ಎನ್.
ವ್ಯವಸ್ಥಾಪಕ ನಿರ್ದೇಶಕರು
ಕರ್ನಾಟಕ ಪಠ್ಯಪುಸ್ತಕ ಸಂಘ (ರಿ)
ಬೆಂಗಳೂರು–85

CONTENTS

CHAPTER 1

NUMBER SYSTEMS

1.1 Introduction

In your earlier classes, you have learnt about the number line and how to represent various types of numbers on it (see Fig. 1.1).

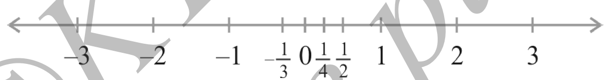


Fig. 1.1 : The number line

Just imagine you start from zero and go on walking along this number line in the positive direction. As far as your eyes can see, there are numbers, numbers and numbers!

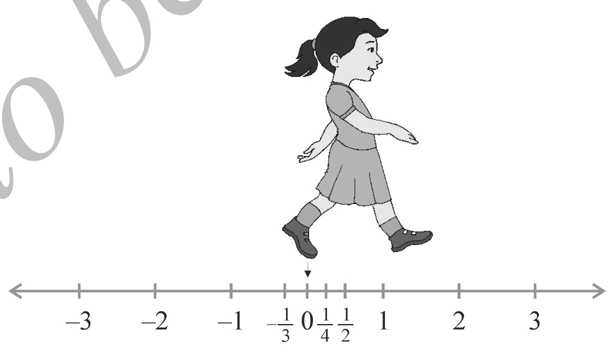
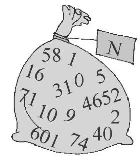


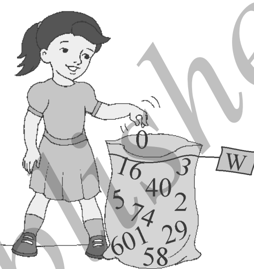
Fig. 1.2

Now suppose you start walking along the number line, and collecting some of the numbers. Get a bag ready to store them!

You might begin with picking up only natural numbers like 1, 2, 3, and so on. You know that this list goes on for ever. (Why is this true?) So, now your bag contains infinitely many natural numbers! Recall that we denote this collection by the symbol **N**.

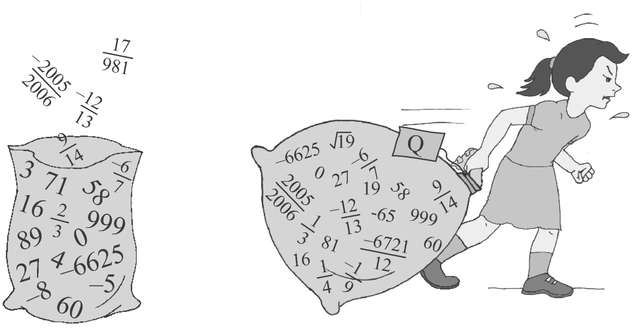


Now turn and walk all the way back, pick up zero and put it into the bag. You now have the collection of *whole numbers* which is denoted by the symbol **W.**



Now, stretching in front of you are many, many negative integers. Put all the negative integers into your bag. What is your new collection? Recall that it is the collection of all *integers*, and it is denoted by the symbol **Z**.

Are there some numbers still left on the line? Of course! There are numbers like $\frac{1}{2}$, $\frac{3}{4}$, or even $\frac{-2005}{2006}$. If you put all such numbers also into the bag, it will now be the



collection of *rational numbers.* The collection of rational numbers is denoted by **Q**. 'Rational' comes from the word 'ratio', and Q comes from the word 'quotient'.

You may recall the definition of rational numbers:

A number 'r' is called a *rational number,* if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. (Why do we insist that $q \neq 0$?)

Notice that all the numbers now in the bag can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, –25 can be written as $\frac{-25}{1}$; here $p = -25$ and $q = 1$. Therefore, the rational numbers also include the natural numbers, whole numbers and integers.

You also know that the rational numbers do not have a unique representation in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{1}{2} = \frac{2}{4} = \frac{10}{20} = \frac{25}{50} = \frac{47}{94}$, and so on. These are *equivalent rational numbers* (*or fractions*). However, when we say that $\frac{p}{q}$ is a rational number, or when we represent $\frac{p}{q}$ on the number line, we assume that $q \neq 0$ and that p and q have no common factors other than 1 (that is, p and q are *co-prime*). So, on the number line, among the infinitely many fractions equivalent to $\frac{1}{2}$, we will choose $\frac{1}{2}$ to represent all of them.

Now, let us solve some examples about the different types of numbers, which you have studied in earlier classes.

Example 1 : Are the following statements true or false? Give reasons for your answers.

(i) Every whole number is a natural number.

(ii) Every integer is a rational number.

(iii) Every rational number is an integer.

Solution : (i) False, because zero is a whole number but not a natural number.

(ii) True, because every integer m can be expressed in the form $\frac{m}{1}$, and so it is a rational number.

(iii) False, because $\frac{3}{5}$ is not an integer.

Example 2 : Find five rational numbers between 1 and 2.

We can approach this problem in at least two ways.

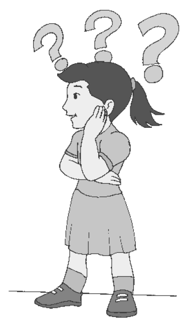
Solution 1 : Recall that to find a rational number between r and s, you can add r and s and divide the sum by 2, that is $\frac{r+s}{2}$ lies between r and s. So, $\frac{3}{2}$ is a number between 1 and 2. You can proceed in this manner to find four more rational numbers between 1 and 2. These four numbers are $\frac{5}{4}, \frac{11}{8}, \frac{13}{8}$ and $\frac{7}{4}$.

Solution 2 : The other option is to find all the five rational numbers in one step. Since we want five numbers, we write 1 and 2 as rational numbers with denominator 5 + 1, i.e., $1 = \frac{6}{6}$ and $2 = \frac{12}{6}$. Then you can check that $\frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}$ and $\frac{11}{6}$ are all rational numbers between 1 and 2. So, the five numbers are $\frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}$ and $\frac{11}{6}$.

Remark : Notice that in Example 2, you were asked to find five rational numbers between 1 and 2. But, you must have realised that in fact there are infinitely many rational numbers between 1 and 2. In general, **there are infinitely many rational numbers between any two given rational numbers**.

Let us take a look at the number line again. Have you picked up all the numbers? Not, yet. The fact is that there are infinitely many more numbers left on the number line! There are gaps in between the places of the numbers you picked up, and not just one or two but infinitely many. The amazing thing is that there are infinitely many numbers lying between any two of these gaps too!

So we are left with the following questions:



1. What are the numbers, that are left on the number line, called?
2. How do we recognise them? That is, how do we distinguish them from the rationals (rational numbers)?

These questions will be answered in the next section.

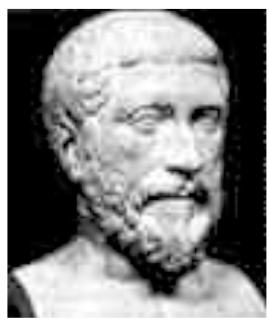
EXERCISE 1.1

1. Is zero a rational number? Can you write it in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$?
2. Find six rational numbers between 3 and 4.
3. Find five rational numbers between $\frac{3}{5}$ and $\frac{4}{5}$.
4. State whether the following statements are true or false. Give reasons for your answers.
   (i) Every natural number is a whole number.
   (ii) Every integer is a whole number.
   (iii) Every rational number is a whole number.

1.2 Irrational Numbers

We saw, in the previous section, that there may be numbers on the number line that are not rationals. In this section, we are going to investigate these numbers. So far, all the numbers you have come across, are of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. So, you may ask: are there numbers which are not of this form? There are indeed such numbers.

The Pythagoreans in Greece, followers of the famous mathematician and philosopher Pythagoras, were the first to discover the numbers which were not rationals, around 400 BC. These numbers are called *irrational numbers* (*irrationals*), because they cannot be written in the form of a ratio of integers. There are many myths surrounding the discovery of irrational numbers by the Pythagorean, Hippacus of Croton. In all the myths, Hippacus has an unfortunate end, either for discovering that $\sqrt{2}$ is irrational or for disclosing the secret about $\sqrt{2}$ to people outside the secret Pythagorean sect!



Pythagoras
(569 BCE – 479 BCE)
Fig. 1.3

Let us formally define these numbers.

A number 's' is called *irrational,* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

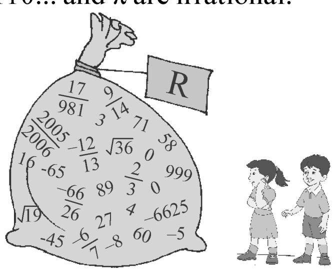
You already know that there are infinitely many rationals. It turns out that there are infinitely many irrational numbers too. Some examples are:

$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, 0.10110111011110...$

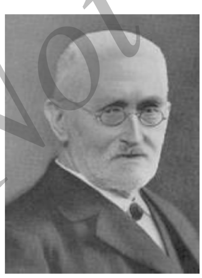
Remark : Recall that when we use the symbol $\sqrt{\ }$, we assume that it is the positive square root of the number. So $\sqrt{4} = 2$, though both 2 and –2 are square roots of 4.

Some of the irrational numbers listed above are familiar to you. For example, you have already come across many of the square roots listed above and the number π.

The Pythagoreans proved that $\sqrt{2}$ is irrational. Later in approximately 425 BC, Theodorus of Cyrene showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ and $\sqrt{17}$ are also irrationals. Proofs of irrationality of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., shall be discussed in Class X. As to π, it was known to various cultures for thousands of years, it was proved to be irrational by Lambert and Legendre only in the late 1700s. In the next section, we will discuss why 0.10110111011110... and π are irrational.



Let us return to the questions raised at the end of the previous section. Remember the bag of rational numbers. If we now put all irrational numbers into the bag, will there be any number left on the number line? The answer is no! It turns out that the collection of all rational numbers and irrational numbers together make up what we call the collection of *real numbers*, which is denoted by **R**. Therefore, a real number is either rational or irrational. So, we can say that **every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number**. This is why we call the number line, the *real number line*.



In the 1870s two German mathematicians, Cantor and Dedekind, showed that : Corresponding to every real number, there is a point on the real number line, and corresponding to every point on the number line, there exists a unique real number.

R. Dedekind (1831-1916)
Fig. 1.4

G. Cantor (1845-1918)
Fig. 1.5

Let us see how we can locate some of the irrational numbers on the number line.

Example 3 : Locate $\sqrt{2}$ on the number line.

Solution : It is easy to see how the Greeks might have discovered $\sqrt{2}$. Consider a unit square OABC, with each side 1 unit in length (see Fig. 1.6). Then you can see by the Pythagoras theorem that OB = $\sqrt{1^2 + 1^2} = \sqrt{2}$. How do we represent $\sqrt{2}$ on the number line?

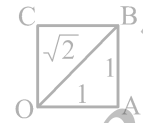


Fig. 1.6

This is easy. Transfer Fig. 1.6 onto the number line making sure that the vertex O coincides with zero (see Fig. 1.7).

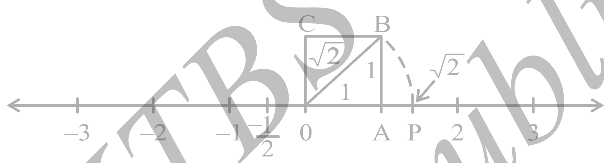


Fig. 1.7

We have just seen that OB = $\sqrt{2}$. Using a compass with centre O and radius OB, draw an arc intersecting the number line at the point P. Then P corresponds to $\sqrt{2}$ on the number line.

Example 4 : Locate $\sqrt{3}$ on the number line.

Solution : Let us return to Fig. 1.7.

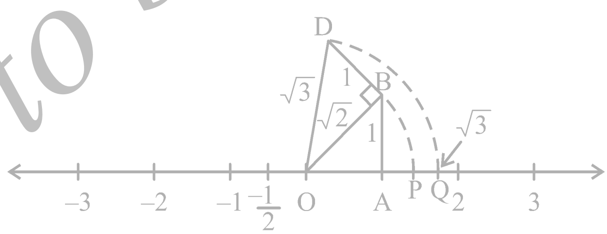


Fig. 1.8

Construct BD of unit length perpendicular to OB (as in Fig. 1.8). Then using the Pythagoras theorem, we see that OD = $\sqrt{\left(\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$. Using a compass, with centre O and radius OD, draw an arc which intersects the number line at the point Q. Then Q corresponds to $\sqrt{3}$.

In the same way, you can locate $\sqrt{n}$ for any positive integer n, after $\sqrt{n-1}$ has been located.

EXERCISE 1.2

1. State whether the following statements are true or false. Justify your answers.
   (i) Every irrational number is a real number.
   (ii) Every point on the number line is of the form $\sqrt{m}$, where m is a natural number.
   (iii) Every real number is an irrational number.
2. Are the square roots of all positive integers irrational? If not, give an example of the square root of a number that is a rational number.
3. Show how $\sqrt{5}$ can be represented on the number line.
4. **Classroom activity (Constructing the 'square root spiral') :** Take a large sheet of paper and construct the 'square root spiral' in the following fashion. Start with a point O and draw a line segment OP_1 of unit length. Draw a line segment P_1P_2 perpendicular to OP_1 of unit length (see Fig. 1.9). Now draw a line segment P_2P_3 perpendicular to OP_2. Then draw a line segment P_3P_4 perpendicular to OP_3. Continuing in this manner, you can get the line segment $P_{n-1}P_n$ by drawing a line segment of unit length perpendicular to OP_{n-1}. In this manner, you will have created the points P_2, P_3,...., P_n,... ., and joined them to create a beautiful spiral depicting $\sqrt{2}, \sqrt{3}, \sqrt{4}, ...$

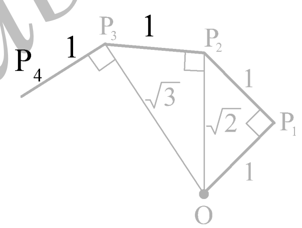


Fig. 1.9 : Constructing square root spiral

1.3 Real Numbers and their Decimal Expansions

In this section, we are going to study rational and irrational numbers from a different point of view. We will look at the decimal expansions of real numbers and see if we can use the expansions to distinguish between rationals and irrationals. We will also explain how to visualise the representation of real numbers on the number line using their decimal expansions. Since rationals are more familiar to us, let us start with them. Let us take three examples : $\frac{10}{3}, \frac{7}{8}, \frac{1}{7}$.

Pay special attention to the remainders and see if you can find any pattern.

Example 5 : Find the decimal expansions of $\frac{10}{3}$, $\frac{7}{8}$ and $\frac{1}{7}$.

Solution :

$$
\begin{array}{r|l}
 & 3.333... \\ \hline
3 & 10 \\
 & \;9 \\ \hline
 & 10 \\
 & \;9 \\ \hline
 & 10 \\
 & \;9 \\ \hline
 & 10 \\
 & \;9 \\ \hline
 & \;1
\end{array}
\qquad
\begin{array}{r|l}
 & 0.875 \\ \hline
8 & 7.0 \\
 & 64 \\ \hline
 & \;\;60 \\
 & \;\;56 \\ \hline
 & \;\;\;\;40 \\
 & \;\;\;\;40 \\ \hline
 & \;\;\;\;\;\;0
\end{array}
\qquad
\begin{array}{r|l}
 & 0.142857... \\ \hline
7 & 1.0 \\
 & \;\;7 \\ \hline
 & \;\;30 \\
 & \;\;28 \\ \hline
 & \;\;\;\;20 \\
 & \;\;\;\;14 \\ \hline
 & \;\;\;\;\;\;60 \\
 & \;\;\;\;\;\;56 \\ \hline
 & \;\;\;\;\;\;\;\;40 \\
 & \;\;\;\;\;\;\;\;35 \\ \hline
 & \;\;\;\;\;\;\;\;\;\;50 \\
 & \;\;\;\;\;\;\;\;\;\;49 \\ \hline
 & \;\;\;\;\;\;\;\;\;\;\;\;1
\end{array}
$$

Remainders : 1, 1, 1, 1, 1...
Divisor : 3

Remainders : 6, 4, 0
Divisor : 8

Remainders : 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1,...
Divisor : 7

What have you noticed? You should have noticed at least three things:

(i) The remainders either become 0 after a certain stage, or start repeating themselves.

(ii) The number of entries in the repeating string of remainders is less than the divisor (in $\frac{1}{3}$ one number repeats itself and the divisor is 3, in $\frac{1}{7}$ there are six entries 326451 in the repeating string of remainders and 7 is the divisor).

(iii) If the remainders repeat, then we get a repeating block of digits in the quotient (for $\frac{1}{3}$, 3 repeats in the quotient and for $\frac{1}{7}$, we get the repeating block 142857 in the quotient).

Although we have noticed this pattern using only the examples above, it is true for all rationals of the form $\frac{p}{q}$ $(q \neq 0)$. On division of p by q, two main things happen – either the remainder becomes zero or never becomes zero and we get a repeating string of remainders. Let us look at each case separately.

Case (i) : The remainder becomes zero

In the example of $\frac{7}{8}$, we found that the remainder becomes zero after some steps and the decimal expansion of $\frac{7}{8} = 0.875$. Other examples are $\frac{1}{2} = 0.5$, $\frac{639}{250} = 2.556$. In all these cases, the decimal expansion terminates or ends after a finite number of steps. We call the decimal expansion of such numbers *terminating*.

Case (ii) : The remainder never becomes zero

In the examples of $\frac{1}{3}$ and $\frac{1}{7}$, we notice that the remainders repeat after a certain stage forcing the decimal expansion to go on for ever. In other words, we have a repeating block of digits in the quotient. We say that this expansion is non-terminating recurring. For example, $\frac{1}{3} = 0.3333...$ and $\frac{1}{7} = 0.142857142857142857...$

The usual way of showing that 3 repeats in the quotient of $\frac{1}{3}$ is to write it as $0.\bar{3}$.

Similarly, since the block of digits 142857 repeats in the quotient of $\frac{1}{7}$, we write $\frac{1}{7}$ as $0.\overline{142857}$, where the bar above the digits indicates the block of digits that repeats. Also 3.57272... can be written as $3.5\overline{72}$. So, all these examples give us *non-terminating recurring* (*repeating*) decimal expansions.

Thus, we see that the decimal expansion of rational numbers have only two choices: either they are terminating or non-terminating recurring.

Now suppose, on the other hand, on your walk on the number line, you come across a number like 3.142678 whose decimal expansion is terminating or a number like 1.272727... that is, $1.\overline{27}$, whose decimal expansion is non-terminating recurring, can you conclude that it is a rational number? The answer is yes!

We will not prove it but illustrate this fact with a few examples. The terminating cases are easy.

Example 6 : Show that 3.142678 is a rational number. In other words, express 3.142678 in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : We have $3.142678 = \frac{3142678}{1000000}$, and hence is a rational number.

Now, let us consider the case when the decimal expansion is non-terminating recurring.

Example 7 : Show that $0.3333... = 0.\overline{3}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Since we do not know what $0.\overline{3}$ is , let us call it 'x' and so

$$x = 0.3333...$$

Now here is where the trick comes in. Look at

$$10\,x = 10 \times (0.333...) = 3.333...$$

Now, $\quad 3.3333... = 3 + x$, since $x = 0.3333...$

Therefore, $\quad 10\,x = 3 + x$

Solving for x, we get

$$9x = 3, \text{ i.e., } x = \frac{1}{3}$$

Example 8 : Show that $1.272727... = 1.\overline{27}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Let $x = 1.272727...$ Since two digits are repeating, we multiply x by 100 to get

$$100\,x = 127.2727...$$

So, $\quad 100\,x = 126 + 1.272727... = 126 + x$

Therefore, $\quad 100\,x - x = 126, \text{ i.e., } 99\,x = 126$

i.e., $$x = \frac{126}{99} = \frac{14}{11}$$

You can check the reverse that $\frac{14}{11} = 1.\overline{27}$.

Example 9 : Show that $0.2353535... = 0.2\overline{35}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Let $x = 0.2\overline{35}$. Over here, note that 2 does not repeat, but the block 35 repeats. Since two digits are repeating, we multiply x by 100 to get

$$100\,x = 23.53535...$$

So, $$100\,x = 23.3 + 0.23535... = 23.3 + x$$

Therefore, $$99\,x = 23.3$$

i.e., $$99\,x = \frac{233}{10}, \text{ which gives } x = \frac{233}{990}$$

You can also check the reverse that $\frac{233}{990} = 0.2\overline{35}$.

So, every number with a non-terminating recurring decimal expansion can be expressed in the form $\frac{p}{q}$ $(q \neq 0)$, where p and q are integers. Let us summarise our results in the following form :

The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.

So, now we know what the decimal expansion of a rational number can be. What about the decimal expansion of irrational numbers? Because of the property above, we can conclude that their decimal expansions are *non-terminating non-recurring*.

So, the property for irrational numbers, similar to the property stated above for rational numbers, is

The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.

Recall s = 0.10110111011110... from the previous section. Notice that it is non-terminating and non-recurring. Therefore, from the property above, it is irrational. Moreover, notice that you can generate infinitely many irrationals similar to s.

What about the famous irrationals $\sqrt{2}$ and π? Here are their decimal expansions up to a certain stage.

$$\sqrt{2} = 1.41421356237309504880168872420 96...$$

$$\pi = 3.14159265358979323846264338327950...$$

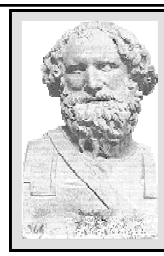
(Note that, we often take $\frac{22}{7}$ as an approximate value for π, but $\pi \neq \frac{22}{7}$.)

Over the years, mathematicians have developed various techniques to produce more and more digits in the decimal expansions of irrational numbers. For example, you might have learnt to find digits in the decimal expansion of $\sqrt{2}$ by the division method. Interestingly, in the Sulbasutras (rules of chord), a mathematical treatise of the Vedic period (800 BC - 500 BC), you find an approximation of $\sqrt{2}$ as follows:

$$\sqrt{2} = 1 + \frac{1}{3} + \left(\frac{1}{4} \times \frac{1}{3}\right) - \left(\frac{1}{34} \times \frac{1}{4} \times \frac{1}{3}\right) = 1.4142156$$

Notice that it is the same as the one given above for the first five decimal places. The history of the hunt for digits in the decimal expansion of π is very interesting.

The Greek genius Archimedes was the first to compute digits in the decimal expansion of π. He showed $3.140845 < \pi < 3.142857$. Aryabhatta (476 – 550 C.E.), the great Indian mathematician and astronomer, found the value of π correct to four decimal places (3.1416). Using high speed computers and advanced algorithms, π has been computed to over 1.24 trillion decimal places!



Archimedes (287 BCE – 212 BCE)
Fig. 1.10

Now, let us see how to obtain irrational numbers.

Example 10 : Find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$.

Solution : We saw that $\frac{1}{7} = 0.\overline{142857}$. So, you can easily calculate $\frac{2}{7} = 0.\overline{285714}$.

To find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$, we find a number which is non-terminating non-recurring lying between them. Of course, you can find infinitely many such numbers.

An example of such a number is 0.150150015000150000...

EXERCISE 1.3

1. Write the following in decimal form and say what kind of decimal expansion each has :

   (i) $\frac{36}{100}$ (ii) $\frac{1}{11}$ (iii) $4\frac{1}{8}$

   (iv) $\frac{3}{13}$ (v) $\frac{2}{11}$ (vi) $\frac{329}{400}$

2. You know that $\frac{1}{7} = 0.\overline{142857}$. Can you predict what the decimal expansions of $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ are, without actually doing the long division? If so, how?

   [**Hint :** Study the remainders while finding the value of $\frac{1}{7}$ carefully.]

3. Express the following in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

   (i) $0.\overline{6}$ (ii) $0.4\overline{7}$ (iii) $0.\overline{001}$

4. Express 0.99999 .... in the form $\frac{p}{q}$. Are you surprised by your answer? With your teacher and classmates discuss why the answer makes sense.

5. What can the maximum number of digits be in the repeating block of digits in the decimal expansion of $\frac{1}{17}$? Perform the division to check your answer.

6. Look at several examples of rational numbers in the form $\frac{p}{q}$ $(q \neq 0)$, where p and q are integers with no common factors other than 1 and having terminating decimal representations (expansions). Can you guess what property q must satisfy?

7. Write three numbers whose decimal expansions are non-terminating non-recurring.

8. Find three different irrational numbers between the rational numbers $\frac{5}{7}$ and $\frac{9}{11}$.

9. Classify the following numbers as rational or irrational :

   (i) $\sqrt{23}$ (ii) $\sqrt{225}$ (iii) 0.3796

   (iv) 7.478478... (v) 1.101001000100001...

1.4 Representing Real Numbers on the Number Line

In the previous section, you have seen that any real number has a decimal expansion. This helps us to represent it on the number line. Let us see how.

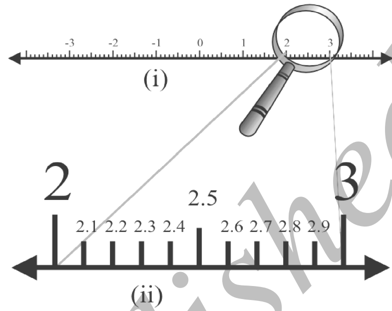


Fig. 1.11

Suppose we want to locate 2.665 on the number line. We know that this lies between 2 and 3.

So, let us look closely at the portion of the number line between 2 and 3. Suppose we divide this into 10 equal parts and mark each point of division as in Fig. 1.11 (i). Then the first mark to the right of 2 will represent 2.1, the second 2.2, and so on. You might be finding some difficulty in observing these points of division between 2 and 3 in Fig. 1.11 (i). To have a clear view of the same, you may take a magnifying glass and look at the portion between 2 and 3. It will look like what you see in Fig. 1.11 (ii). Now, 2.665 lies between 2.6 and 2.7. So, let us focus on the portion between 2.6 and 2.7 [See Fig. 1.12(i)]. We imagine to divide this again into ten equal parts. The first mark will represent 2.61, the next 2.62, and so on. To see this clearly, we magnify this as shown in Fig. 1.12 (ii).

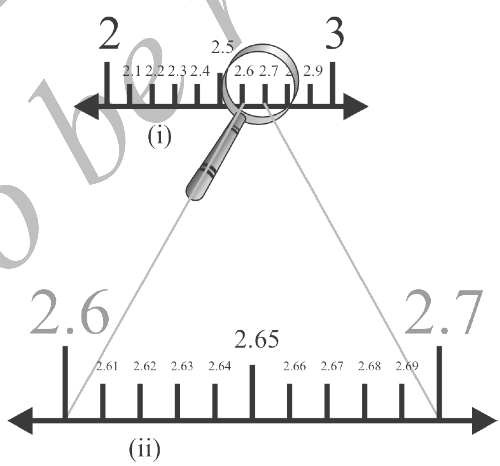


Fig. 1.12

Again, 2.665 lies between 2.66 and 2.67. So, let us focus on this portion of the number line [see Fig. 1.13(i)] and imagine to divide it again into ten equal parts. We magnify it to see it better, as in Fig. 1.13 (ii). The first mark represents 2.661, the next one represents 2.662, and so on. So, 2.665 is the 5th mark in these subdivisions.

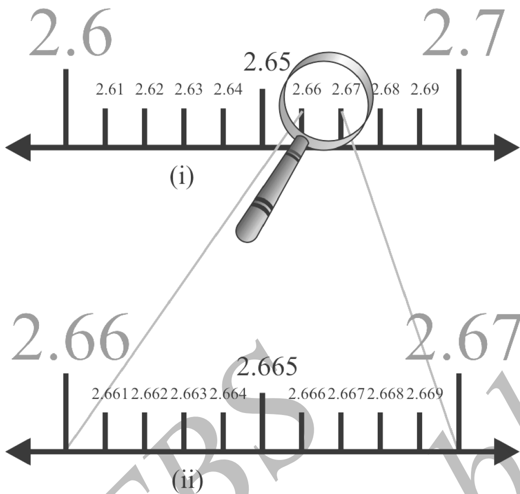


Fig. 1.13

We call this process of visualisation of representation of numbers on the number line, through a magnifying glass, as the **process of successive magnification**.

So, we have seen that it is possible by sufficient successive magnifications to visualise the position (or representation) of a real number with a terminating decimal expansion on the number line.

Let us now try and visualise the position (or representation) of a real number with a non-terminating recurring decimal expansion on the number line. We can look at appropriate intervals through a magnifying glass and by successive magnifications visualise the position of the number on the number line.

Example 11 : Visualize the representation of $5.3\overline{7}$ on the number line upto 5 decimal places, that is, up to 5.37777.

Solution : Once again we proceed by successive magnification, and successively decrease the lengths of the portions of the number line in which $5.3\overline{7}$ is located. First, we see that $5.3\overline{7}$ is located between 5 and 6. In the next step, we locate $5.3\overline{7}$ between 5.3 and 5.4. To get a more accurate visualization of the representation, we divide this portion of the number line into 10 equal parts and use a magnifying glass to visualize that $5.3\overline{7}$ lies between 5.37 and 5.38. To visualize $5.3\overline{7}$ more accurately, we again divide the portion between 5.37 and 5.38 into ten equal parts and use a magnifying glass to visualize that $5.3\overline{7}$ lies between 5.377 and 5.378. Now to visualize $5.3\overline{7}$ still more accurately, we divide the portion between 5.377 an 5.378 into 10 equal parts, and visualize the representation of $5.3\overline{7}$ as in Fig. 1.14 (iv). Notice that $5.3\overline{7}$ is located closer to 5.3778 than to 5.3777 [see Fig 1.14 (iv)].

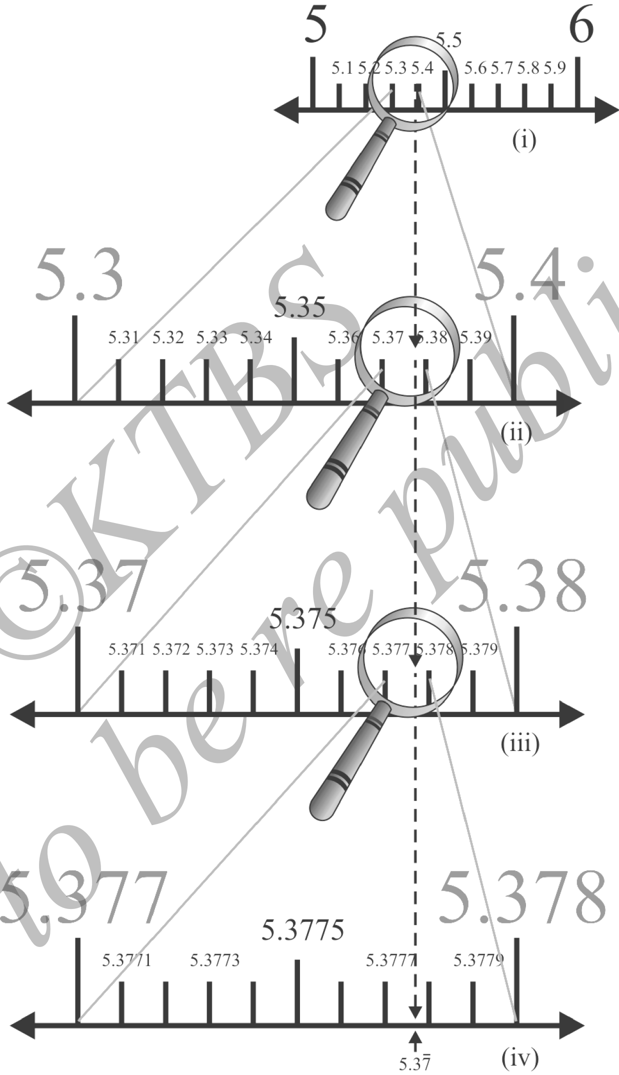


Fig. 1.14

Remark : We can proceed endlessly in this manner, successively viewing through a magnifying glass and simultaneously imagining the decrease in the length of the portion of the number line in which $5.3\overline{7}$ is located. The size of the portion of the line we specify depends on the degree of accuracy we would like for the visualisation of the position of the number on the number line.

You might have realised by now that the same procedure can be used to visualise a real number with a non-terminating non-recurring decimal expansion on the number line.

In the light of the discussions above and visualisations, we can again say that *every real number is represented by a unique point on the number line. Further, every point on the number line represents one and only one real number*.

EXERCISE 1.4

1. Visualise 3.765 on the number line, using successive magnification.
2. Visualise $4.\overline{26}$ on the number line, up to 4 decimal places.

1.5 Operations on Real Numbers

You have learnt, in earlier classes, that rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. Moreover, if we add, subtract, multiply or divide (except by zero) two rational numbers, we still get a rational number (that is, rational numbers are 'closed' with respect to addition, subtraction, multiplication and division). It turns out that irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not *always* irrational. For example, $\left(\sqrt{6}\right)+\left(-\sqrt{6}\right), \left(\sqrt{2}\right)-\left(\sqrt{2}\right), \left(\sqrt{3}\right)\cdot\left(\sqrt{3}\right)$ and $\frac{\sqrt{17}}{\sqrt{17}}$ are rationals.

Let us look at what happens when we add and multiply a rational number with an irrational number. For example, $\sqrt{3}$ is irrational. What about $2+\sqrt{3}$ and $2\sqrt{3}$? Since $\sqrt{3}$ has a non-terminating non-recurring decimal expansion, the same is true for $2+\sqrt{3}$ and $2\sqrt{3}$. Therefore, both $2+\sqrt{3}$ and $2\sqrt{3}$ are also irrational numbers.

Example 12 : Check whether $7\sqrt{5}$, $\frac{7}{\sqrt{5}}$, $\sqrt{2}+21$, $\pi-2$ are irrational numbers or not.

Solution : $\sqrt{5}$ = 2.236... , $\sqrt{2}$ = 1.4142..., π = 3.1415...

Then $7\sqrt{5} = 15.652...$, $\frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{7\sqrt{5}}{5} = 3.1304...$

$\sqrt{2} + 21 = 22.4142...$, $\pi - 2 = 1.1415...$

All these are non-terminating non-recurring decimals. So, all these are irrational numbers.

Now, let us see what generally happens if we add, subtract, multiply, divide, take square roots and even nth roots of these irrational numbers, where n is any natural number. Let us look at some examples.

Example 13 : Add $2\sqrt{2} + 5\sqrt{3}$ and $\sqrt{2} - 3\sqrt{3}$.

Solution : $\left(2\sqrt{2} + 5\sqrt{3}\right) + \left(\sqrt{2} - 3\sqrt{3}\right) = \left(2\sqrt{2} + \sqrt{2}\right) + \left(5\sqrt{3} - 3\sqrt{3}\right)$

$= (2 + 1)\sqrt{2} + (5 - 3)\sqrt{3} = 3\sqrt{2} + 2\sqrt{3}$

Example 14 : Multiply $6\sqrt{5}$ by $2\sqrt{5}$.

Solution : $6\sqrt{5} \times 2\sqrt{5} = 6 \times 2 \times \sqrt{5} \times \sqrt{5} = 12 \times 5 = 60$

Example 15 : Divide $8\sqrt{15}$ by $2\sqrt{3}$.

Solution : $8\sqrt{15} \div 2\sqrt{3} = \frac{8\sqrt{3} \times \sqrt{5}}{2\sqrt{3}} = 4\sqrt{5}$

These examples may lead you to expect the following facts, **which are true**:

(i) The sum or difference of a rational number and an irrational number is irrational.

(ii) The product or quotient of a non-zero rational number with an irrational number is irrational.

(iii) If we add, subtract, multiply or divide two irrationals, the result may be rational or irrational.

We now turn our attention to the operation of taking square roots of real numbers. Recall that, if a is a natural number, then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$. The same definition can be extended for positive real numbers.

Let $a > 0$ be a real number. Then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$.

In Section 1.2, we saw how to represent $\sqrt{n}$ for any positive integer n on the number line. We now show how to find $\sqrt{x}$ for any given positive real number x geometrically. For example, let us find it for $x = 3.5$, i.e., we find $\sqrt{3.5}$ geometrically.

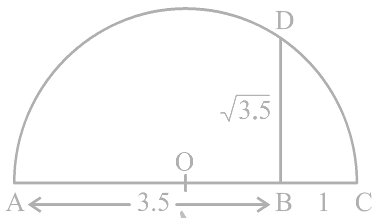


Fig. 1.15

Mark the distance 3.5 units from a fixed point A on a given line to obtain a point B such that AB = 3.5 units (see Fig. 1.15). From B, mark a distance of 1 unit and mark the new point as C. Find the mid-point of AC and mark that point as O. Draw a semicircle with centre O and radius OC. Draw a line perpendicular to AC passing through B and intersecting the semicircle at D. Then, BD = $\sqrt{3.5}$.

More generally, to find $\sqrt{x}$, for any positive real number x, we mark B so that AB = x units, and, as in Fig. 1.16, mark C so that BC = 1 unit. Then, as we have done for the case $x = 3.5$, we find BD = $\sqrt{x}$ (see Fig. 1.16). We can prove this result using the Pythagoras Theorem.

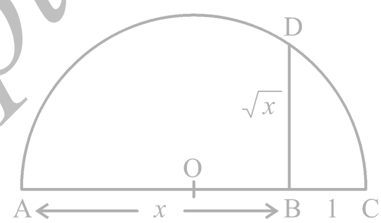


Fig. 1.16

Notice that, in Fig. 1.16, Δ OBD is a right-angled triangle. Also, the radius of the circle is $\dfrac{x+1}{2}$ units.

Therefore, OC = OD = OA = $\dfrac{x+1}{2}$ units.

Now, OB = $x - \left(\dfrac{x+1}{2}\right) = \dfrac{x-1}{2}$.

So, by the Pythagoras Theorem, we have

$$BD^2 = OD^2 - OB^2 = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = \frac{4x}{4} = x.$$

This shows that BD = $\sqrt{x}$.

This construction gives us a visual, and geometric way of showing that $\sqrt{x}$ exists for all real numbers $x > 0$. If you want to know the position of $\sqrt{x}$ on the number line, then let us treat the line BC as the number line, with B as zero, C as 1, and so on. Draw an arc with centre B and radius BD, which intersects the number line in E (see Fig. 1.17). Then, E represents $\sqrt{x}$.

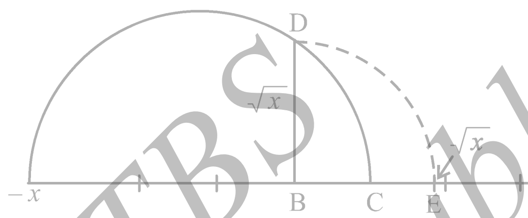


Fig. 1.17

We would like to now extend the idea of square roots to cube roots, fourth roots, and in general nth roots, where n is a positive integer. Recall your understanding of square roots and cube roots from earlier classes.

What is $\sqrt[3]{8}$? Well, we know it has to be some positive number whose cube is 8, and you must have guessed $\sqrt[3]{8}$ = 2. Let us try $\sqrt[5]{243}$. Do you know some number b such that $b^5 = 243$? The answer is 3. Therefore, $\sqrt[5]{243}$ = 3.

From these examples, can you define $\sqrt[n]{a}$ for a real number $a > 0$ and a positive integer n?

Let $a > 0$ be a real number and n be a positive integer. Then $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$. Note that the symbol '$\sqrt{\ }$' used in $\sqrt{2}, \sqrt[3]{8}, \sqrt[n]{a}$, etc. is called the *radical sign*.

We now list some identities relating to square roots, which are useful in various ways. You are already familiar with some of these from your earlier classes. The remaining ones follow from the distributive law of multiplication over addition of real numbers, and from the identity $(x + y)(x - y) = x^2 - y^2$, for any real numbers x and y.

Let a and b be positive real numbers. Then

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(iii) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b$ (iv) $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right)=a^2-b$

(v) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{d}\right)=\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(vi) $\left(\sqrt{a}+\sqrt{b}\right)^2=a+2\sqrt{ab}+b$

Let us look at some particular cases of these identities.

Example 16 : Simplify the following expressions:

(i) $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)$ (ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)$

(iii) $\left(\sqrt{3}+\sqrt{7}\right)^2$ (iv) $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)$

Solution : (i) $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)=10+5\sqrt{5}+2\sqrt{7}+\sqrt{35}$

(ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)=5^2-\left(\sqrt{5}\right)^2=25-5=20$

(iii) $\left(\sqrt{3}+\sqrt{7}\right)^2=\left(\sqrt{3}\right)^2+2\sqrt{3}\sqrt{7}+\left(\sqrt{7}\right)^2=3+2\sqrt{21}+7=10+2\sqrt{21}$

(iv) $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)=\left(\sqrt{11}\right)^2-\left(\sqrt{7}\right)^2=11-7=4$

Remark : Note that 'simplify' in the example above has been used to mean that the expression should be written as the sum of a rational and an irrational number.

We end this section by considering the following problem. Look at $\frac{1}{\sqrt{2}}\cdot$ Can you tell where it shows up on the number line? You know that it is irrational. May be it is easier to handle if the denominator is a rational number. Let us see, if we can 'rationalise' the denominator, that is, to make the denominator into a rational number. To do so, we need the identities involving square roots. Let us see how.

Example 17 : Rationalise the denominator of $\frac{1}{\sqrt{2}}\cdot$

Solution : We want to write $\frac{1}{\sqrt{2}}$ as an equivalent expression in which the denominator is a rational number. We know that $\sqrt{2}\cdot\sqrt{2}$ is rational. We also know that multiplying

$\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ will give us an equivalent expression, since $\frac{\sqrt{2}}{\sqrt{2}} = 1$. So, we put these two facts together to get

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

In this form, it is easy to locate $\frac{1}{\sqrt{2}}$ on the number line. It is half way between 0 and $\sqrt{2}$!

Example 18 : Rationalise the denominator of $\frac{1}{2+\sqrt{3}}$.

Solution : We use the Identity (iv) given earlier. Multiply and divide $\frac{1}{2+\sqrt{3}}$ by $2-\sqrt{3}$ to get $\frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$.

Example 19 : Rationalise the denominator of $\frac{5}{\sqrt{3}-\sqrt{5}}$.

Solution : Here we use the Identity (iii) given earlier.

So, $\frac{5}{\sqrt{3}-\sqrt{5}} = \frac{5}{\sqrt{3}-\sqrt{5}} \times \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}} = \frac{5\left(\sqrt{3}+\sqrt{5}\right)}{3-5} = \left(\frac{-5}{2}\right)\left(\sqrt{3}+\sqrt{5}\right)$

Example 20 : Rationalise the denominator of $\frac{1}{7+3\sqrt{2}}$.

Solution : $\frac{1}{7+3\sqrt{2}} = \frac{1}{7+3\sqrt{2}} \times \left(\frac{7-3\sqrt{2}}{7-3\sqrt{2}}\right) = \frac{7-3\sqrt{2}}{49-18} = \frac{7-3\sqrt{2}}{31}$

So, when the denominator of an expression contains a term with a square root (or a number under a radical sign), the process of converting it to an equivalent expression whose denominator is a rational number is called *rationalising the denominator*.

EXERCISE 1.5

1. Classify the following numbers as rational or irrational:

(i) $2-\sqrt{5}$ (ii) $\left(3+\sqrt{23}\right)-\sqrt{23}$ (iii) $\frac{2\sqrt{7}}{7\sqrt{7}}$

(iv) $\frac{1}{\sqrt{2}}$ (v) 2π

2. Simplify each of the following expressions:

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$ (ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2$ (iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

3. Recall, π is defined as the ratio of the circumference (say c) of a circle to its diameter (say d). That is, $\pi = \frac{c}{d}\cdot$ This seems to contradict the fact that π is irrational. How will you resolve this contradiction?

4. Represent $\sqrt{9.3}$ on the number line.

5. Rationalise the denominators of the following:

(i) $\frac{1}{\sqrt{7}}$ (ii) $\frac{1}{\sqrt{7}-\sqrt{6}}$

(iii) $\frac{1}{\sqrt{5}+\sqrt{2}}$ (iv) $\frac{1}{\sqrt{7}-2}$

1.6 Laws of Exponents for Real Numbers

Do you remember how to simplify the following?

(i) $17^2 \cdot 17^5 =$ (ii) $(5^2)^7 =$

(iii) $\frac{23^{10}}{23^7} =$ (iv) $7^3 \cdot 9^3 =$

Did you get these answers? They are as follows:

(i) $17^2 \cdot 17^5 = 17^7$ (ii) $(5^2)^7 = 5^{14}$

(iii) $\frac{23^{10}}{23^7} = 23^3$ (iv) $7^3 \cdot 9^3 = 63^3$

To get these answers, you would have used the following laws of exponents, which you have learnt in your earlier classes. (Here a, n and m are natural numbers. Remember, a is called the base and m and n are the exponents.)

(i) $a^m \cdot a^n = a^{m+n}$ (ii) $(a^m)^n = a^{mn}$

(iii) $\dfrac{a^m}{a^n} = a^{m-n}, m > n$ (iv) $a^m b^m = (ab)^m$

What is $(a)^0$? Yes, it is 1! So you have learnt that $(a)^0 = 1$. So, using (iii), we can get $\dfrac{1}{a^n} = a^{-n}$. We can now extend the laws to negative exponents too.

So, for example :

(i) $17^2 \cdot 17^{-5} = 17^{-3} = \dfrac{1}{17^3}$ (ii) $(5^2)^{-7} = 5^{-14}$

(iii) $\dfrac{23^{-10}}{23^7} = 23^{-17}$ (iv) $(7)^{-3} \cdot (9)^{-3} = (63)^{-3}$

Suppose we want to do the following computations:

(i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}$ (ii) $\left(3^{\frac{1}{5}}\right)^4$

(iii) $\dfrac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$ (iv) $13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}}$

How would we go about it? It turns out that we can extend the laws of exponents that we have studied earlier, even when the base is a positive real number and the exponents are rational numbers. (Later you will study that it can further to be extended when the exponents are real numbers.) But before we state these laws, and to even make sense of these laws, we need to first understand what, for example $4^{\frac{3}{2}}$ is. So, we have some work to do!

In Section 1.4, we defined $\sqrt[n]{a}$ for a real number $a > 0$ as follows:

Let $a > 0$ be a real number and n a positive integer. Then $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$.

In the language of exponents, we define $\sqrt[n]{a} = a^{\frac{1}{n}}$. So, in particular, $\sqrt[3]{2} = 2^{\frac{1}{3}}$.

There are now two ways to look at $4^{\frac{3}{2}}$.

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

$$4^{\frac{3}{2}} = \left(4^3\right)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = 8$$

Therefore, we have the following definition:

Let $a > 0$ be a real number. Let m and n be integers such that m and n have no common factors other than 1, and $n > 0$. Then,

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

We now have the following extended laws of exponents:

Let $a > 0$ be a real number and p and q be rational numbers. Then, we have

(i) $a^p \,.\, a^q = a^{p+q}$ (ii) $(a^p)^q = a^{pq}$

(iii) $\dfrac{a^p}{a^q} = a^{p-q}$ (iv) $a^p b^p = (ab)^p$

You can now use these laws to answer the questions asked earlier.

Example 21 : Simplify (i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}$ (ii) $\left(3^{\frac{1}{5}}\right)^4$

(iii) $\dfrac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$ (iv) $13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}}$

Solution :

(i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}} = 2^{\left(\frac{2}{3}+\frac{1}{3}\right)} = 2^{\frac{3}{3}} = 2^1 = 2$ (ii) $\left(3^{\frac{1}{5}}\right)^4 = 3^{\frac{4}{5}}$

(iii) $\dfrac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}} = 7^{\left(\frac{1}{5}-\frac{1}{3}\right)} = 7^{\frac{3-5}{15}} = 7^{\frac{-2}{15}}$ (iv) $13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}} = (13 \times 17)^{\frac{1}{5}} = 221^{\frac{1}{5}}$

EXERCISE 1.6

1. Find : (i) $64^{\frac{1}{2}}$ (ii) $32^{\frac{1}{5}}$ (iii) $125^{\frac{1}{3}}$

2. Find : (i) $9^{\frac{3}{2}}$ (ii) $32^{\frac{2}{5}}$ (iii) $16^{\frac{3}{4}}$ (iv) $125^{\frac{-1}{3}}$

3. Simplify : (i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{5}}$ (ii) $\left(\dfrac{1}{3^3}\right)^7$ (iii) $\dfrac{11^{\frac{1}{2}}}{11^{\frac{1}{4}}}$ (iv) $7^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$

1.7 Summary

In this chapter, you have studied the following points:

1. A number r is called a rational number, if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

2. A number s is called a irrational number, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

3. The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.

4. The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.

5. All the rational and irrational numbers make up the collection of real numbers.

6. There is a unique real number corresponding to every point on the number line. Also, corresponding to each real number, there is a unique point on the number line.

7. If r is rational and s is irrational, then $r + s$ and $r - s$ are irrational numbers, and rs and $\frac{r}{s}$ are irrational numbers, $r \neq 0$.

8. For positive real numbers a and b, the following identities hold:

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(iii) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$ (iv) $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$

(v) $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

9. To rationalise the denominator of $\frac{1}{\sqrt{a} + b}$, we multiply this by $\frac{\sqrt{a} - b}{\sqrt{a} - b}$, where a and b are integers.

10. Let $a > 0$ be a real number and p and q be rational numbers. Then

(i) $a^p \cdot a^q = a^{p+q}$ (ii) $(a^p)^q = a^{pq}$

(iii) $\frac{a^p}{a^q} = a^{p-q}$ (iv) $a^p b^p = (ab)^p$

CHAPTER 2

INTRODUCTION TO EUCLID'S GEOMETRY

2.1 Introduction

The word 'geometry' comes form the Greek words 'geo', meaning the 'earth', and 'metrein', meaning 'to measure'. Geometry appears to have originated from the need for measuring land. This branch of mathematics was studied in various forms in every ancient civilisation, be it in Egypt, Babylonia, China, India, Greece, the Incas, etc. The people of these civilisations faced several practical problems which required the development of geometry in various ways.

For example, whenever the river Nile overflowed, it wiped out the boundaries between the adjoining fields of different land owners. After such flooding, these boundaries had to be redrawn. For this purpose, the Egyptians developed a number of geometric techniques and rules for calculating simple areas and also for doing simple constructions. The knowledge of geometry was also used by them for computing volumes of granaries, and for constructing canals and pyramids. They also knew the correct formula to find the volume of a truncated pyramid (see Fig. 2.1).You know that a pyramid is a solid figure, the base of which is a triangle, or square, or some other polygon, and its side faces are triangles converging to a point at the top.

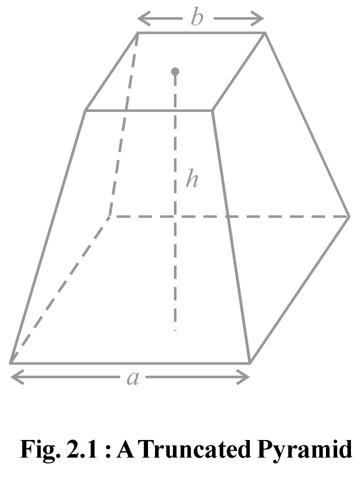


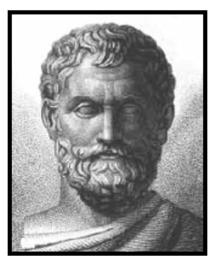
Fig. 2.1 : A Truncated Pyramid

In the Indian subcontinent, the excavations at Harappa and Mohenjo-Daro, etc. show that the Indus Valley Civilisation (about 3000 BCE) made extensive use of geometry. It was a highly organised society. The cities were highly developed and very well planned. For example, the roads were parallel to each other and there was an underground drainage system. The houses had many rooms of different types. This shows that the town dwellers were skilled in mensuration and practical arithmetic. The bricks used for constructions were kiln fired and the ratio length : breadth : thickness, of the bricks was found to be 4 : 2 : 1.

In ancient India, the *Sulbasutras* (800 BCE to 500 BCE) were the manuals of geometrical constructions. The geometry of the Vedic period originated with the construction of altars (or *vedis*) and fireplaces for performing Vedic rites. The location of the sacred fires had to be in accordance to the clearly laid down instructions about their shapes and areas, if they were to be effective instruments. Square and circular altars were used for household rituals, while altars whose shapes were combinations of rectangles, triangles and trapeziums were required for public worship. The *sriyantra* (given in the *Atharvaveda*) consists of nine interwoven isosceles triangles. These triangles are arranged in such a way that they produce 43 subsidiary triangles. Though accurate geometric methods were used for the constructions of altars, the principles behind them were not discussed.

These examples show that geometry was being developed and applied everywhere in the world. But this was happening in an unsystematic manner. What is interesting about these developments of geometry in the ancient world is that they were passed on from one generation to the next, either orally or through palm leaf messages, or by other ways. Also, we find that in some civilisations like Babylonia, geometry remained a very practical oriented discipline, as was the case in India and Rome. The geometry developed by Egyptians mainly consisted of the statements of results. There were no general rules of the procedure. In fact, Babylonians and Egyptians used geometry mostly for practical purposes and did very little to develop it as a systematic science. But in civilisations like Greece, the emphasis was on the *reasoning* behind why certain constructions work. The Greeks were interested in establishing the truth of the statements they discovered using deductive reasoning (see Appendix 1).

A Greek mathematician, Thales is credited with giving the first known proof. This proof was of the statement that a circle is bisected (i.e., cut into two equal parts) by its diameter. One of Thales' most famous pupils was Pythagoras (572 BCE), whom you have heard about. Pythagoras and his group discovered many geometric properties and developed the theory of geometry to a great extent. This process continued till 300 BCE. At that time Euclid, a teacher of mathematics at Alexandria in Egypt, collected all the known work and arranged it in his famous treatise,



Thales
(640 BCE – 546 BCE)
Fig. 2.2

called 'Elements'. He divided the 'Elements' into thirteen chapters, each called a book. These books influenced the whole world's understanding of geometry for generations to come.

In this chapter, we shall discuss Euclid's approach to geometry and shall try to link it with the present day geometry.

Euclid (325 BCE – 265 BCE)
Fig. 2.3

2.2 Euclid's Definitions, Axioms and Postulates

The Greek mathematicians of Euclid's time thought of geometry as an abstract model of the world in which they lived. The notions of point, line, plane (or surface) and so on were derived from what was seen around them. From studies of the space and solids in the space around them, an abstract geometrical notion of a solid object was developed. A solid has shape, size, position, and can be moved from one place to another. Its boundaries are called **surfaces.** They separate one part of the space from another, and are said to have no thickness. The boundaries of the surfaces are **curves** or straight **lines.** These lines end in **points**.

Consider the three steps from solids to points (solids-surfaces-lines-points). In each step we lose one extension, also called a **dimension**. So, a solid has three dimensions, a surface has two, a line has one and a point has none. Euclid summarised these statements as definitions. He began his exposition by listing 23 definitions in Book 1 of the 'Elements'. A few of them are given below :

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The ends of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The edges of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.

If you carefully study these definitions, you find that some of the terms like part, breadth, length, evenly, etc. need to be further explained clearly. For example, consider his definition of a point. In this definition, 'a part' needs to be defined. Suppose if you define 'a part' to be that which occupies 'area', again 'an area' needs to be defined. So, to define one thing, you need to define many other things, and you may get a long chain of definitions without an end. For such reasons, mathematicians agree to leave some geometric terms *undefined*. However, we do have a intuitive feeling for the geometric concept of a point than what the 'definition' above gives us. So, we represent a point as a dot, even though a dot has some dimension.

A similar problem arises in Definition 2 above, since it refers to breadth and length, neither of which has been defined. Because of this, a few terms are kept undefined while developing any course of study. So, in geometry, we *take a point, a line and a plane* (*in Euclid's words a plane surface*) *as undefined terms*. The only thing is that we can represent them intuitively, or explain them with the help of 'physical models'.

Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually 'obvious universal truths'. He divided them into two types: axioms and postulates. He used the term '**postulate**' for the assumptions that were specific to geometry. Common notions (often called **axioms**), on the other hand, were assumptions used throughout mathematics and not specifically linked to geometry. For details about axioms and postulates, refer to Appendix 1. Some of **Euclid's axioms**, not in his order, are given below :

(1) Things which are equal to the same thing are equal to one another.

(2) If equals are added to equals, the wholes are equal.

(3) If equals are subtracted from equals, the remainders are equal.

(4) Things which coincide with one another are equal to one another.

(5) The whole is greater than the part.

(6) Things which are double of the same things are equal to one another.

(7) Things which are halves of the same things are equal to one another.

These 'common notions' refer to magnitudes of some kind. The first common notion could be applied to plane figures. For example, if an area of a triangle equals the area of a rectangle and the area of the rectangle equals that of a square, then the area of the triangle also equals the area of the square.

Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot be compared. For example, a line cannot be added to a rectangle, nor can an angle be compared to a pentagon.

The 4th axiom given above seems to say that if two things are identical (that is, they are the same), then they are equal. In other words, everything equals itself. It is the justification of the principle of superposition. Axiom (5) gives us the definition of 'greater than'. For example, if a quantity B is a part of another quantity A, then A can be written as the sum of B and some third quantity C. Symbolically, A > B means that there is some C such that A = B + C.

Now let us discuss **Euclid's five postulates.** They are :

Postulate 1 : *A straight line may be drawn from any one point to any other point.*

Note that this postulate tells us that at least one straight line passes through two distinct points, but it does not say that there cannot be more than one such line. However, in his work, Euclid has frequently assumed, without mentioning, that there is a *unique* line joining two distinct points. We state this result in the form of an axiom as follows:

Axiom 2.1 : *Given two distinct points, there is a unique line that passes through them.*

How many lines passing through P also pass through Q (see Fig. 2.4)? Only one, that is, the line PQ. How many lines passing through Q also pass through P? Only one, that is, the line PQ. Thus, the statement above is self-evident, and so is taken as an axiom.

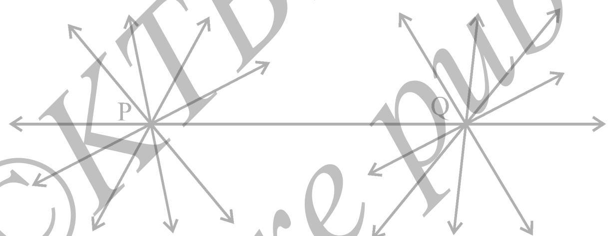


Fig. 2.4

Postulate 2 : *A terminated line can be produced indefinitely.*

Note that what we call a line segment now-a-days is what Euclid called a terminated line. So, according to the present day terms, the second postulate says that a line segment can be extended on either side to form a line (see Fig. 2.5).

Fig. 2.5

Postulate 3 : *A circle can be drawn with any centre and any radius.*

Postulate 4 : *All right angles are equal to one another.*

Postulate 5 : *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.*

For example, the line PQ in Fig. 2.6 falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ.

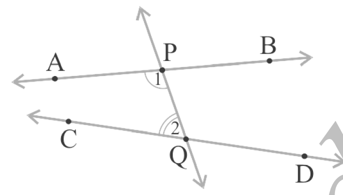


Fig. 2.6

A brief look at the five postulates brings to your notice that Postulate 5 is far more complex than any other postulate. On the other hand, Postulates 1 through 4 are so simple and obvious that these are taken as 'self-evident truths'. However, it is not possible to prove them. So, these statements are accepted without any proof (see Appendix 1). Because of its complexity, the fifth postulate will be given more attention in the next section.

Now-a-days, 'postulates' and 'axioms' are terms that are used interchangeably and in the same sense. 'Postulate' is actually a verb. When we say "let us postulate", we mean, "let us make some statement based on the observed phenomenon in the Universe". Its truth/validity is checked afterwards. If it is true, then it is accepted as a 'Postulate'.

A system of axioms is called **consistent** (see Appendix 1), if it is impossible to deduce from these axioms a statement that contradicts any axiom or previously proved statement. So, when any system of axioms is given, it needs to be ensured that the system is consistent.

After Euclid stated his postulates and axioms, he used them to prove other results. Then using these results, he proved some more results by applying deductive reasoning. The statements that were proved are called **propositions or theorems**. Euclid deduced 465 propositions in a logical chain using his axioms, postulates, definitions and theorems proved earlier in the chain. In the next few chapters on geometry, you will be using these axioms to prove some theorems.

Now, let us see in the following examples how Euclid used his axioms and postulates for proving some of the results:

Example 1 : If A, B and C are three points on a line, and B lies between A and C (see Fig. 2.7), then prove that AB + BC = AC.

Fig. 2.7

Solution : In the figure given above, AC coincides with AB + BC.

Also, Euclid's Axiom (4) says that things which coincide with one another are equal to one another. So, it can be deduced that

$$AB + BC = AC$$

Note that in this solution, it has been assumed that there is a unique line passing through two points.

Example 2 : Prove that an equilateral triangle can be constructed on any given line segment.

Solution : In the statement above, a line segment of any length is given, say AB [see Fig. 5.8(i)].

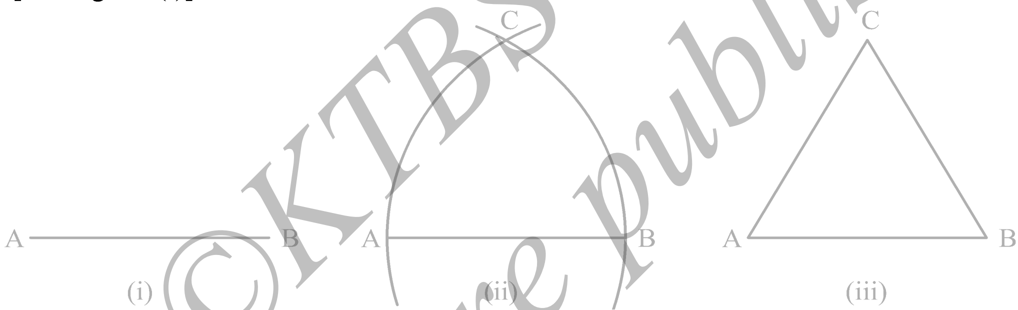


Fig. 2.8

Here, you need to do some construction. Using Euclid's Postulate 3, you can draw a circle with point A as the centre and AB as the radius [see Fig. 2.8(ii)]. Similarly, draw another circle with point B as the centre and BA as the radius. The two circles meet at a point, say C. Now, draw the line segments AC and BC to form Δ ABC [see Fig. 2.8 (iii)].

So, you have to prove that this triangle is equilateral, i.e., AB = AC = BC.

Now, AB = AC, since they are the radii of the same circle (1)

Similarly, AB = BC (Radii of the same circle) (2)

From these two facts, and Euclid's axiom that things which are equal to the same thing are equal to one another, you can conclude that AB = BC = AC.

So, Δ ABC is an equilateral triangle.

Note that here Euclid has assumed, without mentioning anywhere, that the two circles drawn with centres A and B will meet each other at a point.

Now we prove a theorem, which is frequently used in different results:

Theorem 2.1 : *Two distinct lines cannot have more than one point in common.*

Proof : Here we are given two lines *l* and *m*. We need to prove that they have only one point in common.

For the time being, let us suppose that the two lines intersect in two distinct points, say P and Q. So, you have two lines passing through two distinct points P and Q. But this assumption clashes with the axiom that only one line can pass through two distinct points. So, the assumption that we started with, that two lines can pass through two distinct points is wrong.

From this, what can we conclude? We are forced to conclude that two distinct lines cannot have more than one point in common.

EXERCISE 2.1

1. Which of the following statements are true and which are false? Give reasons for your answers.
   (i) Only one line can pass through a single point.
   (ii) There are an infinite number of lines which pass through two distinct points.
   (iii) A terminated line can be produced indefinitely on both the sides.
   (iv) If two circles are equal, then their radii are equal.
   (v) In Fig. 2.9, if AB = PQ and PQ = XY, then AB = XY.

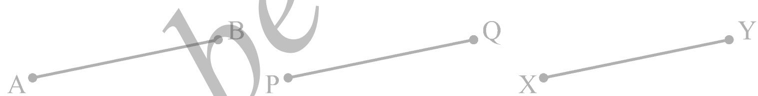


Fig. 2.9

2. Give a definition for each of the following terms. Are there other terms that need to be defined first? What are they, and how might you define them?
   (i) parallel lines (ii) perpendicular lines (iii) line segment
   (iv) radius of a circle (v) square
3. Consider two 'postulates' given below:
   (i) Given any two distinct points A and B, there exists a third point C which is in between A and B.
   (ii) There exist at least three points that are not on the same line.

   Do these postulates contain any undefined terms? Are these postulates consistent? Do they follow from Euclid's postulates? Explain.

4. If a point C lies between two points A and B such that AC = BC, then prove that $AC = \frac{1}{2}AB$. Explain by drawing the figure.

5. In Question 4, point C is called a mid-point of line segment AB. Prove that every line segment has one and only one mid-point.

6. In Fig. 2.10, if AC = BD, then prove that AB = CD.

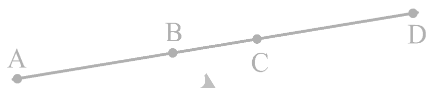


Fig. 2.10

7. Why is Axiom 5, in the list of Euclid's axioms, considered a 'universal truth'? (Note that the question is not about the fifth postulate.)

2.3 Equivalent Versions of Euclid's Fifth Postulate

Euclid's fifth postulate is very significant in the history of mathematics. Recall it again from Section 2.2. We see that by implication, no intersection of lines will take place when the sum of the measures of the interior angles on the same side of the falling line is exactly 180°. There are several equivalent versions of this postulate. One of them is 'Playfair's Axiom' (given by a Scottish mathematician John Playfair in 1729), as stated below:

'*For every line l and for every point* P *not lying on l, there exists a unique line m passing through* P *and parallel to l*'.

From Fig. 2.11, you can see that of all the lines passing through the point P, only line *m* is parallel to line *l*.

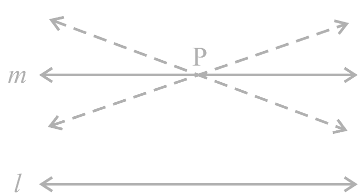


Fig. 2.11

This result can also be stated in the following form:

Two distinct intersecting lines cannot be parallel to the same line.

Euclid did not require his fifth postulate to prove his first 28 theorems. Many mathematicians, including him, were convinced that the fifth postulate is actually a theorem that can be proved using just the first four postulates and other axioms. However, all attempts to prove the fifth postulate as a theorem have failed. But these efforts have led to a great achievement – the creation of several other geometries. These geometries are quite different from Euclidean geometry. They are called *non-Euclidean geometries*. Their creation is considered a landmark in the history of thought because till then everyone had believed that Euclid's was the only geometry and the world itself was Euclidean. Now the geometry of the universe we live in has been shown to be a non-Euclidean geometry. In fact, it is called *spherical geometry*. In spherical geometry, lines are not straight. They are parts of great circles (i.e., circles obtained by the intersection of a sphere and planes passing through the centre of the sphere).

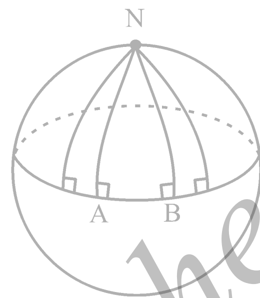


Fig. 2.12

In Fig. 2.12, the lines AN and BN (which are parts of great circles of a sphere) are perpendicular to the same line AB. But they are meeting each other, though the sum of the angles on the same side of line AB is not less than two right angles (in fact, it is $90° + 90° = 180°$). Also, note that the sum of the angles of the triangle NAB is greater than $180°$, as $\angle A + \angle B = 180°$. Thus, Euclidean geometry is valid only for the figures in the plane. On the curved surfaces, it fails.

Now, let us consider an example.

Example 3 : Consider the following statement : There exists a pair of straight lines that are everywhere equidistant from one another. Is this statement a direct consequence of Euclid's fifth postulate? Explain.

Solution : Take any line l and a point P not on l. Then, by Playfair's axiom, which is equivalent to the fifth postulate, we know that there is a unique line m through P which is parallel to l.

Now, the *distance of a point from a line* is the length of the perpendicular from the point to the line. This distance will be the same for any point on m from l and any point on l from m. So, these two lines are everywhere equidistant from one another.

Remark : The geometry that you will be studying in the next few chapters is Euclidean Geometry. However, the axioms and theorems used by us may be different from those of Euclid's.

EXERCISE 2.2

1. How would you rewrite Euclid's fifth postulate so that it would be easier to understand?
2. Does Euclid's fifth postulate imply the existence of parallel lines? Explain.

2.4 Summary

In this chapter, you have studied the following points:

1. Though Euclid defined a point, a line, and a plane, the definitions are not accepted by mathematicians. Therefore, these terms are now taken as undefined.
2. Axioms or postulates are the assumptions which are obvious universal truths. They are not proved.
3. Theorems are statements which are proved, using definitions, axioms, previously proved statements and deductive reasoning.
4. Some of Euclid's axioms were :

   (1) Things which are equal to the same thing are equal to one another.

   (2) If equals are added to equals, the wholes are equal.

   (3) If equals are subtracted from equals, the remainders are equal.

   (4) Things which coincide with one another are equal to one another.

   (5) The whole is greater than the part.

   (6) Things which are double of the same things are equal to one another.

   (7) Things which are halves of the same things are equal to one another.
5. Euclid's postulates were :

   Postulate 1 : A straight line may be drawn from any one point to any other point.

   Postulate 2 : A terminated line can be produced indefinitely.

   Postulate 3 : A circle can be drawn with any centre and any radius.

   Postulate 4 : All right angles are equal to one another.

   Postulate 5 : If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.
6. Two equivalent versions of Euclid's fifth postulate are:

   (i) 'For every line l and for every point P not lying on l, there exists a unique line m passing through P and parallel to l'.

   (ii) Two distinct intersecting lines cannot be parallel to the same line.
7. All the attempts to prove Euclid's fifth postulate using the first 4 postulates failed. But they led to the discovery of several other geometries, called non-Euclidean geometries.

287UIV

CHAPTER 3

LINES AND ANGLES

3.1 Introduction

In Chapter 2, you have studied that a minimum of two points are required to draw a line. You have also studied some axioms and, with the help of these axioms, you proved some other statements. In this chapter, you will study the properties of the angles formed when two lines intersect each other, and also the properties of the angles formed when a line intersects two or more parallel lines at distinct points. Further you will use these properties to prove some statements using deductive reasoning (see Appendix 1). You have already verified these statements through some activities in the earlier classes.

In your daily life, you see different types of angles formed between the edges of plane surfaces. For making a similar kind of model using the plane surfaces, you need to have a thorough knowledge of angles. For instance, suppose you want to make a model of a hut to keep in the school exhibition using bamboo sticks. Imagine how you would make it? You would keep some of the sticks parallel to each other, and some sticks would be kept slanted. Whenever an architect has to draw a plan for a multistoried building, she has to draw intersecting lines and parallel lines at different angles. Without the knowledge of the properties of these lines and angles, do you think she can draw the layout of the building?

In science, you study the properties of light by drawing the ray diagrams. For example, to study the refraction property of light when it enters from one medium to the other medium, you use the properties of intersecting lines and parallel lines. When two or more forces act on a body, you draw the diagram in which forces are represented by directed line segments to study the net effect of the forces on the body. At that time, you need to know the relation between the angles when the rays (or line segments) are parallel to or intersect each other. To find the height of a tower or to find the distance of a ship from the light house, one needs to know the angle formed between the horizontal and the line of sight. Plenty of other examples can be given where lines and angles are used. In the subsequent chapters of geometry, you will be using these properties of lines and angles to deduce more and more useful properties.

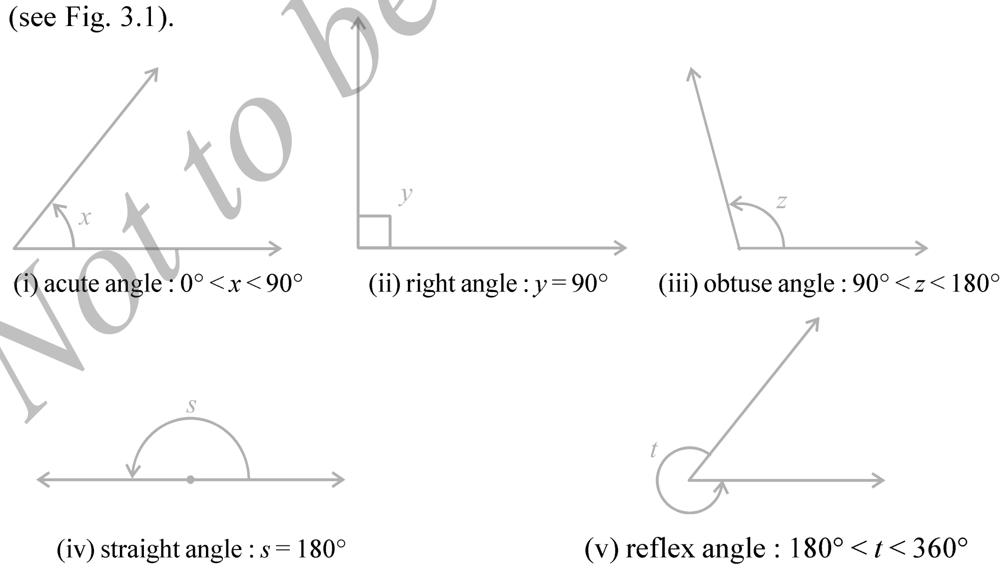
Let us first revise the terms and definitions related to lines and angles learnt in earlier classes.

3.2 Basic Terms and Definitions

Recall that a part (or portion) of a line with two end points is called a **line-segment** and a part of a line with one end point is called a **ray**. Note that the line segment AB is denoted by $\overline{\mathrm{AB}}$, and its length is denoted by AB. The ray AB is denoted by $\overrightarrow{\mathrm{AB}}$, and a line is denoted by $\overleftrightarrow{\mathrm{AB}}$. However, **we will not use these symbols**, and will denote the line segment AB, ray AB, length AB and line AB by the same symbol, AB. The meaning will be clear from the context. Sometimes small letters *l*, *m*, *n*, etc. will be used to denote lines.

If three or more points lie on the same line, they are called **collinear points**; otherwise they are called **non-collinear points**.

Recall that an **angle** is formed when two rays originate from the same end point. The rays making an angle are called the **arms** of the angle and the end point is called the **vertex** of the angle. You have studied different types of angles, such as acute angle, right angle, obtuse angle, straight angle and reflex angle in earlier classes (see Fig. 3.1).



(i) acute angle : $0° < x < 90°$ (ii) right angle : $y = 90°$ (iii) obtuse angle : $90° < z < 180°$

(iv) straight angle : $s = 180°$ (v) reflex angle : $180° < t < 360°$

Fig. 3.1 : Types of Angles

An **acute** angle measures between 0° and 90°, whereas a **right angle** is exactly equal to 90°. An angle greater than 90° but less than 180° is called an **obtuse angle**. Also, recall that a **straight angle** is equal to 180°. An angle which is greater than 180° but less than 360° is called a **reflex angle**. Further, two angles whose sum is 90° are called **complementary angles**, and two angles whose sum is 180° are called **supplementary angles**.

You have also studied about adjacent angles in the earlier classes (see Fig. 3.2). Two angles are **adjacent**, if they have a common vertex, a common arm and their non-common arms are on different sides of the common arm. In Fig. 3.2, $\angle$ ABD and $\angle$ DBC are adjacent angles. Ray BD is their common arm and point B is their common vertex. Ray BA and ray BC are non common arms. Moreover, when two angles are adjacent, then their sum is always equal to the angle formed by the two non-common arms. So, we can write

$$\angle \text{ABC} = \angle \text{ABD} + \angle \text{DBC}.$$

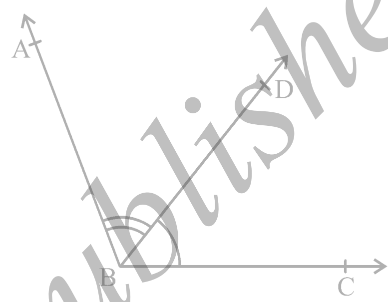


Fig. 3.2 : Adjacent angles

Note that $\angle$ ABC and $\angle$ ABD are not adjacent angles. Why? Because their non-common arms BD and BC lie on the same side of the common arm BA.

If the non-common arms BA and BC in Fig. 3.2, form a line then it will look like Fig. 3.3. In this case, $\angle$ ABD and $\angle$ DBC are called **linear pair of angles**.

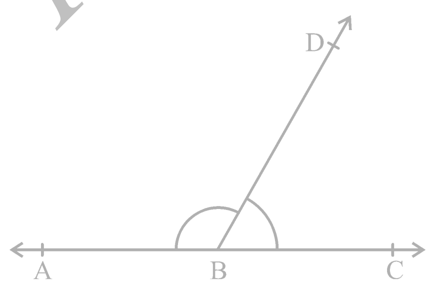


Fig. 3.3 : Linear pair of angles

You may also recall the **vertically opposite angles** formed when two lines, say AB and CD, intersect each other, say at the point O (see Fig. 3.4). There are two pairs of vertically opposite angles.

One pair is $\angle$AOD and $\angle$BOC. Can you find the other pair?

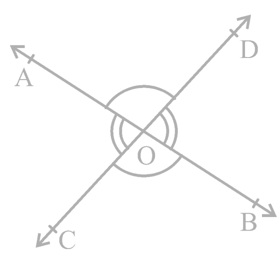
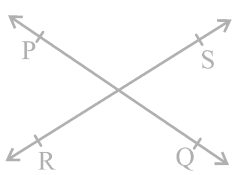


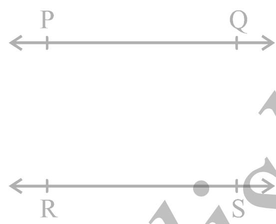
Fig. 3.4 : Vertically opposite angles

3.3 Intersecting Lines and Non-intersecting Lines

Draw two different lines PQ and RS on a paper. You will see that you can draw them in two different ways as shown in Fig. 3.5 (i) and Fig. 3.5 (ii).



(i) Intersecting lines



(ii) Non-intersecting (parallel) lines

Fig. 3.5 : Different ways of drawing two lines

Recall the notion of a line, that it extends indefinitely in both directions. Lines PQ and RS in Fig. 3.5 (i) are intersecting lines and in Fig. 3.5 (ii) are parallel lines. Note that the lengths of the common perpendiculars at different points on these parallel lines is the same. This equal length is called the *distance between two parallel lines*.

3.4 Pairs of Angles

In Section 3.2, you have learnt the definitions of some of the pairs of angles such as complementary angles, supplementary angles, adjacent angles, linear pair of angles, etc. Can you think of some relations between these angles? Now, let us find out the relation between the angles formed when a ray stands on a line. Draw a figure in which a ray stands on a line as shown in Fig. 3.6. Name the line as AB and the ray as OC. What are the angles formed at the point O? They are $\angle$ AOC, $\angle$ BOC and $\angle$ AOB.

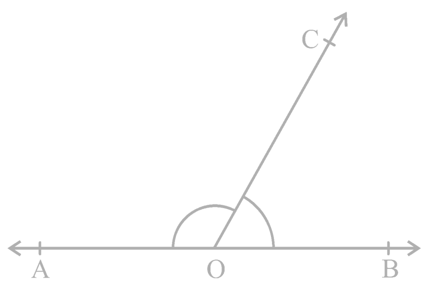


Fig. 3.6 : Linear pair of angles

Can we write $\angle$ AOC + $\angle$ BOC = $\angle$ AOB? (1)

Yes! (Why? Refer to adjacent angles in Section 6.2)

What is the measure of $\angle$ AOB? It is 180°. (Why?) (2)

From (1) and (2), can you say that $\angle$ AOC + $\angle$ BOC = 180°? Yes! (Why?)

From the above discussion, we can state the following Axiom:

Axiom 3.1 : *If a ray stands on a line, then the sum of two adjacent angles so formed is* 180°.

Recall that when the sum of two adjacent angles is 180°, then they are called a **linear pair of angles**.

In Axiom 3.1, it is given that 'a ray stands on a line'. From this 'given', we have concluded that 'the sum of two adjacent angles so formed is 180°'. Can we write Axiom 3.1 the other way? That is, take the 'conclusion' of Axiom 3.1 as 'given' and the 'given' as the 'conclusion'. So it becomes:

(A) If the sum of two adjacent angles is 180°, then a ray stands on a line (that is, the non-common arms form a line).

Now you see that the Axiom 6.1 and statement (A) are in a sense the reverse of each others. We call each as converse of the other. We do not know whether the statement (A) is true or not. Let us check. Draw adjacent angles of different measures as shown in Fig. 3.7. Keep the ruler along one of the non-common arms in each case. Does the other non-common arm also lie along the ruler?

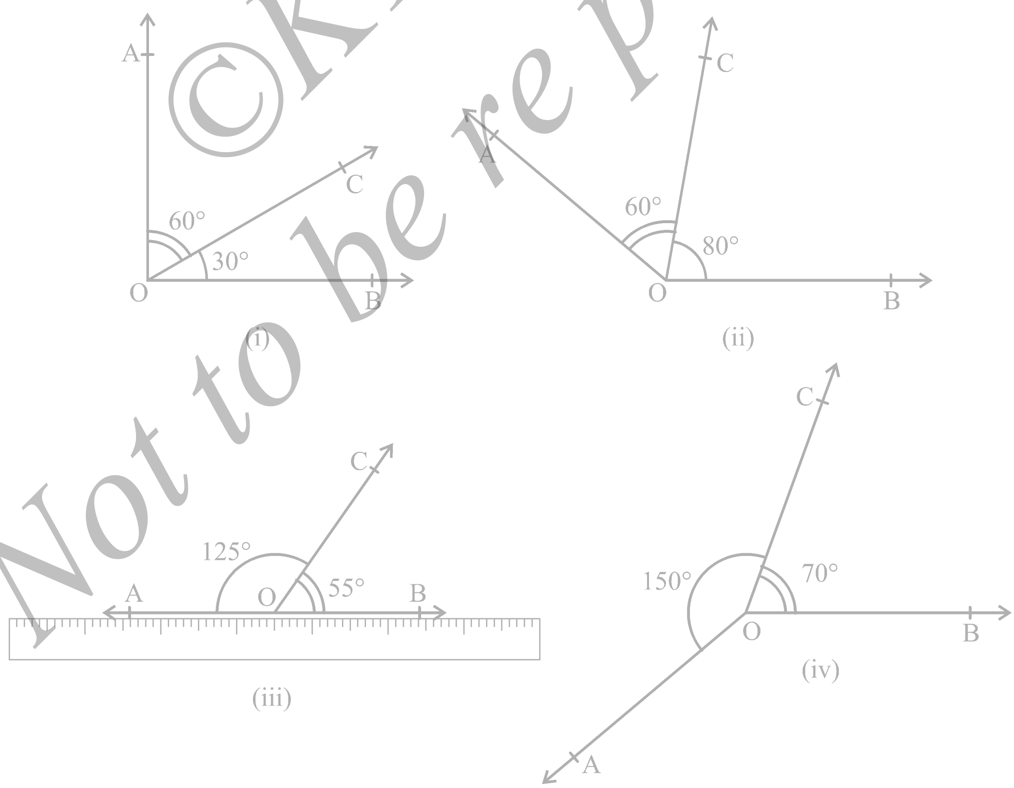


Fig. 3.7 : Adjacent angles with different measures

You will find that only in Fig. 3.7 (iii), both the non-common arms lie along the ruler, that is, points A, O and B lie on the same line and ray OC stands on it. Also see that $\angle$ AOC + $\angle$ COB = 125° + 55° = 180°. From this, you may conclude that statement (A) is true. So, you can state in the form of an axiom as follows:

Axiom 3.2 : *If the sum of two adjacent angles is* 180°, *then the non-common arms of the angles form a line.*

For obvious reasons, the two axioms above together is called the **Linear Pair Axiom**.

Let us now examine the case when two lines intersect each other.

Recall, from earlier classes, that when two lines intersect, the vertically opposite angles are equal. Let us prove this result now. See Appendix 1 for the ingredients of a proof, and keep those in mind while studying the proof given below.

Theorem 3.1 : *If two lines intersect each other, then the vertically opposite angles are equal.*

Proof : In the statement above, it is given that 'two lines intersect each other'. So, let AB and CD be two lines intersecting at O as shown in Fig. 3.8. They lead to two pairs of vertically opposite angles, namely,

(i) $\angle$ AOC and $\angle$ BOD (ii) $\angle$ AOD and $\angle$ BOC.

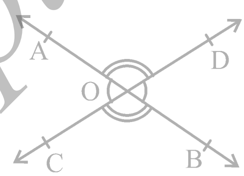


Fig. 3.8 : Vertically opposite angles

We need to prove that $\angle$ AOC = $\angle$ BOD and $\angle$ AOD = $\angle$ BOC.

Now, ray OA stands on line CD.

Therefore, $\angle$ AOC + $\angle$ AOD = 180° (Linear pair axiom) (1)

Can we write $\angle$ AOD + $\angle$ BOD = 180°? Yes! (Why?) (2)

From (1) and (2), we can write

$$\angle \text{AOC} + \angle \text{AOD} = \angle \text{AOD} + \angle \text{BOD}$$

This implies that $\angle$ AOC = $\angle$ BOD (Refer Section 2.2, Axiom 3)

Similarly, it can be proved that $\angle$AOD = $\angle$BOC

Now, let us do some examples based on Linear Pair Axiom and Theorem 3.1.

Example 1 : In Fig. 3.9, lines PQ and RS intersect each other at point O. If $\angle$ POR : $\angle$ ROQ = 5 : 7, find all the angles.

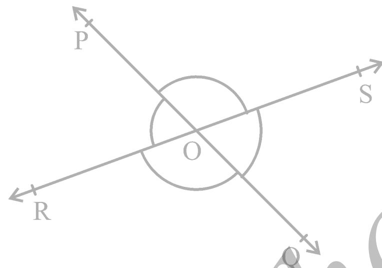


Fig. 3.9

Solution : $\angle$ POR + $\angle$ ROQ = 180° (Linear pair of angles)

But $\angle$ POR : $\angle$ ROQ = 5 : 7 (Given)

Therefore, $\angle \text{POR} = \frac{5}{12} \times 180° = 75°$

Similarly, $\angle \text{ROQ} = \frac{7}{12} \times 180° = 105°$

Now, $\angle$ POS = $\angle$ROQ = 105° (Vertically opposite angles)

and $\angle$ SOQ = $\angle$POR = 75° (Vertically opposite angles)

Example 2 : In Fig. 3.10, ray OS stands on a line POQ. Ray OR and ray OT are angle bisectors of $\angle$ POS and $\angle$ SOQ, respectively. If $\angle$ POS = x, find $\angle$ ROT.

Solution : Ray OS stands on the line POQ.

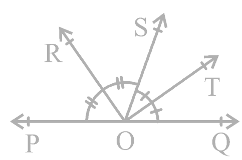


Fig. 3.10

Therefore, $\angle$ POS + $\angle$ SOQ = 180°

But, $\angle$ POS = x

Therefore, x + $\angle$ SOQ = 180°

So, $\angle$ SOQ = 180° – x

Now, ray OR bisects $\angle$ POS, therefore,

$$\angle \text{ROS} = \frac{1}{2} \times \angle \text{POS}$$

$$= \frac{1}{2} \times x = \frac{x}{2}$$

Similarly, $$\angle \text{SOT} = \frac{1}{2} \times \angle \text{SOQ}$$

$$= \frac{1}{2} \times (180° - x)$$

$$= 90° - \frac{x}{2}$$

Now, $\angle$ ROT = $\angle$ ROS + $\angle$ SOT

$$= \frac{x}{2} + 90° - \frac{x}{2}$$

$$= 90°$$

Example 3 : In Fig. 3.11, OP, OQ, OR and OS are four rays. Prove that $\angle$ POQ + $\angle$ QOR + $\angle$ SOR + $\angle$ POS = 360°.

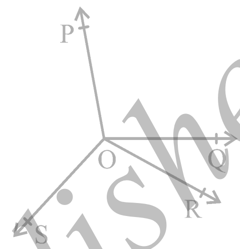


Fig. 3.11

Solution : In Fig. 3.11, you need to produce any of the rays OP, OQ, OR or OS backwards to a point. Let us produce ray OQ backwards to a point T so that TOQ is a line (see Fig. 3.12).

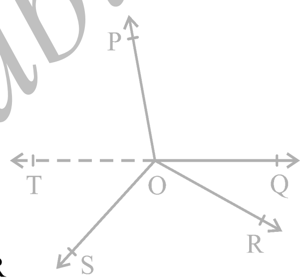


Fig. 3.12

Now, ray OP stands on line TOQ.

Therefore, $\angle$ TOP + $\angle$ POQ = 180° (1)

(Linear pair axiom)

Similarly, ray OS stands on line TOQ.

Therefore, $\angle$ TOS + $\angle$ SOQ = 180° (2)

But $\angle$ SOQ = $\angle$ SOR + $\angle$ QOR

So, (2) becomes

$\angle$ TOS + $\angle$ SOR + $\angle$ QOR = 180° (3)

Now, adding (1) and (3), you get

$\angle$ TOP + $\angle$ POQ + $\angle$ TOS + $\angle$ SOR + $\angle$ QOR = 360° (4)

But $\angle$ TOP + $\angle$ TOS = $\angle$ POS

Therefore, (4) becomes

$\angle$ POQ + $\angle$ QOR + $\angle$ SOR + $\angle$ POS = 360°

EXERCISE 3.1

1. In Fig. 3.13, lines AB and CD intersect at O. If $\angle$ AOC + $\angle$ BOE = 70° and $\angle$ BOD = 40°, find $\angle$ BOE and reflex $\angle$ COE.

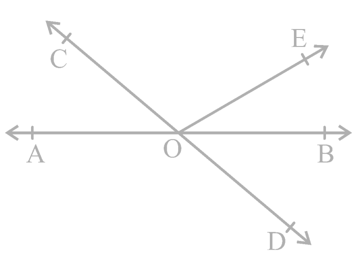


Fig. 3.13

2. In Fig. 3.14, lines XY and MN intersect at O. If $\angle$ POY = 90° and $a : b = 2 : 3$, find c.

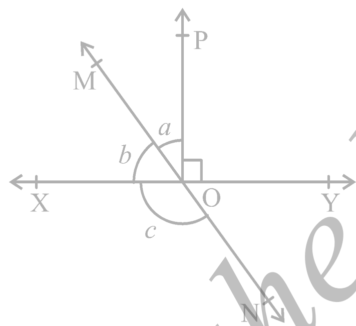


Fig. 3.14

3. In Fig. 3.15, $\angle$ PQR = $\angle$ PRQ, then prove that $\angle$ PQS = $\angle$ PRT.

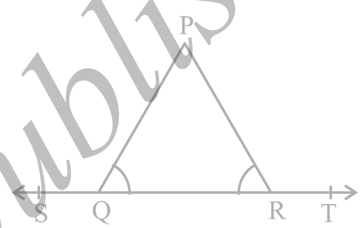


Fig. 3.15

4. In Fig. 3.16, if $x + y = w + z$, then prove that AOB is a line.

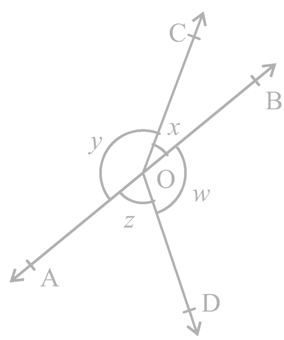


Fig. 3.16

5. In Fig. 3.17, POQ is a line. Ray OR is perpendicular to line PQ. OS is another ray lying between rays OP and OR. Prove that

$\angle \text{ROS} = \frac{1}{2}(\angle \text{QOS} - \angle \text{POS})$.

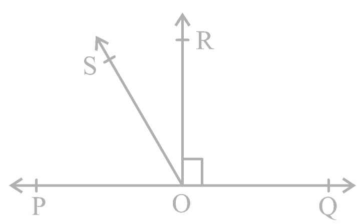


Fig. 3.17

6. It is given that $\angle$ XYZ = 64° and XY is produced to point P. Draw a figure from the given information. If ray YQ bisects $\angle$ ZYP, find $\angle$ XYQ and reflex $\angle$ QYP.

3.5 Parallel Lines and a Transversal

Recall that a line which intersects two or more lines at distinct points is called a **transversal** (see Fig. 3.18). Line *l* intersects lines *m* and *n* at points P and Q respectively. Therefore, line *l* is a transversal for lines *m* and *n*. Observe that four angles are formed at each of the points P and Q.

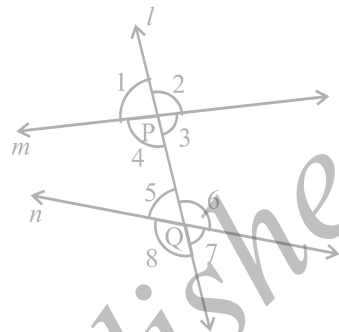


Fig. 3.18

Let us name these angles as ∠ 1, ∠ 2, . . ., ∠8 as shown in Fig. 3.18.

∠ 1, ∠ 2, ∠ 7 and ∠ 8 are called **exterior angles**, while ∠ 3, ∠ 4, ∠ 5 and ∠ 6 are called **interior angles**.

Recall that in the earlier classes, you have named some pairs of angles formed when a transversal intersects two lines. These are as follows:

(a) **Corresponding angles :**

(i) ∠ 1 and ∠ 5 (ii) ∠ 2 and ∠ 6

(iii) ∠ 4 and ∠ 8 (iv) ∠ 3 and ∠ 7

(b) **Alternate interior angles :**

(i) ∠ 4 and ∠ 6 (ii) ∠ 3 and ∠ 5

(c) **Alternate exterior angles:**

(i) ∠ 1 and ∠ 7 (ii) ∠ 2 and ∠ 8

(d) **Interior angles on the same side of the transversal:**

(i) ∠ 4 and ∠ 5 (ii) ∠ 3 and ∠ 6

Interior angles on the same side of the transversal are also referred to as **consecutive interior** angles or **allied** angles or **co-interior** angles. Further, many a times, we simply use the words alternate angles for alternate interior angles.

Now, let us find out the relation between the angles in these pairs when line *m* is parallel to line *n*. You know that the ruled lines of your notebook are parallel to each other. So, with ruler and pencil, draw two parallel lines along any two of these lines and a transversal to intersect them as shown in Fig. 3.19.

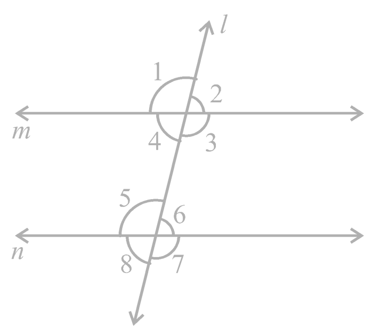


Fig. 3.19

Now, measure any pair of corresponding angles and find out the relation between them. You may find that : $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 4 = \angle 8$ and $\angle 3 = \angle 7$. From this, you may conclude the following axiom.

Axiom 3.3 : *If a transversal intersects two parallel lines, then each pair of corresponding angles is equal.*

Axiom 3.3 is also referred to as the **corresponding angles axiom**. Now, let us discuss the converse of this axiom which is as follows:

If a transversal intersects two lines such that a pair of corresponding angles is equal, then the two lines are parallel.

Does this statement hold true? It can be verified as follows: Draw a line AD and mark points B and C on it. At B and C, construct $\angle$ ABQ and $\angle$ BCS equal to each other as shown in Fig. 3.20 (i).

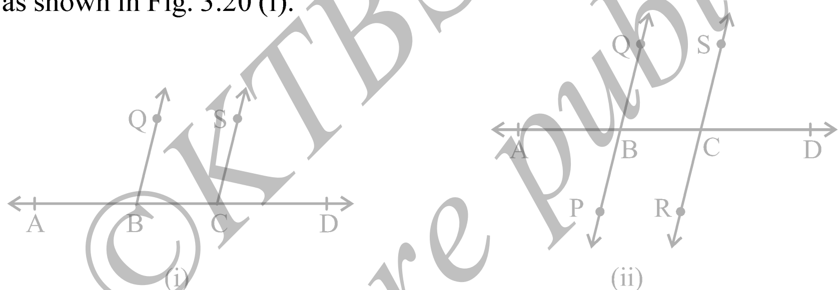


Fig. 3.20

Produce QB and SC on the other side of AD to form two lines PQ and RS [see Fig. 3.20 (ii)]. You may observe that the two lines do not intersect each other. You may also draw common perpendiculars to the two lines PQ and RS at different points and measure their lengths. You will find it the same everywhere. So, you may conclude that the lines are parallel. Therefore, the converse of corresponding angles axiom is also true. So, we have the following axiom:

Axiom 3.4 : *If a transversal intersects two lines such that a pair of corresponding angles is equal, then the two lines are parallel to each other.*

Can we use corresponding angles axiom to find out the relation between the alternate interior angles when a transversal intersects two parallel lines? In Fig. 3.21, transveral PS intersects parallel lines AB and CD at points Q and R respectively.

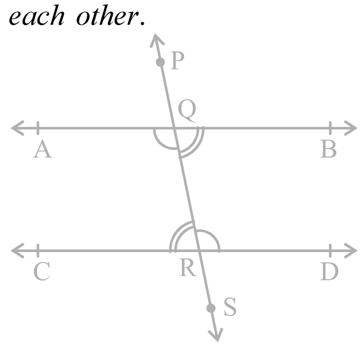


Fig. 3.21

Is $\angle$ BQR = $\angle$ QRC and $\angle$ AQR = $\angle$ QRD?

You know that $\angle$ PQA = $\angle$ QRC (1)

(Corresponding angles axiom)

Is $\angle$ PQA = $\angle$ BQR? Yes! (Why ?) (2)

So, from (1) and (2), you may conclude that

$$\angle \text{BQR} = \angle \text{QRC}.$$

Similarly, $\angle$ AQR = $\angle$ QRD.

This result can be stated as a theorem given below:

Theorem 3.2 : *If a transversal intersects two parallel lines, then each pair of alternate interior angles is equal.*

Now, using the converse of the corresponding angles axiom, can we show the two lines parallel if a pair of alternate interior angles is equal? In Fig. 3.22, the transversal PS intersects lines AB and CD at points Q and R respectively such that $\angle$ BQR = $\angle$ QRC.

Is AB || CD?

$\angle$ BQR = $\angle$ PQA (Why?) (1)

But, $\angle$ BQR = $\angle$ QRC (Given) (2)

So, from (1) and (2), you may conclude that

$\angle$ PQA = $\angle$ QRC

But they are corresponding angles.

So, AB || CD (Converse of corresponding angles axiom)

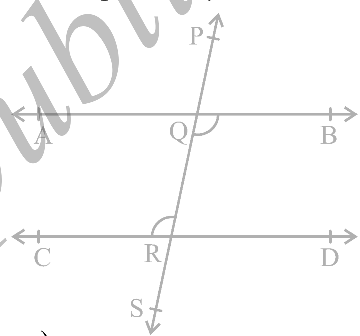


Fig. 3.22

This result can be stated as a theorem given below:

Theorem 3.3 : *If a transversal intersects two lines such that a pair of alternate interior angles is equal, then the two lines are parallel.*

In a similar way, you can obtain the following two theorems related to interior angles on the same side of the transversal.

Theorem 3.4 : *If a transversal intersects two parallel lines, then each pair of interior angles on the same side of the transversal is supplementary.*

Theorem 3.5 : *If a transversal intersects two lines such that a pair of interior angles on the same side of the transversal is supplementary, then the two lines are parallel.*

You may recall that you have verified all the above axioms and theorems in earlier classes through activities. You may repeat those activities here also.

3.6 Lines Parallel to the Same Line

If two lines are parallel to the same line, will they be parallel to each other? Let us check it. See Fig. 3.23 in which line $m \parallel$ line l and line $n \parallel$ line l.

Let us draw a line t transversal for the lines, l, m and n. It is given that line $m \parallel$ line l and line $n \parallel$ line l.

Therefore, $\angle 1 = \angle 2$ and $\angle 1 = \angle 3$

(Corresponding angles axiom)

So, $\angle 2 = \angle 3$ (Why?)

But $\angle 2$ and $\angle 3$ are corresponding angles and they are equal.

Therefore, you can say that

Line $m \parallel$ Line n

(Converse of corresponding angles axiom)

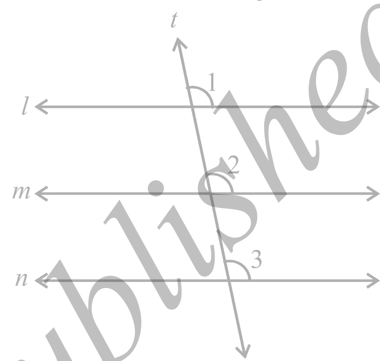


Fig. 3.23

This result can be stated in the form of the following theorem:

Theorem 3.6 : *Lines which are parallel to the same line are parallel to each other.*

Note : The property above can be extended to more than two lines also.

Now, let us solve some examples related to parallel lines.

Example 4 : In Fig. 3.24, if PQ || RS, $\angle$ MXQ = 135° and $\angle$ MYR = 40°, find $\angle$ XMY.

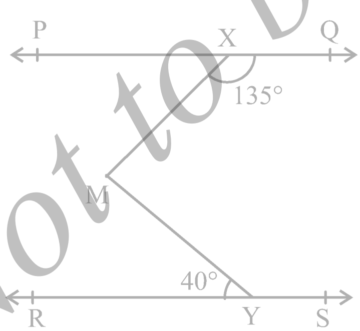


Fig. 3.24

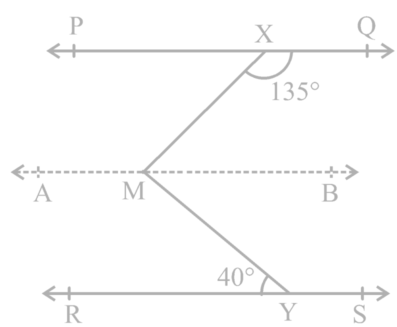


Fig. 3.25

Solution : Here, we need to draw a line AB parallel to line PQ, through point M as shown in Fig. 3.25. Now, AB || PQ and PQ || RS.

Therefore, AB || RS (Why?)

Now, $\angle$ QXM + $\angle$ XMB = 180°

(AB || PQ, Interior angles on the same side of the transversal XM)

But $\angle$ QXM = 135°

So, 135° + $\angle$ XMB = 180°

Therefore, $\angle$ XMB = 45° (1)

Now, $\angle$ BMY = $\angle$ MYR (AB || RS, Alternate angles)

Therefore, $\angle$ BMY = 40° (2)

Adding (1) and (2), you get

$\angle$ XMB + $\angle$ BMY = 45° + 40°

That is, $\angle$ XMY = 85°

Example 5 : If a transversal intersects two lines such that the bisectors of a pair of corresponding angles are parallel, then prove that the two lines are parallel.

Solution : In Fig. 3.26, a transversal AD intersects two lines PQ and RS at points B and C respectively. Ray BE is the bisector of $\angle$ ABQ and ray CG is the bisector of $\angle$ BCS; and BE || CG.

We are to prove that PQ || RS.

It is given that ray BE is the bisector of $\angle$ ABQ.

Therefore, $\angle \text{ABE} = \frac{1}{2} \angle \text{ABQ}$ (1)

Similarly, ray CG is the bisector of $\angle$ BCS.

Therefore, $\angle \text{BCG} = \frac{1}{2} \angle \text{BCS}$ (2)

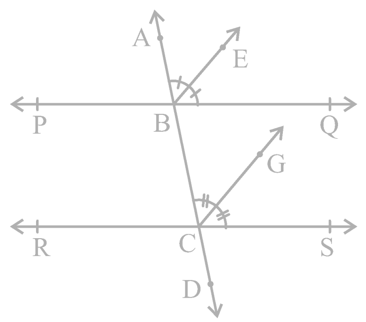


Fig. 3.26

But BE || CG and AD is the transversal.

Therefore, $\angle$ ABE = $\angle$ BCG

(Corresponding angles axiom) (3)

Substituting (1) and (2) in (3), you get

$$\frac{1}{2} \angle \text{ABQ} = \frac{1}{2} \angle \text{BCS}$$

That is, $\angle$ ABQ = $\angle$ BCS

But, they are the corresponding angles formed by transversal AD with PQ and RS; and are equal.

Therefore, PQ || RS

(Converse of corresponding angles axiom)

Example 6 : In Fig. 3.27, AB || CD and CD || EF. Also EA ⊥ AB. If ∠ BEF = 55°, find the values of x, y and z.

Solution : $y + 55° = 180°$

(Interior angles on the same side of the transversal ED)

Therefore, $y = 180° – 55° = 125°$

Again $x = y$

(AB || CD, Corresponding angles axiom)

Therefore $x = 125°$

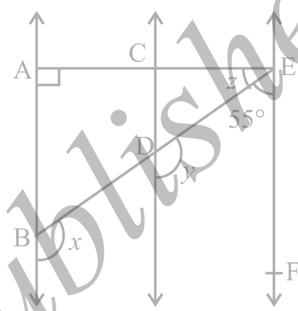


Fig. 3.27

Now, since AB || CD and CD || EF, therefore, AB || EF.

So, ∠ EAB + ∠ FEA = 180° (Interior angles on the same side of the transversal EA)

Therefore, $90° + z + 55° = 180°$

Which gives $z = 35°$

EXERCISE 3.2

1. In Fig. 3.28, find the values of x and y and then show that AB || CD.

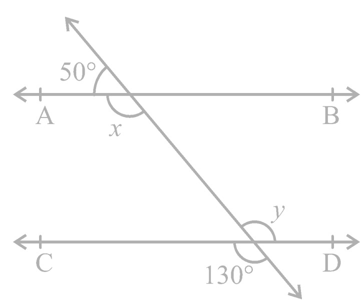


Fig. 3.28

2. In Fig. 3.29, if AB || CD, CD || EF and $y : z = 3 : 7$, find x.

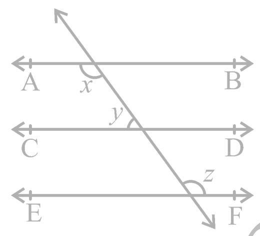


Fig. 3.29

3. In Fig. 3.30, if AB || CD, EF ⊥ CD and ∠ GED = 126°, find ∠ AGE, ∠ GEF and ∠ FGE.

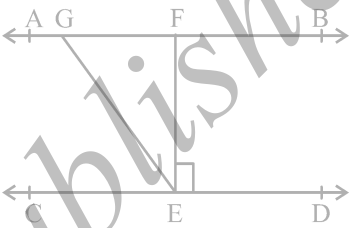


Fig. 3.30

4. In Fig. 3.31, if PQ || ST, ∠ PQR = 110° and ∠ RST = 130°, find ∠ QRS.

[**Hint :** Draw a line parallel to ST through point R.]

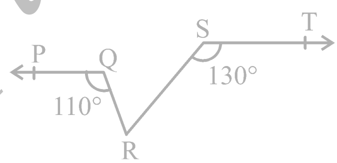


Fig. 3.31

5. In Fig. 3.32, if AB || CD, ∠ APQ = 50° and ∠ PRD = 127°, find x and y.

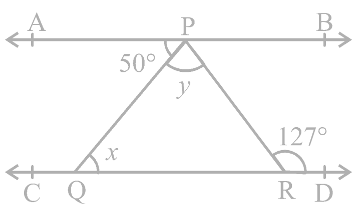


Fig. 3.32

6. In Fig. 3.33, PQ and RS are two mirrors placed parallel to each other. An incident ray AB strikes the mirror PQ at B, the reflected ray moves along the path BC and strikes the mirror RS at C and again reflects back along CD. Prove that AB || CD.

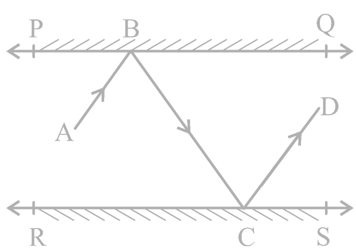


Fig. 3.33

3.7 Angle Sum Property of a Triangle

In the earlier classes, you have studied through activities that the sum of all the angles of a triangle is 180°. We can prove this statement using the axioms and theorems related to parallel lines.

Theorem 3.7 : *The sum of the angles of a triangle is 180º.*

Proof : Let us see what is given in the statement above, that is, the hypothesis and what we need to prove. We are given a triangle PQR and $\angle 1$, $\angle 2$ and $\angle 3$ are the angles of Δ PQR (see Fig. 3.34).

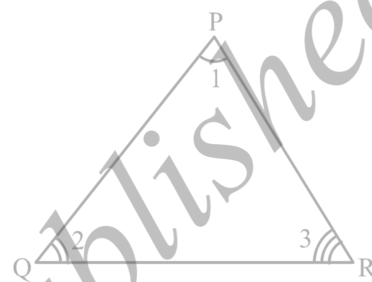


Fig. 3.34

We need to prove that $\angle 1 + \angle 2 + \angle 3 = 180°$. Let us draw a line XPY parallel to QR through the opposite vertex P, as shown in Fig. 3.35, so that we can use the properties related to parallel lines.

Now, XPY is a line.

Therefore, $\angle 4 + \angle 1 + \angle 5 = 180°$ (1)

But XPY || QR and PQ, PR are transversals.

So, $\angle 4 = \angle 2$ and $\angle 5 = \angle 3$

(Pairs of alternate angles)

Substituting $\angle 4$ and $\angle 5$ in (1), we get

$$\angle 2 + \angle 1 + \angle 3 = 180°$$

That is, $\angle 1 + \angle 2 + \angle 3 = 180°$

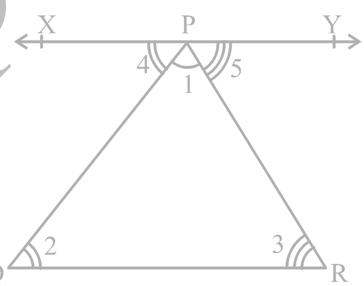


Fig. 3.35

Recall that you have studied about the formation of an exterior angle of a triangle in the earlier classes (see Fig. 3.36). Side QR is produced to point S, $\angle$ PRS is called an exterior angle of ΔPQR.

Is $\angle 3 + \angle 4 = 180°$? (Why?) (1)

Also, see that

$\angle 1 + \angle 2 + \angle 3 = 180°$ (Why?) (2)

From (1) and (2), you can see that

$$\angle 4 = \angle 1 + \angle 2.$$

This result can be stated in the form of a theorem as given below:

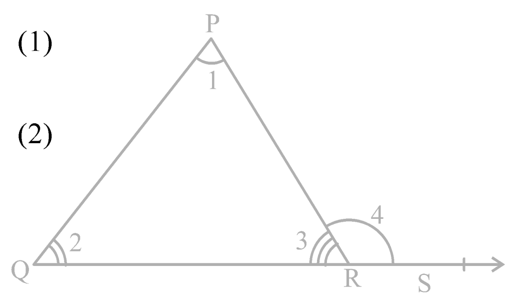


Fig. 3.36

Theorem 3.8 : *If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.*

It is obvious from the above theorem that an *exterior angle of a triangle is greater than either of its interior apposite angles.*

Now, let us take some examples based on the above theorems.

Example 7 : In Fig. 3.37, if QT $\perp$ PR, $\angle$ TQR = 40° and $\angle$ SPR = 30°, find x and y.

Solution : In Δ TQR, 90° + 40° + x = 180°

(Angle sum property of a triangle)

Therefore, $x = 50°$

Now, $y = \angle \text{SPR} + x$ (Theorem 3.8)

Therefore, $y = 30° + 50°$

$= 80°$

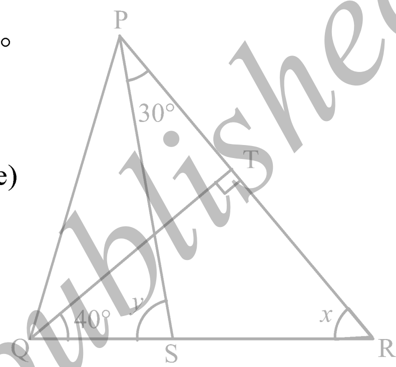


Fig. 3.37

Example 8 : In Fig. 3.38, the sides AB and AC of ΔABC are produced to points E and D respectively. If bisectors BO and CO of $\angle$ CBE and $\angle$ BCD respectively meet at point O, then prove that $\angle \text{BOC} = 90° - \frac{1}{2} \angle \text{BAC}$.

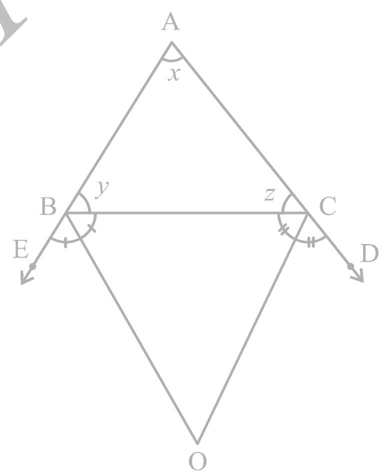


Fig. 3.38

Solution : Ray BO is the bisector of $\angle$ CBE.

Therefore, $\angle \text{CBO} = \frac{1}{2} \angle \text{CBE}$

$= \frac{1}{2}(180° - y)$

$= 90° - \frac{y}{2}$ (1)

Similarly, ray CO is the bisector of $\angle$ BCD.

Therefore, $\angle \text{BCO} = \frac{1}{2} \angle \text{BCD}$

$= \frac{1}{2}(180° - z)$

$= 90° - \frac{z}{2}$ (2)

In Δ BOC, $\angle$ BOC + $\angle$ BCO + $\angle$ CBO = 180° (3)

Substituting (1) and (2) in (3), you get

$$\angle \text{BOC} + 90° - \frac{z}{2} + 90° - \frac{y}{2} = 180°$$

So, $\angle \text{BOC} = \dfrac{z}{2} + \dfrac{y}{2}$

or, $\angle \text{BOC} = \dfrac{1}{2}(y + z)$ (4)

But, $x + y + z = 180°$ (Angle sum property of a triangle)

Therefore, $y + z = 180° - x$

Therefore, (4) becomes

$$\angle \text{BOC} = \frac{1}{2}(180° - x)$$

$$= 90° - \frac{x}{2}$$

$$= 90° - \frac{1}{2}\angle \text{BAC}$$

EXERCISE 3.3

1. In Fig. 3.39, sides QP and RQ of Δ PQR are produced to points S and T respectively. If $\angle$ SPR = 135° and $\angle$ PQT = 110°, find $\angle$ PRQ.
2. In Fig. 3.40, $\angle$ X = 62°, $\angle$ XYZ = 54°. If YO and ZO are the bisectors of $\angle$ XYZ and $\angle$ XZY respectively of Δ XYZ, find $\angle$ OZY and $\angle$ YOZ.
3. In Fig. 3.41, if AB || DE, $\angle$ BAC = 35° and $\angle$ CDE = 53°, find $\angle$ DCE.

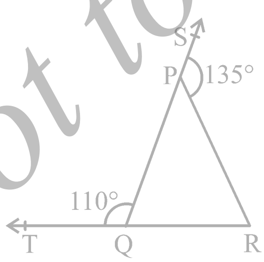


Fig. 3.39

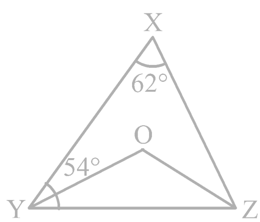


Fig. 3.40

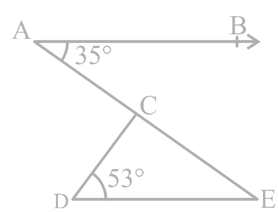


Fig. 3.41

4. In Fig. 3.42, if lines PQ and RS intersect at point T, such that $\angle$ PRT = 40°, $\angle$ RPT = 95° and $\angle$ TSQ = 75°, find $\angle$ SQT.

5. In Fig. 3.43, if PQ $\perp$ PS, PQ $\parallel$ SR, $\angle$ SQR = 28° and $\angle$ QRT = 65°, then find the values of x and y.

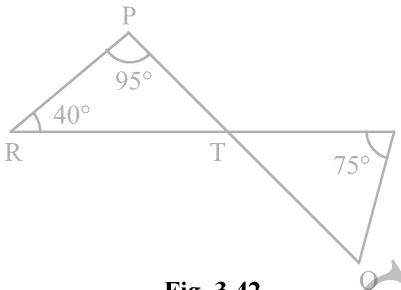


Fig. 3.42

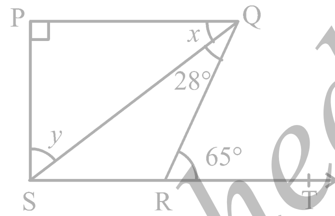


Fig. 3.43

6. In Fig. 3.44, the side QR of Δ PQR is produced to a point S. If the bisectors of $\angle$ PQR and $\angle$ PRS meet at point T, then prove that $\angle \text{QTR} = \frac{1}{2} \angle \text{QPR}$.

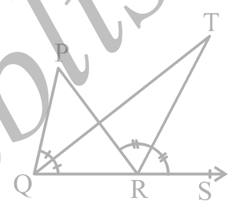


Fig. 3.44

3.8 Summary

In this chapter, you have studied the following points:

1. If a ray stands on a line, then the sum of the two adjacent angles so formed is 180° and vice-versa. This property is called as the Linear pair axiom.
2. If two lines intersect each other, then the vertically opposite angles are equal.
3. If a transversal intersects two parallel lines, then
   (i) each pair of corresponding angles is equal,
   (ii) each pair of alternate interior angles is equal,
   (iii) each pair of interior angles on the same side of the transversal is supplementary.
4. If a transversal intersects two lines such that, either
   (i) any one pair of corresponding angles is equal, or
   (ii) any one pair of alternate interior angles is equal, or
   (iii) any one pair of interior angles on the same side of the transversal is supplementary, then the lines are parallel.
5. Lines which are parallel to a given line are parallel to each other.
6. The sum of the three angles of a triangle is 180°.
7. If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

CHAPTER 4

POLYNOMIALS

4.1 Introduction

You have studied algebraic expressions, their addition, subtraction, multiplication and division in earlier classes. You also have studied how to factorise some algebraic expressions. You may recall the algebraic identities :

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

and $$x^2 - y^2 = (x + y)(x - y)$$

and their use in factorisation. In this chapter, we shall start our study with a particular type of algebraic expression, called *polynomial,* and the terminology related to it. We shall also study the *Remainder Theorem* and *Factor Theorem* and their use in the factorisation of polynomials. In addition to the above, we shall study some more algebraic identities and their use in factorisation and in evaluating some given expressions.

4.2 Polynomials in One Variable

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x, y, z, etc. to denote variables. Notice that $2x$, $3x$, $-x$, $-\frac{1}{2}x$ are algebraic expressions. All these expressions are of the form (a constant) × x. Now suppose we want to write an expression which is (a constant) × (a variable) and we do not know what the constant is. In such cases, we write the constant as a, b, c, etc. So the expression will be ax, say.

However, there is a difference between a letter denoting a constant and a letter denoting a variable. The values of the constants remain the same throughout a particular situation, that is, the values of the constants do not change in a given problem, but the value of a variable can keep changing.

Now, consider a square of side 3 units (see Fig. 4.1). What is its perimeter? You know that the perimeter of a square is the sum of the lengths of its four sides. Here, each side is 3 units. So, its perimeter is 4×3, i.e., 12 units. What will be the perimeter if each side of the square is 10 units? The perimeter is 4×10, i.e., 40 units. In case the length of each side is x units (see Fig. 4.2), the perimeter is given by $4x$ units. So, as the length of the side varies, the perimeter varies.

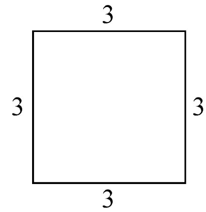


Fig. 4.1

Can you find the area of the square PQRS? It is $x \times x = x^2$ square units. x^2 is an algebraic expression. You are also familiar with other algebraic expressions like $2x$, $x^2 + 2x$, $x^3 - x^2 + 4x + 7$. Note that, all the algebraic expressions we have considered so far have only whole numbers as the exponents of the variable. Expressions of this form are called *polynomials in one variable*. In the examples above, the variable is x. For instance, $x^3 - x^2 + 4x + 7$ is a polynomial in x. Similarly, $3y^2 + 5y$ is a polynomial in the variable y and $t^2 + 4$ is a polynomial in the variable t.

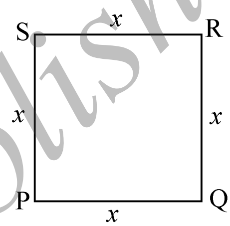


Fig. 4.2

In the polynomial $x^2 + 2x$, the expressions x^2 and $2x$ are called the **terms** of the polynomial. Similarly, the polynomial $3y^2 + 5y + 7$ has three terms, namely, $3y^2$, $5y$ and 7. Can you write the terms of the polynomial $-x^3 + 4x^2 + 7x - 2$? This polynomial has 4 terms, namely, $-x^3$, $4x^2$, $7x$ and -2.

Each term of a polynomial has a **coefficient**. So, in $-x^3 + 4x^2 + 7x - 2$, the coefficient of x^3 is -1, the coefficient of x^2 is 4, the coefficient of x is 7 and -2 is the coefficient of x^0 (Remember, $x^0 = 1$). Do you know the coefficient of x in $x^2 - x + 7$? It is -1.

2 is also a polynomial. In fact, 2, -5, 7, etc. are examples of *constant polynomials*. The constant polynomial 0 is called the **zero polynomial**. This plays a very important role in the collection of all polynomials, as you will see in the higher classes.

Now, consider algebraic expressions such as $x + \dfrac{1}{x}$, $\sqrt{x} + 3$ and $\sqrt[3]{y} + y^2$. Do you know that you can write $x + \dfrac{1}{x} = x + x^{-1}$? Here, the exponent of the second term, i.e., x^{-1} is -1, which is not a whole number. So, this algebraic expression is not a polynomial.

Again, $\sqrt{x} + 3$ can be written as $x^{\frac{1}{2}} + 3$. Here the exponent of x is $\dfrac{1}{2}$, which is not a whole number. So, is $\sqrt{x} + 3$ a polynomial? No, it is not. What about $\sqrt[3]{y} + y^2$? It is also not a polynomial (Why?).

If the variable in a polynomial is x, we may denote the polynomial by $p(x)$, or $q(x)$, or $r(x)$, etc. So, for example, we may write :

$$p(x) = 2x^2 + 5x - 3$$
$$q(x) = x^3 - 1$$
$$r(y) = y^3 + y + 1$$
$$s(u) = 2 - u - u^2 + 6u^5$$

A polynomial can have any (finite) number of terms. For instance, $x^{150} + x^{149} + ... + x^2 + x + 1$ is a polynomial with 151 terms.

Consider the polynomials $2x$, 2, $5x^3$, $-5x^2$, y and u^4. Do you see that each of these polynomials has only one term? Polynomials having only one term are called *monomials* ('mono' means 'one').

Now observe each of the following polynomials:

$p(x) = x + 1$, $\quad q(x) = x^2 - x$, $\quad r(y) = y^{30} + 1$, $\quad t(u) = u^{43} - u^2$

How many terms are there in each of these? Each of these polynomials has only two terms. Polynomials having only two terms are called *binomials* ('bi' means 'two').

Similarly, polynomials having only three terms are called *trinomials* ('tri' means 'three'). Some examples of trinomials are

$p(x) = x + x^2 + \pi$, $\quad q(x) = \sqrt{2} + x - x^2$,

$r(u) = u + u^2 - 2$, $\quad t(y) = y^4 + y + 5$.

Now, look at the polynomial $p(x) = 3x^7 - 4x^6 + x + 9$. What is the term with the highest power of x ? It is $3x^7$. The exponent of x in this term is 7. Similarly, in the polynomial $q(y) = 5y^6 - 4y^2 - 6$, the term with the highest power of y is $5y^6$ and the exponent of y in this term is 6. We call the highest power of the variable in a polynomial as the *degree of the polynomial*. So, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5y^6 - 4y^2 - 6$ is 6. **The degree of a non-zero constant polynomial is zero**.

Example 1 : Find the degree of each of the polynomials given below:

(i) $x^5 - x^4 + 3$ $\quad$ (ii) $2 - y^2 - y^3 + 2y^8$ $\quad$ (iii) 2

Solution : (i) The highest power of the variable is 5. So, the degree of the polynomial is 5.

(ii) The highest power of the variable is 8. So, the degree of the polynomial is 8.

(iii) The only term here is 2 which can be written as $2x^0$. So the exponent of x is 0. Therefore, the degree of the polynomial is 0.

Now observe the polynomials $p(x) = 4x + 5$, $q(y) = 2y$, $r(t) = t + \sqrt{2}$ and $s(u) = 3 - u$. Do you see anything common among all of them? The degree of each of these polynomials is one. A polynomial of degree one is called a *linear polynomial*. Some more linear polynomials in one variable are $2x - 1$, $\sqrt{2}\,y + 1$, $2 - u$. Now, try and find a linear polynomial in x with 3 terms? You would not be able to find it because a linear polynomial in x can have at most two terms. So, any linear polynomial in x will be of the form $ax + b$, where a and b are constants and $a \neq 0$ (why?). Similarly, $ay + b$ is a linear polynomial in y.

Now consider the polynomials :

$$2x^2 + 5,\ 5x^2 + 3x + \pi,\ x^2 \text{ and } x^2 + \frac{2}{5}x$$

Do you agree that they are all of degree two? A polynomial of degree two is called a *quadratic polynomial*. Some examples of a quadratic polynomial are $5 - y^2$, $4y + 5y^2$ and $6 - y - y^2$. Can you write a quadratic polynomial in one variable with four different terms? You will find that a quadratic polynomial in one variable will have at most 3 terms. If you list a few more quadratic polynomials, you will find that any quadratic polynomial in x is of the form $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants. Similarly, quadratic polynomial in y will be of the form $ay^2 + by + c$, provided $a \neq 0$ and a, b, c are constants.

We call a polynomial of degree three a *cubic polynomial*. Some examples of a cubic polynomial in x are $4x^3$, $2x^3 + 1$, $5x^3 + x^2$, $6x^3 - x$, $6 - x^3$, $2x^3 + 4x^2 + 6x + 7$. How many terms do you think a cubic polynomial in one variable can have? It can have at most 4 terms. These may be written in the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c and d are constants.

Now, that you have seen what a polynomial of degree 1, degree 2, or degree 3 looks like, can you write down a polynomial in one variable of degree n for any natural number n? A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

In particular, if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (all the constants are zero), we get the **zero polynomial**, which is denoted by 0. What is the degree of the zero polynomial? The degree of the zero polynomial is *not defined*.

So far we have dealt with polynomials in one variable only. We can also have polynomials in more than one variable. For example, $x^2 + y^2 + xyz$ (where variables are x, y and z) is a polynomial in three variables. Similarly $p^2 + q^{10} + r$ (where the variables are p, q and r), $u^3 + v^2$ (where the variables are u and v) are polynomials in three and two variables, respectively. You will be studying such polynomials in detail later.

EXERCISE 4.1

1. Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer.

(i) $4x^2 - 3x + 7$ (ii) $y^2 + \sqrt{2}$ (iii) $3\sqrt{t} + t\sqrt{2}$ (iv) $y + \frac{2}{y}$

(v) $x^{10} + y^3 + t^{50}$

2. Write the coefficients of x^2 in each of the following:

(i) $2 + x^2 + x$ (ii) $2 - x^2 + x^3$ (iii) $\frac{\pi}{2}x^2 + x$ (iv) $\sqrt{2}x - 1$

3. Give one example each of a binomial of degree 35, and of a monomial of degree 100.

4. Write the degree of each of the following polynomials:

(i) $5x^3 + 4x^2 + 7x$ (ii) $4 - y^2$

(iii) $5t - \sqrt{7}$ (iv) 3

5. Classify the following as linear, quadratic and cubic polynomials:

(i) $x^2 + x$ (ii) $x - x^3$ (iii) $y + y^2 + 4$ (iv) $1 + x$

(v) $3t$ (vi) r^2 (vii) $7x^3$

4.3 Zeroes of a Polynomial

Consider the polynomial $p(x) = 5x^3 - 2x^2 + 3x - 2$.

If we replace x by 1 everywhere in $p(x)$, we get

$$\begin{aligned} p(1) &= 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2 \\ &= 5 - 2 + 3 - 2 \\ &= 4 \end{aligned}$$

So, we say that the value of $p(x)$ at $x = 1$ is 4.

Similarly, $p(0) = 5(0)^3 - 2(0)^2 + 3(0) - 2$

$= -2$

Can you find $p(-1)$?

Example 2 : Find the value of each of the following polynomials at the indicated value of variables:

(i) $p(x) = 5x^2 - 3x + 7$ at $x = 1$.

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$ at $y = 2$.

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$ at $t = a$.

Solution : (i) $p(x) = 5x^2 - 3x + 7$

The value of the polynomial $p(x)$ at $x = 1$ is given by

$$p(1) = 5(1)^2 - 3(1) + 7$$
$$= 5 - 3 + 7 = 9$$

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$

The value of the polynomial $q(y)$ at $y = 2$ is given by

$$q(2) = 3(2)^3 - 4(2) + \sqrt{11} = 24 - 8 + \sqrt{11} = 16 + \sqrt{11}$$

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$

The value of the polynomial $p(t)$ at $t = a$ is given by

$$p(a) = 4a^4 + 5a^3 - a^2 + 6$$

Now, consider the polynomial $p(x) = x - 1$.

What is $p(1)$? Note that : $p(1) = 1 - 1 = 0$.

As $p(1) = 0$, we say that 1 is a *zero* of the polynomial $p(x)$.

Similarly, you can check that 2 is a *zero* of $q(x)$, where $q(x) = x - 2$.

In general, we say that a *zero* of a polynomial $p(x)$ is a number c such that $p(c) = 0$.

You must have observed that the zero of the polynomial $x - 1$ is obtained by equating it to 0, i.e., $x - 1 = 0$, which gives $x = 1$. We say $p(x) = 0$ is a polynomial equation and 1 is the *root of the polynomial* equation $p(x) = 0$. So we say 1 is the zero of the polynomial $x - 1$, or a *root* of the polynomial equation $x - 1 = 0$.

Now, consider the constant polynomial 5. Can you tell what its zero is? It has no zero because replacing x by any number in $5x^0$ still gives us 5. In fact, *a non-zero constant polynomial has no zero*. What about the zeroes of the zero polynomial? By convention, *every real number is a zero of the zero polynomial*.

Example 3 : Check whether –2 and 2 are zeroes of the polynomial $x + 2$.

Solution : Let $p(x) = x + 2$.

Then $p(2) = 2 + 2 = 4$, $p(-2) = -2 + 2 = 0$

Therefore, –2 is a zero of the polynomial $x + 2$, but 2 is not.

Example 4 : Find a zero of the polynomial $p(x) = 2x + 1$.

Solution : Finding a zero of $p(x)$, is the same as solving the equation

$$p(x) = 0$$

Now, $2x + 1 = 0$ gives us $x = -\frac{1}{2}$

So, $-\frac{1}{2}$ is a zero of the polynomial $2x + 1$.

Now, if $p(x) = ax + b$, $a \neq 0$, is a linear polynomial, how can we find a zero of $p(x)$? Example 4 may have given you some idea. Finding a zero of the polynomial $p(x)$, amounts to solving the polynomial equation $p(x) = 0$.

Now, $p(x) = 0$ means $ax + b = 0, a \neq 0$

So, $ax = -b$

i.e., $x = -\frac{b}{a}$.

So, $x = -\frac{b}{a}$ is the only zero of $p(x)$, i.e., a *linear polynomial has one and only one zero.*

Now we can say that 1 is *the* zero of $x - 1$, and -2 is *the* zero of $x + 2$.

Example 5 : Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

Solution : Let $p(x) = x^2 - 2x$

Then $p(2) = 2^2 - 4 = 4 - 4 = 0$

and $p(0) = 0 - 0 = 0$

Hence, 2 and 0 are both zeroes of the polynomial $x^2 - 2x$.

Let us now list our observations:

(i) A zero of a polynomial need not be 0.

(ii) 0 may be a zero of a polynomial.

(iii) Every linear polynomial has one and only one zero.

(iv) A polynomial can have more than one zero.

EXERCISE 4.2

1. Find the value of the polynomial $5x - 4x^2 + 3$ at

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$

2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:

(i) $p(y) = y^2 - y + 1$ (ii) $p(t) = 2 + t + 2t^2 - t^3$

(iii) $p(x) = x^3$ (iv) $p(x) = (x - 1)(x + 1)$

3. Verify whether the following are zeroes of the polynomial, indicated against them.

(i) $p(x) = 3x + 1,\ x = -\frac{1}{3}$ (ii) $p(x) = 5x - \pi,\ x = \frac{4}{5}$

(iii) $p(x) = x^2 - 1,\ x = 1, -1$ (iv) $p(x) = (x + 1)(x - 2),\ x = -1, 2$

(v) $p(x) = x^2,\ x = 0$ (vi) $p(x) = lx + m,\ x = -\frac{m}{l}$

(vii) $p(x) = 3x^2 - 1,\ x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ (viii) $p(x) = 2x + 1,\ x = \frac{1}{2}$

4. Find the zero of the polynomial in each of the following cases:

(i) $p(x) = x + 5$ (ii) $p(x) = x - 5$ (iii) $p(x) = 2x + 5$

(iv) $p(x) = 3x - 2$ (v) $p(x) = 3x$ (vi) $p(x) = ax,\ a \neq 0$

(vii) $p(x) = cx + d,\ c \neq 0,\ c, d$ are real numbers.

4.4 Remainder Theorem

Let us consider two numbers 15 and 6. You know that when we divide 15 by 6, we get the quotient 2 and remainder 3. Do you remember how this fact is expressed? We write 15 as

$$15 = (6 \times 2) + 3$$

We observe that the *remainder* 3 is less than the *divisor* 6. Similarly, if we divide 12 by 6, we get

$$12 = (6 \times 2) + 0$$

What is the remainder here? Here the remainder is 0, and we say that 6 is a *factor* of 12 or 12 is a *multiple* of 6.

Now, the question is: can we divide one polynomial by another? To start with, let us try and do this when the divisor is a monomial. So, let us divide the polynomial $2x^3 + x^2 + x$ by the monomial x.

We have $$(2x^3 + x^2 + x) \div x = \frac{2x^3}{x} + \frac{x^2}{x} + \frac{x}{x}$$

$$= 2x^2 + x + 1$$

In fact, you may have noticed that x is common to each term of $2x^3 + x^2 + x$. So we can write $2x^3 + x^2 + x$ as $x(2x^2 + x + 1)$.

We say that x and $2x^2 + x + 1$ are *factors* of $2x^3 + x^2 + x$, and $2x^3 + x^2 + x$ is a *multiple* of x as well as a multiple of $2x^2 + x + 1$.

Consider another pair of polynomials $3x^2 + x + 1$ and x.

Here, $(3x^2 + x + 1) \div x = (3x^2 \div x) + (x \div x) + (1 \div x)$.

We see that we cannot divide 1 by x to get a polynomial term. So in this case we stop here, and note that 1 is the remainder. Therefore, we have

$$3x^2 + x + 1 = \{x \times (3x + 1)\} + 1$$

In this case, $3x + 1$ is the quotient and 1 is the remainder. Do you think that x is a factor of $3x^2 + x + 1$? Since the remainder is not zero, it is not a factor.

Now let us consider an example to see how we can divide a polynomial by any non-zero polynomial.

Example 6 : Divide $p(x)$ by $g(x)$, where $p(x) = x + 3x^2 - 1$ and $g(x) = 1 + x$.

Solution : We carry out the process of division by means of the following steps:

Step 1 : We write the dividend $x + 3x^2 - 1$ and the divisor $1 + x$ in the standard form, i.e., after arranging the terms in the descending order of their degrees. So, the dividend is $3x^2 + x - 1$ and divisor is $x + 1$.

Step 2 : We divide the first term of the dividend by the first term of the divisor, i.e., we divide $3x^2$ by x, and get $3x$. This gives us the first term of the quotient.

$$\frac{3x^2}{x} = 3x = \text{first term of quotient}$$

Step 3 : We multiply the divisor by the first term of the quotient, and subtract this product from the dividend, i.e., we multiply $x + 1$ by $3x$ and subtract the product $3x^2 + 3x$ from the dividend $3x^2 + x - 1$. This gives us the remainder as $-2x - 1$.

$$\begin{array}{r|l} & 3x \\ \hline x+1 & 3x^2 + x - 1 \\ & 3x^2 + 3x \\ & - \quad - \\ \hline & \quad -2x - 1 \end{array}$$

Step 4 : We treat the remainder $-2x - 1$ as the new dividend. The divisor remains the same. We repeat Step 2 to get the next term of the quotient, i.e., we divide the first term $-2x$ of the (new) dividend by the first term x of the divisor and obtain -2. Thus, -2 is the second term in the quotient.

$$\frac{-2x}{x} = -2$$

$= \text{second term of quotient}$

New Quotient $= 3x - 2$

Step 5 : We multiply the divisor by the second term of the quotient and subtract the product from the dividend. That is, we multiply $x + 1$ by -2 and subtract the product $-2x - 2$ from the dividend $-2x - 1$. This gives us 1 as the remainder.

$$
\begin{array}{l|l}
(x+1)(-2) & -2x - 1 \\
= -2x - 2 & -2x - 2 \\
 & \underline{+ \quad\;\; +} \\
 & \quad\quad +1
\end{array}
$$

This process continues till the remainder is 0 or the degree of the new dividend is less than the degree of the divisor. At this stage, this new dividend becomes the remainder and the sum of the quotients gives us the whole quotient.

Step 6 : Thus, the quotient in full is $3x - 2$ and the remainder is 1.

Let us look at what we have done in the process above as a whole:

$$
\begin{array}{r|l}
 & 3x - 2 \\ \hline
x + 1 & 3x^2 + x - 1 \\
 & 3x^2 + 3x \\
 & \underline{- \quad\;\; -} \\
 & \quad\quad -2x - 1 \\
 & \quad\quad -2x - 2 \\
 & \quad\quad \underline{+ \quad\;\; +} \\
 & \quad\quad\quad\quad 1
\end{array}
$$

Notice that $3x^2 + x - 1 = (x + 1)(3x - 2) + 1$

i.e., **Dividend = (Divisor × Quotient) + Remainder**

In general, if $p(x)$ and $g(x)$ are two polynomials such that degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that:

$$p(x) = g(x)q(x) + r(x),$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$. Here we say that $p(x)$ divided by $g(x)$, gives $q(x)$ as quotient and $r(x)$ as remainder.

In the example above, the divisor was a linear polynomial. In such a situation, let us see if there is any link between the remainder and certain values of the dividend.

In $p(x) = 3x^2 + x - 1$, if we replace x by -1, we have

$$p(-1) = 3(-1)^2 + (-1) - 1 = 1$$

So, the remainder obtained on dividing $p(x) = 3x^2 + x - 1$ by $x + 1$ is the same as the value of the polynomial $p(x)$ at the zero of the polynomial $x + 1$, i.e., -1.

Let us consider some more examples.

Example 7 : Divide the polynomial $3x^4 - 4x^3 - 3x - 1$ by $x - 1$.

Solution : By long division, we have:

$$
\begin{array}{r|l}
 & 3x^3 - x^2 - x - 4 \\
\hline
x - 1 & 3x^4 - 4x^3 - 3x - 1 \\
 & \underset{-}{3x^4} \underset{+}{-} 3x^3 \\
\hline
 & \quad - x^3 \qquad - 3x - 1 \\
 & \quad \underset{+}{-} x^3 \underset{-}{+} x^2 \\
\hline
 & \qquad - x^2 - 3x - 1 \\
 & \qquad \underset{+}{-} x^2 \underset{-}{+} x \\
\hline
 & \qquad\quad - 4x - 1 \\
 & \qquad\quad \underset{+}{-} 4x \underset{-}{+} 4 \\
\hline
 & \qquad\qquad - 5
\end{array}
$$

Here, the remainder is -5. Now, the zero of $x - 1$ is 1. So, putting $x = 1$ in $p(x)$, we see that

$$
\begin{aligned}
p(1) &= 3(1)^4 - 4(1)^3 - 3(1) - 1 \\
&= 3 - 4 - 3 - 1 \\
&= -5, \text{ which is the remainder.}
\end{aligned}
$$

Example 8 : Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.

Solution : By long division,

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x + 1 & x^3 + 1 \\
 & \underset{-}{x^3} \underset{-}{+} x^2 \\
\hline
 & \quad - x^2 \qquad + 1 \\
 & \quad \underset{+}{-} x^2 \underset{+}{-} x \\
\hline
 & \qquad\quad x + 1 \\
 & \qquad\quad \underset{-}{x} \underset{-}{+} 1 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

So, we find that the remainder is 0.

Here $p(x) = x^3 + 1$, and the root of $x + 1 = 0$ is $x = -1$. We see that

$$
\begin{aligned}
p(-1) &= (-1)^3 + 1 \\
&= -1 + 1 \\
&= 0,
\end{aligned}
$$

which is equal to the remainder obtained by actual division.

Is it not a simple way to find the remainder obtained on dividing a polynomial by a *linear polynomial*? We shall now generalise this fact in the form of the following theorem. We shall also show you why the theorem is true, by giving you a proof of the theorem.

Remainder Theorem : *Let $p(x)$ be any polynomial of degree greater than or equal to one and let a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.*

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1. Suppose that when $p(x)$ is divided by $x - a$, the quotient is $q(x)$ and the remainder is $r(x)$, i.e.,

$$p(x) = (x - a)\, q(x) + r(x)$$

Since the degree of $x - a$ is 1 and the degree of $r(x)$ is less than the degree of $x - a$, the degree of $r(x) = 0$. This means that $r(x)$ is a constant, say r.

So, for every value of x, $r(x) = r$.

Therefore, $$p(x) = (x - a)\, q(x) + r$$

In particular, if $x = a$, this equation gives us

$$
\begin{aligned}
p(a) &= (a - a)\, q(a) + r \\
&= r,
\end{aligned}
$$

which proves the theorem.

Let us use this result in another example.

Example 9 : Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Solution : Here, $p(x) = x^4 + x^3 - 2x^2 + x + 1$, and the zero of $x - 1$ is 1.

So, $$
\begin{aligned}
p(1) &= (1)^4 + (1)^3 - 2(1)^2 + 1 + 1 \\
&= 2
\end{aligned}
$$

So, by the Remainder Theorem, 2 is the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Example 10 : Check whether the polynomial $q(t) = 4t^3 + 4t^2 - t - 1$ is a multiple of $2t + 1$.

Solution : As you know, $q(t)$ will be a multiple of $2t + 1$ only, if $2t + 1$ divides $q(t)$ leaving remainder zero. Now, taking $2t + 1 = 0$, we have $t = -\frac{1}{2}\cdot$

Also, $$q\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = -\frac{1}{2} + 1 + \frac{1}{2} - 1 = 0$$

So the remainder obtained on dividing $q(t)$ by $2t + 1$ is 0.

So, $2t + 1$ is a factor of the given polynomial $q(t)$, that is $q(t)$ is a multiple of $2t + 1$.

EXERCISE 2.3

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by

   (i) $x + 1$ (ii) $x - \frac{1}{2}$ (iii) x (iv) $x + \pi$ (v) $5 + 2x$
2. Find the remainder when $x^3 - ax^2 + 6x - a$ is divided by $x - a$.
3. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.

4.5 Factorisation of Polynomials

Let us now look at the situation of Example 10 above more closely. It tells us that since the remainder, $q\left(-\frac{1}{2}\right) = 0$, $(2t + 1)$ is a factor of $q(t)$, i.e., $q(t) = (2t + 1)\, g(t)$ for some polynomial $g(t)$. This is a particular case of the following theorem.

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then (i) $x - a$ is a factor of $p(x)$, if $p(a) = 0$, and (ii) $p(a) = 0$, if $x - a$ is a factor of $p(x)$.

Proof: By the Remainder Theorem, $p(x)=(x - a)\, q(x) + p(a)$.

(i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x)$, which shows that $x - a$ is a factor of $p(x)$.

(ii) Since $x - a$ is a factor of $p(x)$, $p(x) = (x - a)\, g(x)$ for same polynomial $g(x)$. In this case, $p(a) = (a - a)\, g(a) = 0$.

Example 11 : Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.

Solution : The zero of $x + 2$ is -2. Let $p(x) = x^3 + 3x^2 + 5x + 6$ and $s(x) = 2x + 4$

Then, $$p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6$$

$= -8 + 12 - 10 + 6$

$= 0$

So, by the Factor Theorem, $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$.

Again, $s(-2) = 2(-2) + 4 = 0$

So, $x + 2$ is a factor of $2x + 4$. In fact, you can check this without applying the Factor Theorem, since $2x + 4 = 2(x + 2)$.

Example 12 : Find the value of k, if $x - 1$ is a factor of $4x^3 + 3x^2 - 4x + k$.

Solution : As $x - 1$ is a factor of $p(x) = 4x^3 + 3x^2 - 4x + k$, $p(1) = 0$

Now, $p(1) = 4(1)^3 + 3(1)^2 - 4(1) + k$

So, $4 + 3 - 4 + k = 0$

i.e., $k = -3$

We will now use the Factor Theorem to factorise some polynomials of degree 2 and 3. You are already familiar with the factorisation of a quadratic polynomial like $x^2 + lx + m$. You had factorised it by splitting the middle term lx as $ax + bx$ so that $ab = m$. Then $x^2 + lx + m = (x + a)(x + b)$. We shall now try to factorise quadratic polynomials of the type $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants.

Factorisation of the polynomial $ax^2 + bx + c$ **by splitting the middle term** is as follows:

Let its factors be $(px + q)$ and $(rx + s)$. Then

$$ax^2 + bx + c = (px + q)(rx + s) = pr\,x^2 + (ps + qr)\,x + qs$$

Comparing the coefficients of x^2, we get $a = pr$.

Similarly, comparing the coefficients of x, we get $b = ps + qr$.

And, on comparing the constant terms, we get $c = qs$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

Therefore, to factorise $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac. This will be clear from Example 13.

Example 13 : Factorise $6x^2 + 17x + 5$ by splitting the middle term, and by using the Factor Theorem.

Solution 1 : (By splitting method) **:** If we can find two numbers p and q such that $p + q = 17$ and $pq = 6 \times 5 = 30$, then we can get the factors.

So, let us look for the pairs of factors of 30. Some are 1 and 30, 2 and 15, 3 and 10, 5 and 6. Of these pairs, 2 and 15 will give us $p + q = 17$.

$$\begin{aligned}\text{So, } 6x^2 + 17x + 5 &= 6x^2 + (2 + 15)x + 5\\ &= 6x^2 + 2x + 15x + 5\\ &= 2x(3x + 1) + 5(3x + 1)\\ &= (3x + 1)\,(2x + 5)\end{aligned}$$

Solution 2 : (Using the Factor Theorem)

$6x^2 + 17x + 5 = 6\left(x^2 + \frac{17}{6}x + \frac{5}{6}\right) = 6\,p(x)$, say. If a and b are the zeroes of $p(x)$, then $6x^2 + 17x + 5 = 6(x - a)\,(x - b)$. So, $ab = \frac{5}{6}\cdot$ Let us look at some possibilities for a and b. They could be $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{5}{2}, \pm 1$. Now, $p\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{17}{6}\left(\frac{1}{2}\right) + \frac{5}{6} \neq 0$. But $p\left(\frac{-1}{3}\right) = 0$. So, $\left(x + \frac{1}{3}\right)$ is a factor of $p(x)$. Similarly, by trial, you can find that $\left(x + \frac{5}{2}\right)$ is a factor of $p(x)$.

$$\begin{aligned}\text{Therefore,}\qquad 6x^2 + 17x + 5 &= 6\left(x + \frac{1}{3}\right)\left(x + \frac{5}{2}\right)\\ &= 6\left(\frac{3x + 1}{3}\right)\left(\frac{2x + 5}{2}\right)\\ &= (3x + 1)\,(2x + 5)\end{aligned}$$

For the example above, the use of the splitting method appears more efficient. However, let us consider another example.

Example 14 : Factorise $y^2 - 5y + 6$ by using the Factor Theorem.

Solution : Let $p(y) = y^2 - 5y + 6$. Now, if $p(y) = (y - a)\,(y - b)$, you know that the constant term will be ab. So, $ab = 6$. So, to look for the factors of $p(y)$, we look at the factors of 6.

The factors of 6 are 1, 2 and 3.

Now, $\quad p(2) = 2^2 - (5 \times 2) + 6 = 0$

So, $\quad y - 2$ is a factor of $p(y)$.

Also, $\quad p(3) = 3^2 - (5 \times 3) + 6 = 0$

So, $\quad y - 3$ is also a factor of $y^2 - 5y + 6$.

Therefore, $\quad y^2 - 5y + 6 = (y - 2)(y - 3)$

Note that $y^2 - 5y + 6$ can also be factorised by splitting the middle term $-5y$.

Now, let us consider factorising cubic polynomials. Here, the splitting method will not be appropriate to start with. We need to find at least one factor first, as you will see in the following example.

Example 15 : Factorise $x^3 - 23x^2 + 142x - 120$.

Solution : Let $\quad p(x) = x^3 - 23x^2 + 142x - 120$

We shall now look for all the factors of –120. Some of these are ±1, ±2, ±3,

±4, ±5, ±6, ±8, ±10, ±12, ±15, ±20, ±24, ±30, ±60.

By trial, we find that $p(1) = 0$. So $x - 1$ is a factor of $p(x)$.

Now we see that $x^3 - 23x^2 + 142x - 120 = x^3 - x^2 - 22x^2 + 22x + 120x - 120$

$$= x^2(x - 1) - 22x(x - 1) + 120(x - 1) \quad \text{(Why?)}$$

$$= (x - 1)(x^2 - 22x + 120) \quad [\text{Taking } (x - 1) \text{ common}]$$

We could have also got this by dividing $p(x)$ by $x - 1$.

Now $x^2 - 22x + 120$ can be factorised either by splitting the middle term or by using the Factor theorem. By splitting the middle term, we have:

$$\begin{aligned} x^2 - 22x + 120 &= x^2 - 12x - 10x + 120 \\ &= x(x - 12) - 10(x - 12) \\ &= (x - 12)(x - 10) \end{aligned}$$

So, $$x^3 - 23x^2 - 142x - 120 = (x - 1)(x - 10)(x - 12)$$

EXERCISE 4.4

1. Determine which of the following polynomials has $(x + 1)$ a factor :

   (i) $x^3 + x^2 + x + 1$ (ii) $x^4 + x^3 + x^2 + x + 1$

   (iii) $x^4 + 3x^3 + 3x^2 + x + 1$ (iv) $x^3 - x^2 - \left(2 + \sqrt{2}\right)x + \sqrt{2}$

2. Use the Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = 2x^3 + x^2 - 2x - 1,\ g(x) = x + 1$

(ii) $p(x) = x^3 + 3x^2 + 3x + 1,\ g(x) = x + 2$

(iii) $p(x) = x^3 - 4x^2 + x + 6,\ g(x) = x - 3$

3. Find the value of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = x^2 + x + k$ (ii) $p(x) = 2x^2 + kx + \sqrt{2}$

(iii) $p(x) = kx^2 - \sqrt{2}x + 1$ (iv) $p(x) = kx^2 - 3x + k$

4. Factorise :

(i) $12x^2 - 7x + 1$ (ii) $2x^2 + 7x + 3$

(iii) $6x^2 + 5x - 6$ (iv) $3x^2 - x - 4$

5. Factorise :

(i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 - 3x^2 - 9x - 5$

(iii) $x^3 + 13x^2 + 32x + 20$ (iv) $2y^3 + y^2 - 2y - 1$

4.6 Algebraic Identities

From your earlier classes, you may recall that an algebraic identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes:

Identity I : $(x + y)^2 = x^2 + 2xy + y^2$

Identity II : $(x - y)^2 = x^2 - 2xy + y^2$

Identity III : $x^2 - y^2 = (x + y)(x - y)$

Identity IV : $(x + a)(x + b) = x^2 + (a + b)x + ab$

You must have also used some of these algebraic identities to factorise the algebraic expressions. You can also see their utility in computations.

Example 16 : Find the following products using appropriate identities:

(i) $(x + 3)(x + 3)$ (ii) $(x - 3)(x + 5)$

Solution : (i) Here we can use Identity I : $(x + y)^2 = x^2 + 2xy + y^2$. Putting $y = 3$ in it, we get

$$(x + 3)(x + 3) = (x + 3)^2 = x^2 + 2(x)(3) + (3)^2$$
$$= x^2 + 6x + 9$$

(ii) Using Identity IV above, i.e., $(x + a)(x + b) = x^2 + (a + b)x + ab$, we have

$$(x - 3)(x + 5) = x^2 + (-3 + 5)x + (-3)(5)$$
$$= x^2 + 2x - 15$$

Example 17 : Evaluate 105 × 106 without multiplying directly.

Solution :

$$
\begin{aligned}
105 \times 106 &= (100 + 5) \times (100 + 6) \\
&= (100)^2 + (5 + 6)\,(100) + (5 \times 6), \text{ using Identity IV} \\
&= 10000 + 1100 + 30 \\
&= 11130
\end{aligned}
$$

You have seen some uses of the identities listed above in finding the product of some given expressions. These identities are useful in factorisation of algebraic expressions also, as you can see in the following examples.

Example 18 : Factorise:

(i) $49a^2 + 70ab + 25b^2$ (ii) $\dfrac{25}{4}x^2 - \dfrac{y^2}{9}$

Solution : (i) Here you can see that

$$49a^2 = (7a)^2,\ 25b^2 = (5b)^2,\ 70ab = 2(7a)\,(5b)$$

Comparing the given expression with $x^2 + 2xy + y^2$, we observe that $x = 7a$ and $y = 5b$.

Using Identity I, we get

$$49a^2 + 70ab + 25b^2 = (7a + 5b)^2 = (7a + 5b)\,(7a + 5b)$$

(ii) We have $\dfrac{25}{4}x^2 - \dfrac{y^2}{9} = \left(\dfrac{5}{2}x\right)^2 - \left(\dfrac{y}{3}\right)^2$

Now comparing it with Identity III, we get

$$
\begin{aligned}
\frac{25}{4}x^2 - \frac{y^2}{9} &= \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2 \\
&= \left(\frac{5}{2}x + \frac{y}{3}\right)\left(\frac{5}{2}x - \frac{y}{3}\right)
\end{aligned}
$$

So far, all our identities involved products of binomials. Let us now extend the Identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^2$ by using Identity I.

Let $x + y = t$. Then,

$$
\begin{aligned}
(x + y + z)^2 &= (t + z)^2 \\
&= t^2 + 2tz + t^2 && \text{(Using Identity I)} \\
&= (x + y)^2 + 2(x + y)z + z^2 && \text{(Substituting the value of } t\text{)}
\end{aligned}
$$

$$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \qquad \text{(Using Identity I)}$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \qquad \text{(Rearranging the terms)}$$

So, we get the following identity:

Identity V : $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Remark : We call the right hand side expression **the expanded form** of the left hand side expression. Note that the expansion of $(x + y + z)^2$ consists of three square terms and three product terms.

Example 19 : Write $(3a + 4b + 5c)^2$ in expanded form.

Solution : Comparing the given expression with $(x + y + z)^2$, we find that

$$x = 3a,\ y = 4b \text{ and } z = 5c.$$

Therefore, using Identity V, we have

$$(3a + 4b + 5c)^2 = (3a)^2 + (4b)^2 + (5c)^2 + 2(3a)(4b) + 2(4b)(5c) + 2(5c)(3a)$$

$$= 9a^2 + 16b^2 + 25c^2 + 24ab + 40bc + 30ac$$

Example 20 : Expand $(4a – 2b – 3c)^2$.

Solution : Using Identity V, we have

$$(4a – 2b – 3c)^2 = [4a + (–2b) + (–3c)]^2$$

$$= (4a)^2 + (–2b)^2 + (–3c)^2 + 2(4a)(–2b) + 2(–2b)(–3c) + 2(–3c)(4a)$$

$$= 16a^2 + 4b^2 + 9c^2 – 16ab + 12bc – 24ac$$

Example 21 : Factorise $4x^2 + y^2 + z^2 – 4xy – 2yz + 4xz$.

Solution : We have $4x^2 + y^2 + z^2 – 4xy – 2yz + 4xz = (2x)^2 + (–y)^2 + (z)^2 + 2(2x)(–y) + 2(–y)(z) + 2(2x)(z)$

$$= [2x + (–y) + z]^2 \qquad \text{(Using Identity V)}$$

$$= (2x – y + z)^2 = (2x – y + z)(2x – y + z)$$

So far, we have dealt with identities involving second degree terms. Now let us extend Identity I to compute $(x + y)^3$. We have:

$$(x + y)^3 = (x + y)\,(x + y)^2$$

$$= (x + y)(x^2 + 2xy + y^2)$$

$$= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2)$$

$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$= x^3 + y^3 + 3xy(x + y)$$

So, we get the following identity:

Identity VI : $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

Also, by replacing y by $-y$ in the Identity VI, we get

Identity VII : $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$$= x^3 - 3x^2y + 3xy^2 - y^3$$

Example 22 : Write the following cubes in the expanded form:

(i) $(3a + 4b)^3$ (ii) $(5p - 3q)^3$

Solution : (i) Comparing the given expression with $(x + y)^3$, we find that

$$x = 3a \text{ and } y = 4b.$$

So, using Identity VI, we have:

$$(3a + 4b)^3 = (3a)^3 + (4b)^3 + 3(3a)(4b)(3a + 4b)$$
$$= 27a^3 + 64b^3 + 108a^2b + 144ab^2$$

(ii) Comparing the given expression with $(x - y)^3$, we find that

$$x = 5p,\ y = 3q.$$

So, using Identity VII, we have:

$$(5p - 3q)^3 = (5p)^3 - (3q)^3 - 3(5p)(3q)(5p - 3q)$$
$$= 125p^3 - 27q^3 - 225p^2q + 135pq^2$$

Example 23 : Evaluate each of the following using suitable identities:

(i) $(104)^3$ (ii) $(999)^3$

Solution : (i) We have

$$(104)^3 = (100 + 4)^3$$
$$= (100)^3 + (4)^3 + 3(100)(4)(100 + 4)$$

(Using Identity VI)

$$= 1000000 + 64 + 124800$$
$$= 1124864$$

(ii) We have

$$(999)^3 = (1000 - 1)^3$$
$$= (1000)^3 - (1)^3 - 3(1000)(1)(1000 - 1)$$

(Using Identity VII)

$$= 1000000000 - 1 - 2997000$$
$$= 997002999$$

Example 24 : Factorise $8x^3 + 27y^3 + 36x^2y + 54xy^2$

Solution : The given expression can be written as

$$\begin{aligned}&(2x)^3 + (3y)^3 + 3(4x^2)(3y) + 3(2x)(9y^2)\\ &= (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2\\ &= (2x + 3y)^3 \qquad \text{(Using Identity VI)}\\ &= (2x + 3y)(2x + 3y)(2x + 3y)\end{aligned}$$

Now consider $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

On expanding, we get the product as

$$\begin{aligned}&x(x^2 + y^2 + z^2 - xy - yz - zx) + y(x^2 + y^2 + z^2 - xy - yz - zx)\\ &+ z(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + xy^2 + xz^2 - x^2y - xyz - zx^2 + x^2y\\ &+ y^3 + yz^2 - xy^2 - y^2z - xyz + x^2z + y^2z + z^3 - xyz - yz^2 - xz^2\\ &= x^3 + y^3 + z^3 - 3xyz \qquad \text{(On simplification)}\end{aligned}$$

So, we obtain the following identity:

Identity VIII : $\mathbf{x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$

Example 25 : Factorise : $8x^3 + y^3 + 27z^3 - 18xyz$

Solution : Here, we have

$$\begin{aligned}&8x^3 + y^3 + 27z^3 - 18xyz\\ &= (2x)^3 + y^3 + (3z)^3 - 3(2x)(y)(3z)\\ &= (2x + y + 3z)[(2x)^2 + y^2 + (3z)^2 - (2x)(y) - (y)(3z) - (2x)(3z)]\\ &= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6xz)\end{aligned}$$

EXERCISE 4.5

1. Use suitable identities to find the following products:

(i) $(x+4)(x+10)$ (ii) $(x+8)(x-10)$ (iii) $(3x+4)(3x-5)$

(iv) $(y^2 + \frac{3}{2})(y^2 - \frac{3}{2})$ (v) $(3-2x)(3+2x)$

2. Evaluate the following products without multiplying directly:

(i) 103×107 (ii) 95×96 (iii) 104×96

3. Factorise the following using appropriate identities:

(i) $9x^2 + 6xy + y^2$ (ii) $4y^2 - 4y + 1$ (iii) $x^2 - \frac{y^2}{100}$

4. Expand each of the following, using suitable identities:

(i) $(x+2y+4z)^2$ (ii) $(2x-y+z)^2$ (iii) $(-2x+3y+2z)^2$

(iv) $(3a-7b-c)^2$ (v) $(-2x+5y-3z)^2$ (vi) $\left[\frac{1}{4}a-\frac{1}{2}b+1\right]^2$

5. Factorise:

(i) $4x^2+9y^2+16z^2+12xy-24yz-16xz$

(ii) $2x^2+y^2+8z^2-2\sqrt{2}\,xy+4\sqrt{2}\,yz-8xz$

6. Write the following cubes in expanded form:

(i) $(2x+1)^3$ (ii) $(2a-3b)^3$ (iii) $\left[\frac{3}{2}x+1\right]^3$ (iv) $\left[x-\frac{2}{3}y\right]^3$

7. Evaluate the following using suitable identities:

(i) $(99)^3$ (ii) $(102)^3$ (iii) $(998)^3$

8. Factorise each of the following:

(i) $8a^3+b^3+12a^2b+6ab^2$ (ii) $8a^3-b^3-12a^2b+6ab^2$

(iii) $27-125a^3-135a+225a^2$ (iv) $64a^3-27b^3-144a^2b+108ab^2$

(v) $27p^3-\frac{1}{216}-\frac{9}{2}p^2+\frac{1}{4}p$

9. Verify : (i) $x^3+y^3=(x+y)(x^2-xy+y^2)$ (ii) $x^3-y^3=(x-y)(x^2+xy+y^2)$

10. Factorise each of the following:

(i) $27y^3+125z^3$ (ii) $64m^3-343n^3$

[**Hint :** See Question 9.]

11. Factorise : $27x^3+y^3+z^3-9xyz$

12. Verify that $x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$

13. If $x+y+z=0$, show that $x^3+y^3+z^3=3xyz$.

14. Without actually calculating the cubes, find the value of each of the following:

(i) $(-12)^3+(7)^3+(5)^3$

(ii) $(28)^3+(-15)^3+(-13)^3$

15. Give possible expressions for the length and breadth of each of the following rectangles, in which their areas are given:

Area : $25a^2-35a+12$

(i)

Area : $35y^2+13y-12$

(ii)

16. What are the possible expressions for the dimensions of the cuboids whose volumes are given below?

(i)	(ii)
Volume : $3x^2 - 12x$	Volume : $12ky^2 + 8ky - 20k$

4.7 Summary

In this chapter, you have studied the following points:

1. A *polynomial* $p(x)$ in one variable x is an algebraic expression in x of the form

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$,

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

$a_0, a_1, a_2, \ldots, a_n$ are respectively the *coefficients* of $x^0, x, x^2, \ldots, x^n$, and n is called the *degree of the polynomial*. Each of $a_n x^n$, $a_{n-1} x^{n-1}$, ..., a_0, with $a_n \neq 0$, is called a *term* of the polynomial $p(x)$.

2. A polynomial of one term is called a monomial.

3. A polynomial of two terms is called a binomial.

4. A polynomial of three terms is called a trinomial.

5. A polynomial of degree one is called a linear polynomial.

6. A polynomial of degree two is called a quadratic polynomial.

7. A polynomial of degree three is called a cubic polynomial.

8. A real number 'a' is a *zero* of a polynomial $p(x)$ if $p(a) = 0$. In this case, a is also called a *root* of the equation $p(x) = 0$.

9. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero, and every real number is a zero of the zero polynomial.

10. Remainder Theorem : If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.

11. Factor Theorem : $x - a$ is a factor of the polynomial $p(x)$, if $p(a) = 0$. Also, if $x - a$ is a factor of $p(x)$, then $p(a) = 0$.

12. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

13. $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

14. $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

15. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

28QLM6

CHAPTER 5

TRIANGLES

5.1 Introduction

You have studied about triangles and their various properties in your earlier classes. You know that a closed figure formed by three intersecting lines is called a triangle. ('Tri' means 'three'). A triangle has three sides, three angles and three vertices. For example, in triangle ABC, denoted as Δ ABC (see Fig. 5.1); AB, BC, CA are the three sides, $\angle$ A, $\angle$ B, $\angle$ C are the three angles and A, B, C are three vertices.

In Chapter 6, you have also studied some properties of triangles. In this chapter, you will study in details about the congruence of triangles, rules of congruence, some more properties of triangles and inequalities in a triangle. You have already verified most of these properties in earlier classes. We will now prove some of them.

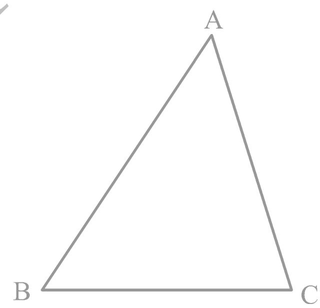


Fig. 5.1

5.2 Congruence of Triangles

You must have observed that two copies of your photographs of the same size are identical. Similarly, two bangles of the same size, two ATM cards issued by the same bank are identical. You may recall that on placing a one rupee coin on another minted in the same year, they cover each other completely.

Do you remember what such figures are called? Indeed they are called **congruent figures** ('congruent' means equal in all respects or figures whose shapes and sizes are both the same).

Now, draw two circles of the same radius and place one on the other. What do you observe? They cover each other completely and we call them as congruent circles.

Repeat this activity by placing one square on the other with sides of the same measure (see Fig. 5.2) or by placing two equilateral triangles of equal sides on each other. You will observe that the squares are congruent to each other and so are the equilateral triangles.

Fig. 5.2

You may wonder why we are studying congruence. You all must have seen the ice tray in your refrigerator. Observe that the moulds for making ice are all congruent. The cast used for moulding in the tray also has congruent depressions (may be all are rectangular or all circular or all triangular). So, whenever identical objects have to be produced, the concept of congruence is used in making the cast.

Sometimes, you may find it difficult to replace the refill in your pen by a new one and this is so when the new refill is not of the same size as the one you want to remove. Obviously, if the two refills are identical or congruent, the new refill fits.

So, you can find numerous examples where congruence of objects is applied in daily life situations.

Can you think of some more examples of congruent figures?

Now, which of the following figures are not congruent to the square in Fig 5.3 (i) :

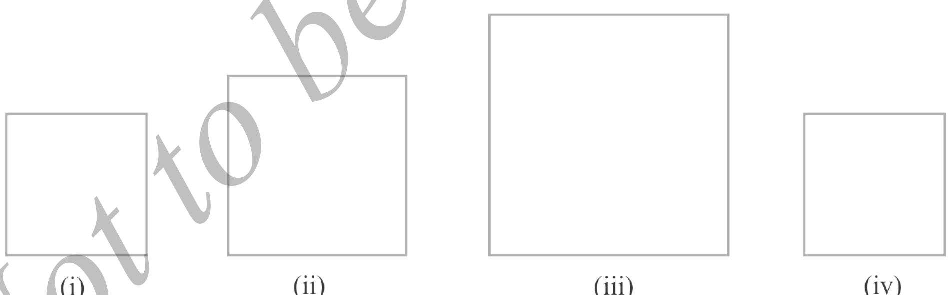


Fig. 5.3

The large squares in Fig. 5.3 (ii) and (iii) are obviously not congruent to the one in Fig 5.3 (i), but the square in Fig 5.3 (iv) is congruent to the one given in Fig 5.3 (i).

Let us now discuss the congruence of two triangles.

You already know that two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle.

Now, which of the triangles given below are congruent to triangle ABC in Fig. 5.4 (i)?

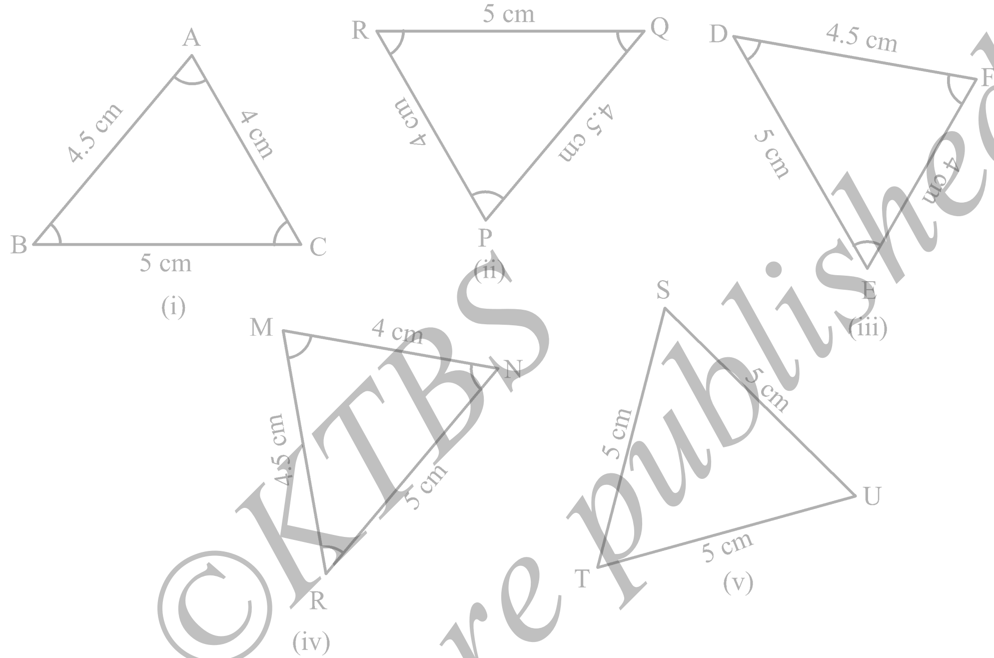


Fig. 5.4

Cut out each of these triangles from Fig. 5.4 (ii) to (v) and turn them around and try to cover Δ ABC. Observe that triangles in Fig. 5.4 (ii), (iii) and (iv) are congruent to Δ ABC while Δ TSU of Fig 5.4 (v) is not congruent to Δ ABC.

If Δ PQR is congruent to Δ ABC, we write Δ PQR ≅ Δ ABC.

Notice that when Δ PQR ≅ Δ ABC, then sides of Δ PQR fall on corresponding equal sides of Δ ABC and so is the case for the angles.

That is, PQ covers AB, QR covers BC and RP covers CA; ∠ P covers ∠ A, ∠ Q covers ∠ B and ∠ R covers ∠ C. Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, R to C and so on which is written as

$$P \leftrightarrow A,\ Q \leftrightarrow B,\ R \leftrightarrow C$$

Note that under this correspondence, Δ PQR ≅ Δ ABC; but it will not be correct to write ΔQRP ≅ Δ ABC.

Similarly, for Fig. 5.4 (iii),

FD ↔ AB, DE ↔ BC and EF ↔ CA

and F ↔ A, D ↔ B and E ↔ C

So, Δ FDE ≅ Δ ABC but writing Δ DEF ≅ Δ ABC is not correct.

Give the correspondence between the triangle in Fig. 5.4 (iv) and Δ ABC.

So, it is necessary to write the correspondence of vertices correctly for writing of congruence of triangles in symbolic form.

Note that in **congruent triangles corresponding parts** are **equal** and we write in short 'CPCT' for *corresponding parts of congruent triangles*.

5.3 Criteria for Congruence of Triangles

In earlier classes, you have learnt four criteria for congruence of triangles. Let us recall them.

Draw two triangles with one side 3 cm. Are these triangles congruent? Observe that they are not congruent (see Fig. 5.5).

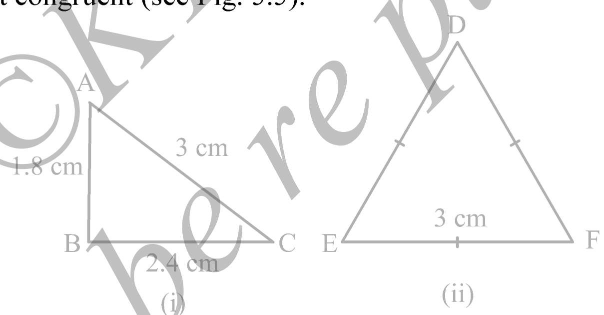


Fig. 5.5

Now, draw two triangles with one side 4 cm and one angle 50° (see Fig. 5.6). Are they congruent?

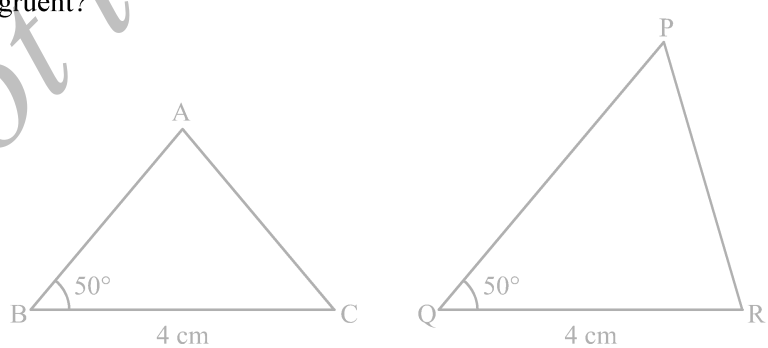


Fig. 5.6

See that these two triangles are not congruent.

Repeat this activity with some more pairs of triangles.

So, equality of one pair of sides or one pair of sides and one pair of angles is not sufficient to give us congruent triangles.

What would happen if the other pair of arms (sides) of the equal angles are also equal?

In Fig 5.7, BC = QR, $\angle$ B = $\angle$ Q and also, AB = PQ. Now, what can you say about congruence of Δ ABC and Δ PQR?

Recall from your earlier classes that, in this case, the two triangles are congruent. Verify this for Δ ABC and Δ PQR in Fig. 5.7.

Repeat this activity with other pairs of triangles. Do you observe that the equality of two sides and the included angle is enough for the congruence of triangles? Yes, it is enough.

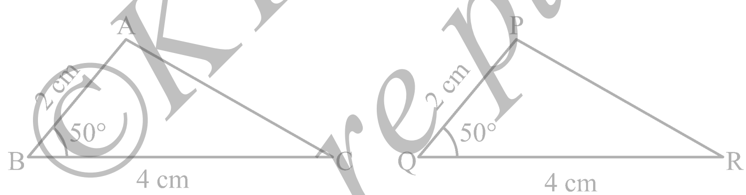


Fig. 5.7

This is the first criterion for congruence of triangles.

Axiom 5.1 (SAS congruence rule) : *Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.*

This result cannot be proved with the help of previously known results and so it is accepted true as an axiom (see Appendix 1).

Let us now take some examples.

Example 1 : In Fig. 5.8, OA = OB and OD = OC. Show that

(i) Δ AOD $\cong$ Δ BOC and (ii) AD || BC.

Solution : (i) You may observe that in Δ AOD and Δ BOC,

$$\left.\begin{array}{l} OA = OB \\ OD = OC \end{array}\right\} \quad \text{(Given)}$$

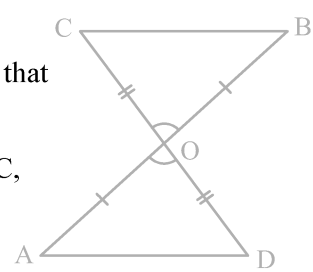


Fig. 5.8

Also, since $\angle$ AOD and $\angle$ BOC form a pair of vertically opposite angles, we have

$$\angle AOD = \angle BOC.$$

So, Δ AOD $\cong$ Δ BOC (by the SAS congruence rule)

(ii) In congruent triangles AOD and BOC, the other corresponding parts are also equal.

So, $\angle$ OAD = $\angle$ OBC and these form a pair of alternate angles for line segments AD and BC.

Therefore, AD $\parallel$ BC.

Example 2 : AB is a line segment and line *l* is its perpendicular bisector. If a point P lies on *l*, show that P is equidistant from A and B.

Solution : Line $l \perp$ AB and passes through C which is the mid-point of AB (see Fig. 5.9). You have to show that PA = PB. Consider Δ PCA and Δ PCB.

We have AC = BC (C is the mid-point of AB)

$\angle$ PCA = $\angle$ PCB = 90° (Given)

PC = PC (Common)

So, Δ PCA $\cong$ Δ PCB (SAS rule)

and so, PA = PB, as they are corresponding sides of congruent triangles.

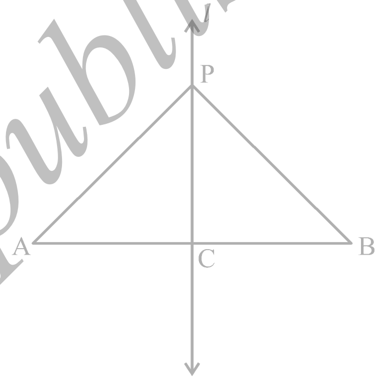


Fig. 5.9

Now, let us construct two triangles, whose sides are 4 cm and 5 cm and one of the angles is 50° and this angle is not included in between the equal sides (see Fig. 5.10). Are the two triangles congruent?

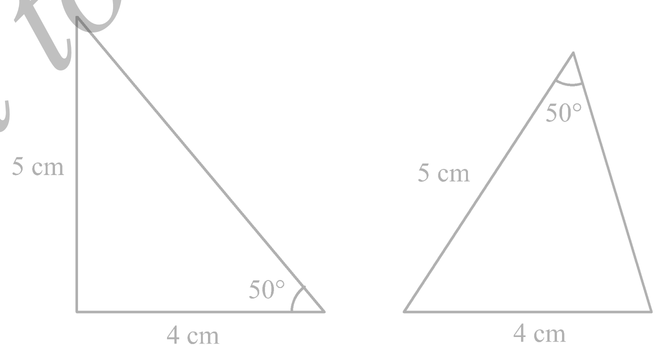


Fig. 5.10

Notice that the two triangles are not congruent.

Repeat this activity with more pairs of triangles. You will observe that for triangles to be congruent, it is very important that the equal angles are included between the pairs of equal sides.

So, SAS *congruence rule holds* but not ASS or SSA rule.

Next, try to construct the two triangles in which two angles are 60° and 45° and the side included between these angles is 4 cm (see Fig. 5.11).

Fig. 5.11

Cut out these triangles and place one triangle on the other. What do you observe? See that one triangle covers the other completely; that is, the two triangles are congruent. Repeat this activity with more pairs of triangles. You will observe that equality of two angles and the included side is sufficient for congruence of triangles.

This result is the **Angle-Side-Angle** criterion for congruence and is written as **ASA** criterion. You have verified this criterion in earlier classes, but let us state and prove this result.

Since this result can be proved, it is called a theorem and to prove it, we use the SAS axiom for congruence.

Theorem 5.1 (ASA congruence rule) : *Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.*

Proof : We are given two triangles ABC and DEF in which:

$$\angle B = \angle E, \angle C = \angle F$$

and $\qquad BC = EF$

We need to prove that $\qquad \Delta ABC \cong \Delta DEF$

For proving the congruence of the two triangles see that three cases arise.

Case (i) : Let AB = DE (see Fig. 5.12).

Now what do you observe? You may observe that

$$AB = DE \quad \text{(Assumed)}$$

$$\angle B = \angle E \quad \text{(Given)}$$

$$BC = EF \quad \text{(Given)}$$

So, $\Delta\ ABC \cong \Delta\ DEF$ (By SAS rule)

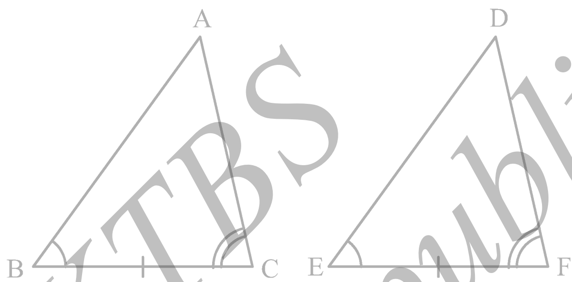


Fig. 5.12

Case (ii) : Let if possible AB > DE. So, we can take a point P on AB such that PB = DE. Now consider Δ PBC and Δ DEF (see Fig. 5.13).

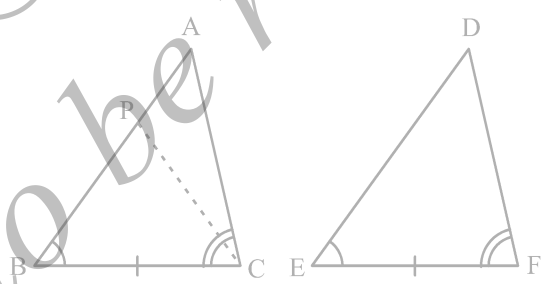


Fig. 5.13

Observe that in Δ PBC and Δ DEF,

$$PB = DE \quad \text{(By construction)}$$

$$\angle B = \angle E \quad \text{(Given)}$$

$$BC = EF \quad \text{(Given)}$$

So, we can conclude that:

Δ PBC ≅ Δ DEF, by the SAS axiom for congruence.

Since the triangles are congruent, their corresponding parts will be equal.

So, $\angle$ PCB = $\angle$ DFE

But, we are given that

$\angle$ ACB = $\angle$ DFE

So, $\angle$ ACB = $\angle$ PCB

Is this possible?

This is possible only if P coincides with A.

or, BA = ED

So, Δ ABC $\cong$ Δ DEF (by SAS axiom)

Case (iii) : If AB < DE, we can choose a point M on DE such that ME = AB and repeating the arguments as given in Case (ii), we can conclude that AB = DE and so, Δ ABC $\cong$ Δ DEF.

Suppose, now in two triangles two pairs of angles and one pair of corresponding sides are equal but the side is not included between the corresponding equal pairs of angles. Are the triangles still congruent? You will observe that they are congruent. Can you reason out why?

You know that the sum of the three angles of a triangle is 180°. So if two pairs of angles are equal, the third pair is also equal (180° – sum of equal angles).

So, two *triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal*. We may call it as the **AAS Congruence Rule**.

Now let us perform the following activity :

Draw triangles with angles 40°, 50° and 90°. How many such triangles can you draw?

In fact, you can draw as many triangles as you want with different lengths of sides (see Fig. 5.14).

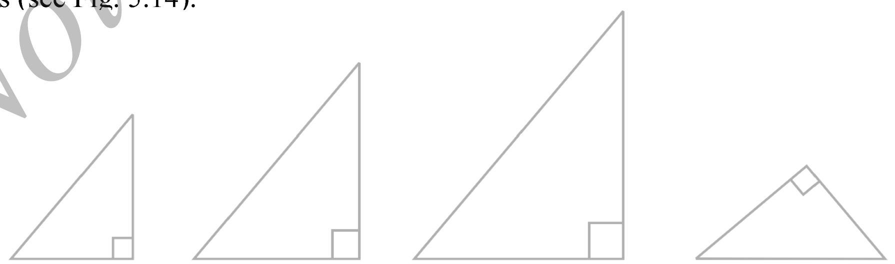


Fig. 5.14

Observe that the triangles may or may not be congruent to each other.

So, equality of three angles is not sufficient for congruence of triangles. Therefore, for congruence of triangles out of three equal parts, one has to be a side.

Let us now take some more examples.

Example 3 : Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD (see Fig. 5.15). Show that (i) $\Delta AOB \cong \Delta DOC$ (ii) O is also the mid-point of BC.

Solution : (i) Consider Δ AOB and Δ DOC.

$\angle ABO = \angle DCO$

(Alternate angles as AB || CD and BC is the transversal)

$\angle AOB = \angle DOC$

(Vertically opposite angles)

$OA = OD$ (Given)

Therefore, $\Delta AOB \cong \Delta DOC$ (AAS rule)

(ii) $OB = OC$ (CPCT)

So, O is the mid-point of BC.

Fig. 5.15

EXERCISE 5.1

1. In quadrilateral ACBD, AC = AD and AB bisects $\angle$ A (see Fig. 5.16). Show that $\Delta ABC \cong \Delta ABD$.

   What can you say about BC and BD?

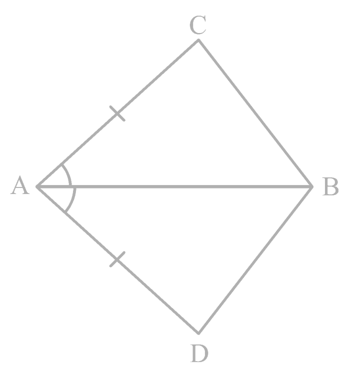


Fig. 5.16

2. ABCD is a quadrilateral in which AD = BC and $\angle$ DAB = $\angle$ CBA (see Fig. 5.17). Prove that

(i) Δ ABD $\cong$ Δ BAC

(ii) BD = AC

(iii) $\angle$ ABD = $\angle$ BAC.

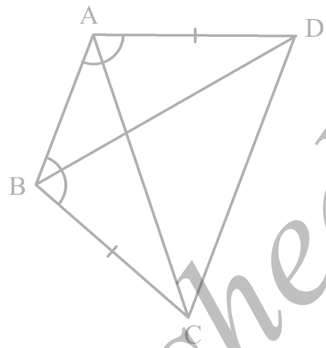


Fig. 5.17

3. AD and BC are equal perpendiculars to a line segment AB (see Fig. 5.18). Show that CD bisects AB.

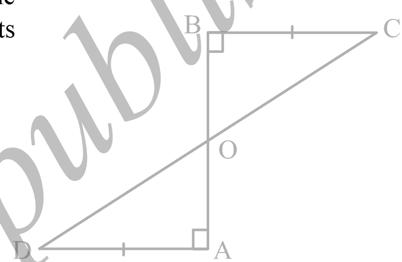


Fig. 5.18

4. l and m are two parallel lines intersected by another pair of parallel lines p and q (see Fig. 5.19). Show that Δ ABC $\cong$ Δ CDA.

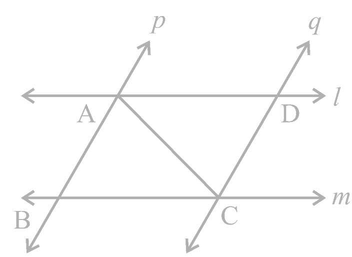


Fig. 5.19

5. Line l is the bisector of an angle $\angle$ A and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle$ A (see Fig. 5.20). Show that:

(i) Δ APB $\cong$ Δ AQB

(ii) BP = BQ or B is equidistant from the arms of $\angle$ A.

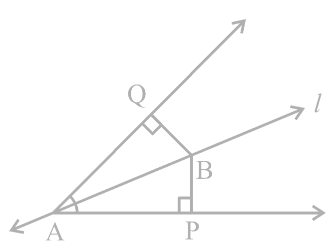


Fig. 5.20

6. In Fig. 5.21, AC = AE, AB = AD and $\angle$ BAD = $\angle$ EAC. Show that BC = DE.

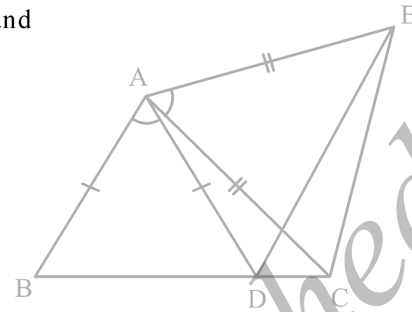


Fig. 5.21

7. AB is a line segment and P is its mid-point. D and E are points on the same side of AB such that $\angle$ BAD = $\angle$ ABE and $\angle$ EPA = $\angle$ DPB (see Fig. 5.22). Show that
   (i) Δ DAP $\cong$ Δ EBP
   (ii) AD = BE

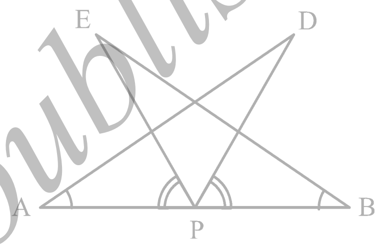


Fig. 5.22

8. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see Fig. 5.23). Show that:
   (i) Δ AMC $\cong$ Δ BMD
   (ii) $\angle$ DBC is a right angle.
   (iii) Δ DBC $\cong$ Δ ACB
   (iv) CM = $\frac{1}{2}$ AB

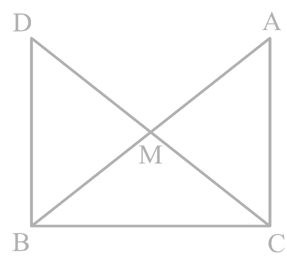


Fig. 5.23

5.4 Some Properties of a Triangle

In the above section you have studied two criteria for congruence of triangles. Let us now apply these results to study some properties related to a triangle whose two sides are equal.

Perform the activity given below:

Construct a triangle in which two sides are equal, say each equal to 3.5 cm and the third side equal to 5 cm (see Fig. 5.24). You have done such constructions in earlier classes.

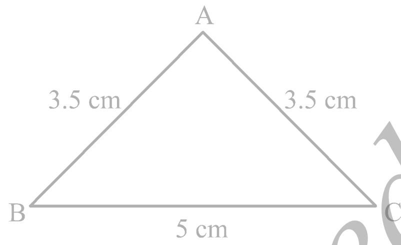


Fig. 5.24

Do you remember what is such a triangle called?

A triangle in which two sides are equal is called an **isosceles triangle**. So, Δ ABC of Fig. 5.24 is an isosceles triangle with AB = AC.

Now, measure $\angle$ B and $\angle$ C. What do you observe?

Repeat this activity with other isosceles triangles with different sides.

You may observe that in each such triangle, the angles opposite to the equal sides are equal.

This is a very important result and is indeed true for any isosceles triangle. It can be proved as shown below.

Theorem 5.2 : *Angles opposite to equal sides of an isosceles triangle are equal.*

This result can be proved in many ways. One of the proofs is given here.

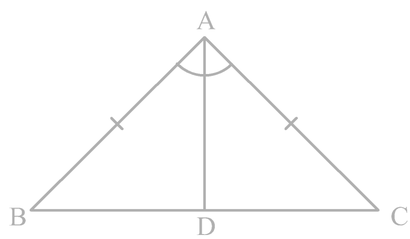


Fig. 5.25

Proof : We are given an isosceles triangle ABC in which AB = AC. We need to prove that $\angle$ B = $\angle$ C.

Let us draw the bisector of $\angle$ A and let D be the point of intersection of this bisector of $\angle$ A and BC (see Fig. 5.25).

In Δ BAD and Δ CAD,

	AB = AC	(Given)
	$\angle$ BAD = $\angle$ CAD	(By construction)
	AD = AD	(Common)
So,	Δ BAD $\cong$ Δ CAD	(By SAS rule)

So, $\angle$ ABD = $\angle$ ACD, since they are corresponding angles of congruent triangles.

So, $\angle$ B = $\angle$ C

Is the converse also true? That is:

If two angles of any triangle are equal, can we conclude that the sides opposite to them are also equal?

Perform the following activity.

Construct a triangle ABC with BC of any length and $\angle$ B = $\angle$ C = 50°. Draw the bisector of $\angle$ A and let it intersect BC at D (see Fig. 5.26).

Cut out the triangle from the sheet of paper and fold it along AD so that vertex C falls on vertex B.

What can you say about sides AC and AB?

Observe that AC covers AB completely

So, AC = AB

Repeat this activity with some more triangles. Each time you will observe that the sides opposite to equal angles are equal. So we have the following:

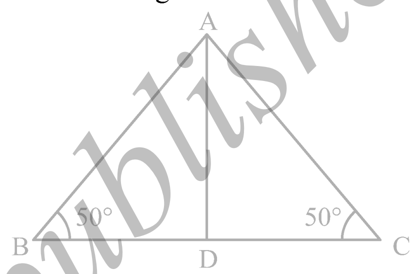


Fig. 5.26

Theorem 5.3 : *The sides opposite to equal angles of a triangle are equal.*

This is the converse of Theorem 5.2.

You can prove this theorem by ASA congruence rule.

Let us take some examples to apply these results.

Example 4 : In Δ ABC, the bisector AD of $\angle$ A is perpendicular to side BC (see Fig. 7.27). Show that AB = AC and Δ ABC is isosceles.

Solution : In ΔABD and ΔACD,

	$\angle$ BAD = $\angle$ CAD	(Given)
	AD = AD	(Common)
	$\angle$ ADB = $\angle$ ADC = 90°	(Given)
So,	Δ ABD ≅ Δ ACD	(ASA rule)
So,	AB = AC	(CPCT)

or, Δ ABC is an isosceles triangle.

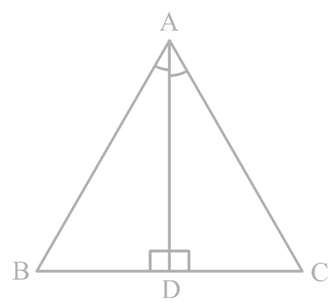


Fig. 5.27

Example 5 : E and F are respectively the mid-points of equal sides AB and AC of Δ ABC (see Fig. 5.28). Show that BF = CE.

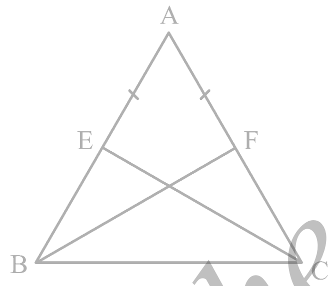


Fig. 5.28

Solution : In Δ ABF and Δ ACE,

AB = AC (Given)

∠ A = ∠ A (Common)

AF = AE (Halves of equal sides)

So, Δ ABF ≅ Δ ACE (SAS rule)

Therefore, BF = CE (CPCT)

Example 6 : In an isosceles triangle ABC with AB = AC, D and E are points on BC such that BE = CD (see Fig. 5.29). Show that AD = AE.

Solution : In Δ ABD and Δ ACE,

AB = AC (Given) (1)

∠ B = ∠ C

(Angles opposite to equal sides) (2)

Also, BE = CD

So, BE – DE = CD – DE

That is, BD = CE (3)

So, Δ ABD ≅ Δ ACE

(Using (1), (2), (3) and SAS rule).

This gives AD = AE (CPCT)

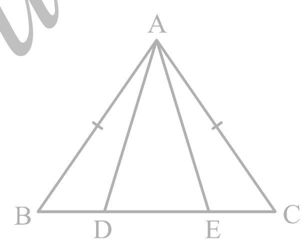


Fig. 5.29

EXERCISE 5.2

1. In an isosceles triangle ABC, with AB = AC, the bisectors of ∠ B and ∠ C intersect each other at O. Join A to O. Show that :

   (i) OB = OC (ii) AO bisects ∠ A

2. In Δ ABC, AD is the perpendicular bisector of BC (see Fig. 5.30). Show that Δ ABC is an isosceles triangle in which AB = AC.

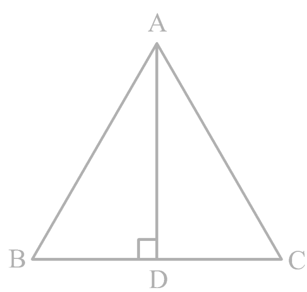


Fig. 5.30

3. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively (see Fig. 5.31). Show that these altitudes are equal.

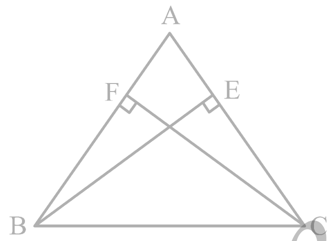


Fig. 5.31

4. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal (see Fig. 5.32). Show that

(i) Δ ABE $\cong$ Δ ACF

(ii) AB = AC, i.e., ABC is an isosceles triangle.

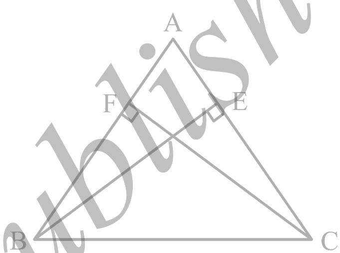


Fig. 5.32

5. ABC and DBC are two isosceles triangles on the same base BC (see Fig. 5.33). Show that $\angle$ ABD = $\angle$ ACD.

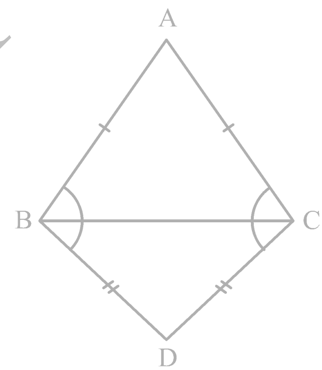


Fig. 5.33

6. ΔABC is an isosceles triangle in which AB = AC. Side BA is produced to D such that AD = AB (see Fig. 5.34). Show that $\angle$ BCD is a right angle.

7. ABC is a right angled triangle in which $\angle$ A = 90° and AB = AC. Find $\angle$ B and $\angle$ C.

8. Show that the angles of an equilateral triangle are 60° each.

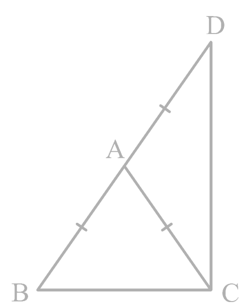


Fig. 5.34

5.5 Some More Criteria for Congruence of Triangles

You have seen earlier in this chapter that equality of three angles of one triangle to three angles of the other is not sufficient for the congruence of the two triangles. You may wonder whether equality of three sides of one triangle to three sides of another triangle is enough for congruence of the two triangles. You have already verified in earlier classes that this is indeed true.

To be sure, construct two triangles with sides 4 cm, 3.5 cm and 4.5 cm (see Fig. 5.35). Cut them out and place them on each other. What do you observe? They cover each other completely, if the equal sides are placed on each other. So, the triangles are congruent.

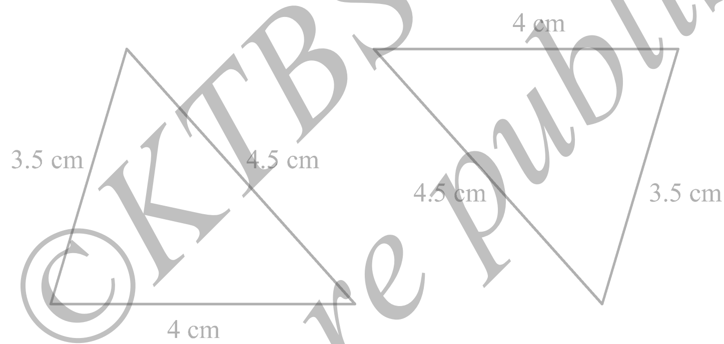


Fig. 5.35

Repeat this activity with some more triangles. We arrive at another rule for congruence.

Theorem 5.4 (SSS congruence rule) : *If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.*

This theorem can be proved using a suitable construction.

You have already seen that in the SAS congruence rule, the pair of equal angles has to be the included angle between the pairs of corresponding pair of equal sides and if this is not so, the two triangles may not be congruent.

Perform this activity:

Construct two right angled triangles with hypotenuse equal to 5 cm and one side equal to 4 cm each (see Fig. 5.36).

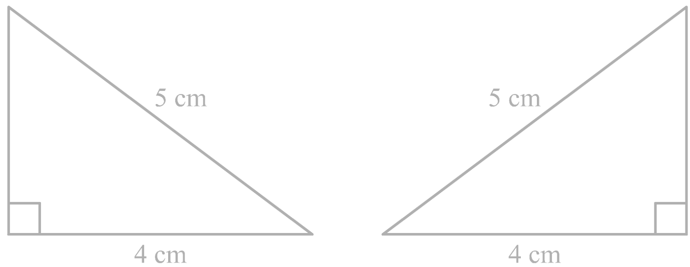


Fig. 5.36

Cut them out and place one triangle over the other with equal side placed on each other. Turn the triangles, if necessary. What do you observe?

The two triangles cover each other completely and so they are congruent. Repeat this activity with other pairs of right triangles. What do you observe?

You will find that two right triangles are congruent if one pair of sides and the hypotenuse are equal. You have verified this in earlier classes.

Note that, the right angle is ***not*** the included angle in this case.

So, you arrive at the following congruence rule:

Theorem 3.5 (RHS congruence rule) : *If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent.*

Note that RHS stands for **Right angle - Hypotenuse - Side.**

Let us now take some examples.

Example 7 : AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B (see Fig. 5.37). Show that the line PQ is the perpendicular bisector of AB.

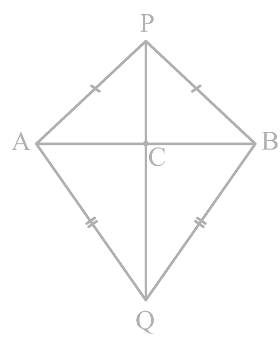


Fig. 5.37

Solution : You are given that PA = PB and QA = QB and you are to show that PQ $\perp$ AB and PQ bisects AB. Let PQ intersect AB at C.

Can you think of two congruent triangles in this figure?

Let us take Δ PAQ and Δ PBQ.

In these triangles,

$$AP = BP \quad \text{(Given)}$$

$$AQ = BQ \quad \text{(Given)}$$

$$PQ = PQ \quad \text{(Common)}$$

So, $\Delta\ PAQ \cong \Delta\ PBQ$ (SSS rule)

Therefore, $\angle\ APQ = \angle\ BPQ$ (CPCT).

Now let us consider Δ PAC and Δ PBC.

You have : $AP = BP$ (Given)

$\angle\ APC = \angle\ BPC$ ($\angle\ APQ = \angle\ BPQ$ proved above)

$PC = PC$ (Common)

So, $\Delta\ PAC \cong \Delta\ PBC$ (SAS rule)

Therefore, $AC = BC$ (CPCT) (1)

and $\angle\ ACP = \angle\ BCP$ (CPCT)

Also, $\angle\ ACP + \angle\ BCP = 180°$ (Linear pair)

So, $2\angle\ ACP = 180°$

or, $\angle\ ACP = 90°$ (2)

From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

[Note that, without showing the congruence of Δ PAQ and Δ PBQ, you cannot show that $\Delta\ PAC \cong \Delta\ PBC$ even though $AP = BP$ (Given)

$PC = PC$ (Common)

and $\angle\ PAC = \angle\ PBC$ (Angles opposite to equal sides in ΔAPB)

It is because these results give us SSA rule which is not always valid or true for congruence of triangles. Also the angle is not included between the equal pairs of sides.]

Let us take some more examples.

Example 8 : P is a point equidistant from two lines *l* and *m* intersecting at point A (see Fig. 5.38). Show that the line AP bisects the angle between them.

Solution : You are given that lines *l* and *m* intersect each other at A. Let $PB \perp l$, $PC \perp m$. It is given that PB = PC.

You are to show that $\angle\ PAB = \angle\ PAC$.

Let us consider Δ PAB and Δ PAC. In these two triangles,

	PB = PC	(Given)
	∠ PBA = ∠ PCA = 90°	(Given)
	PA = PA	(Common)
So,	Δ PAB ≅ Δ PAC	(RHS rule)
So,	∠ PAB = ∠ PAC	(CPCT)

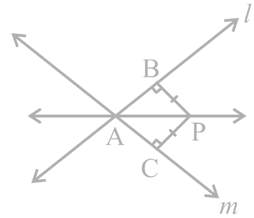


Fig. 5.38

Note that this result is the converse of the result proved in Q.5 of Exercise 5.1.

EXERCISE 5.3

1. Δ ABC and Δ DBC are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC (see Fig. 5.39). If AD is extended to intersect BC at P, show that

(i) Δ ABD ≅ Δ ACD

(ii) Δ ABP ≅ Δ ACP

(iii) AP bisects ∠ A as well as ∠ D.

(iv) AP is the perpendicular bisector of BC.

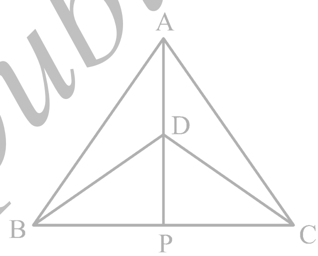


Fig. 5.39

2. AD is an altitude of an isosceles triangle ABC in which AB = AC. Show that

(i) AD bisects BC (ii) AD bisects ∠ A.

3. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of Δ PQR (see Fig. 5.40). Show that:

(i) Δ ABM ≅ Δ PQN

(ii) Δ ABC ≅ Δ PQR

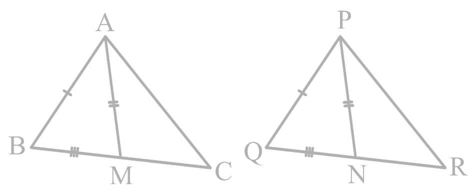


Fig. 5.40

4. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

5. ABC is an isosceles triangle with AB = AC. Draw AP ⊥ BC to show that ∠ B = ∠ C.

5.6 Inequalities in a Triangle

So far, you have been mainly studying the equality of sides and angles of a triangle or triangles. Sometimes, we do come across unequal objects, we need to compare them. For example, line-segment AB is greater in length as compared to line segment CD in Fig. 5.41 (i) and $\angle$ A is greater than $\angle$ B in Fig 5.41 (ii).

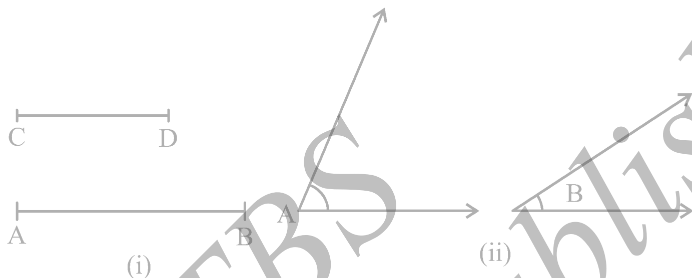


Fig. 5.41

Let us now examine whether there is any relation between unequal sides and unequal angles of a triangle. For this, let us perform the following activity:

Activity : Fix two pins on a drawing board say at B and C and tie a thread to mark a side BC of a triangle.

Fix one end of another thread at C and tie a pencil at the other (free) end . Mark a point A with the pencil and draw Δ ABC (see Fig 5.42). Now, shift the pencil and mark another point A′ on CA beyond A (new position of it)

So, A′C > AC (Comparing the lengths)

Join A′ to B and complete the triangle A′BC. What can you say about $\angle$ A′BC and $\angle$ ABC?

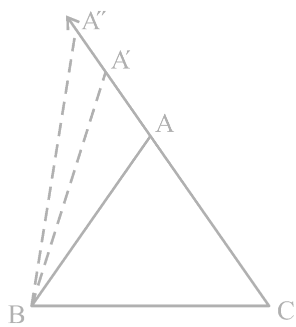


Fig. 5.42

Compare them. What do you observe?

Clearly, $\angle$ A′BC > $\angle$ ABC

Continue to mark more points on CA (extended) and draw the triangles with the side BC and the points marked.

You will observe that as the length of the side AC is increased (by taking different positions of A), the angle opposite to it, that is, $\angle$ B also increases.

Let us now perform another activity :

Activity : Construct a scalene triangle (that is a triangle in which all sides are of different lengths). Measure the lengths of the sides.

Now, measure the angles. What do you observe?

In Δ ABC of Fig 5.43, BC is the longest side and AC is the shortest side.

Also, $\angle$ A is the largest and $\angle$ B is the smallest.

Repeat this activity with some other triangles.

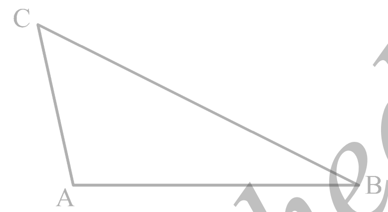


Fig. 5.43

We arrive at a very important result of inequalities in a triangle. It is stated in the form of a theorem as shown below:

Theorem 5.6 : *If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).*

You may prove this theorem by taking a point P on BC such that CA = CP in Fig. 5.43.

Now, let us perform another activity :

Activity : Draw a line-segment AB. With A as centre and some radius, draw an arc and mark different points say P, Q, R, S, T on it.

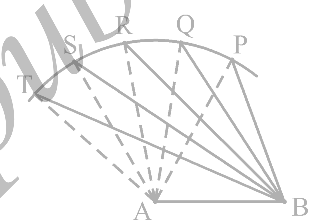


Fig. 5.44

Join each of these points with A as well as with B (see Fig. 5.44). Observe that as we move from P to T, $\angle$ A is becoming larger and larger. What is happening to the length of the side opposite to it? Observe that the length of the side is also increasing; that is $\angle$ TAB > $\angle$ SAB > $\angle$ RAB > $\angle$ QAB > $\angle$ PAB and TB > SB > RB > QB > PB.

Now, draw any triangle with all angles unequal to each other. Measure the lengths of the sides (see Fig. 5.45).

Observe that the side opposite to the largest angle is the longest. In Fig. 5.45, $\angle$ B is the largest angle and AC is the longest side.

Repeat this activity for some more triangles and we see that the converse of Theorem 5.6 is also true. In this way, we arrive at the following theorem:

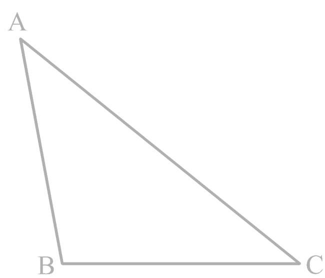


Fig. 5.45

Theorem 5.7 : *In any triangle, the side opposite to the larger (greater) angle is longer.*

This theorem can be proved by the method of contradiction.

Now take a triangle ABC and in it, find AB + BC, BC + AC and AC + AB. What do you observe?

You will observe that $AB + BC > AC$,

$BC + AC > AB$ and $AC + AB > BC$.

Repeat this activity with other triangles and with this you can arrive at the following theorem :

Theorem 5.8 : *The sum of any two sides of a triangle is greater than the third side.*

In Fig. 5.46, observe that the side BA of Δ ABC has been produced to a point D such that AD = AC. Can you show that ∠ BCD > ∠ BDC and BA + AC > BC? Have you arrived at the proof of the above theorem.

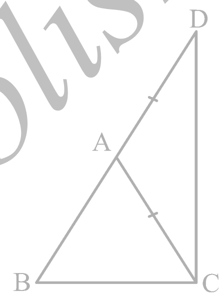


Fig. 5.46

Let us now take some examples based on these results.

Example 9 : D is a point on side BC of Δ ABC such that AD = AC (see Fig. 5.47). Show that AB > AD.

Solution : In Δ DAC,

AD = AC (Given)

So, ∠ ADC = ∠ ACD

(Angles opposite to equal sides)

Now, ∠ ADC is an exterior angle for ΔABD.

So, ∠ ADC > ∠ ABD

or, ∠ ACD > ∠ ABD

or, ∠ ACB > ∠ ABC

So, AB > AC (Side opposite to larger angle in Δ ABC)

or, AB > AD (AD = AC)

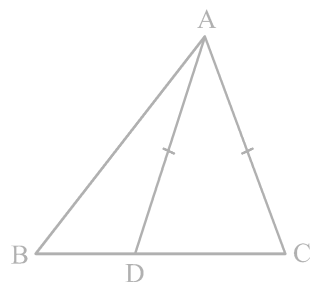


Fig. 5.47

EXERCISE 5.4

1. Show that in a right angled triangle, the hypotenuse is the longest side.

2. In Fig. 5.48, sides AB and AC of Δ ABC are extended to points P and Q respectively. Also, ∠ PBC < ∠ QCB. Show that AC > AB.

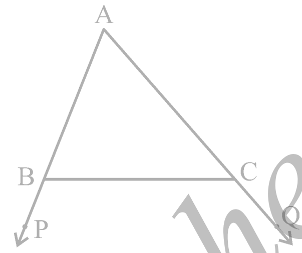


Fig. 5.48

3. In Fig. 5.49, ∠ B < ∠ A and ∠ C < ∠ D. Show that AD < BC.

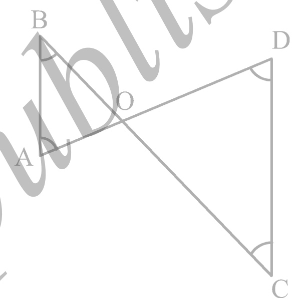


Fig. 5.49

4. AB and CD are respectively the smallest and longest sides of a quadrilateral ABCD (see Fig. 5.50). Show that ∠ A > ∠ C and ∠ B > ∠ D.

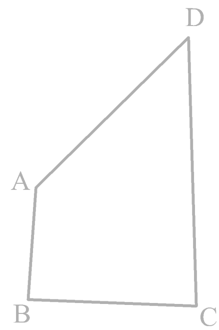


Fig. 5.50

5. In Fig 5.51, PR > PQ and PS bisects ∠ QPR. Prove that ∠ PSR > ∠ PSQ.

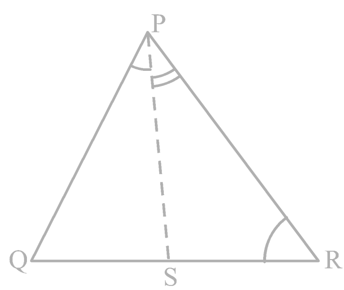


Fig. 5.51

6. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.

EXERCISE 5.5 (Optional)*

1. ABC is a triangle. Locate a point in the interior of Δ ABC which is equidistant from all the vertices of Δ ABC.

2. In a triangle locate a point in its interior which is equidistant from all the sides of the triangle.

3. In a huge park, people are concentrated at three points (see Fig. 5.52):

A : where there are different slides and swings for children,

B : near which a man-made lake is situated,

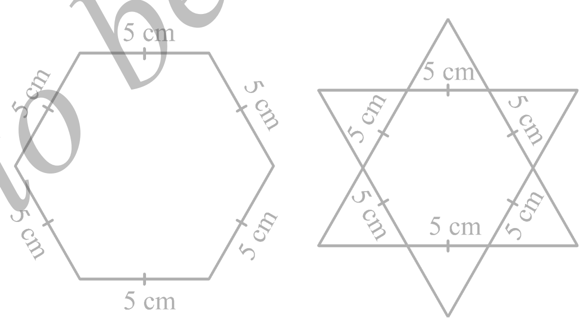
C : which is near to a large parking and exit.

Where should an icecream parlour be set up so that maximum number of persons can approach it?

(*Hint* : The parlour should be equidistant from A, B and C)

Fig. 5.52

4. Complete the hexagonal and star shaped Rangolies [see Fig. 5.53 (i) and (ii)] by filling them with as many equilateral triangles of side 1 cm as you can. Count the number of triangles in each case. Which has more triangles?



(i) **(ii)**

Fig. 5.53

*These exercises are not from examination point of view.

5.7 Summary

In this chapter, you have studied the following points :

1. Two figures are congruent, if they are of the same shape and of the same size.
2. Two circles of the same radii are congruent.
3. Two squares of the same sides are congruent.
4. If two triangles ABC and PQR are congruent under the correspondence $A \leftrightarrow P$, $B \leftrightarrow Q$ and $C \leftrightarrow R$, then symbolically, it is expressed as $\Delta\,ABC \cong \Delta\,PQR$.
5. If two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle, then the two triangles are congruent (SAS Congruence Rule).
6. If two angles and the included side of one triangle are equal to two angles and the included side of the other triangle, then the two triangles are congruent (ASA Congruence Rule).
7. If two angles and one side of one triangle are equal to two angles and the corresponding side of the other triangle, then the two triangles are congruent (AAS Congruence Rule).
8. Angles opposite to equal sides of a triangle are equal.
9. Sides opposite to equal angles of a triangle are equal.
10. Each angle of an equilateral triangle is of $60°$.
11. If three sides of one triangle are equal to three sides of the other triangle, then the two triangles are congruent (SSS Congruence Rule).
12. If in two right triangles, hypotenuse and one side of a triangle are equal to the hypotenuse and one side of other triangle, then the two triangles are congruent (RHS Congruence Rule).
13. In a triangle, angle opposite to the longer side is larger (greater).
14. In a triangle, side opposite to the larger (greater) angle is longer.
15. Sum of any two sides of a triangle is greater than the third side.

28ZHNT

CHAPTER 6

CONSTRUCTIONS

6.1 Introduction

In earlier chapters, the diagrams, which were necessary to prove a theorem or solving exercises were not necessarily precise. They were drawn only to give you a feeling for the situation and as an aid for proper reasoning. However, sometimes one needs an accurate figure, for example - to draw a map of a building to be constructed, to design tools, and various parts of a machine, to draw road maps etc. To draw such figures some basic geometrical instruments are needed. You must be having a geometry box which contains the following:

(i) A graduated scale, on one side of which centimetres and millimetres are marked off and on the other side inches and their parts are marked off.

(ii) A pair of set - squares, one with angles 90°, 60° and 30° and other with angles 90°, 45° and 45°.

(iii) A pair of dividers (or a divider) with adjustments.

(iv) A pair of compasses (or a compass) with provision of fitting a pencil at one end.

(v) A protractor.

Normally, all these instruments are needed in drawing a geometrical figure, such as a triangle, a circle, a quadrilateral, a polygon, etc. with given measurements. But a geometrical construction is the process of drawing a geometrical figure using only two instruments – an *ungraduated ruler*, also called a *straight edge* and a *compass*. In construction where measurements are also required, you may use a graduated scale and protractor also. In this chapter, some basic constructions will be considered. These will then be used to construct certain kinds of triangles.

6.2 Basic Constructions

In Class VI, you have learnt how to construct a circle, the perpendicular bisector of a line segment, angles of 30°, 45°, 60°, 90° and 120°, and the bisector of a given angle, without giving any justification for these constructions. In this section, you will construct some of these, with reasoning behind, why these constructions are valid.

Construction 6.1 : *To construct the bisector of a given angle.*

Given an angle ABC, we want to construct its bisector.

Steps of Construction :

1. Taking B as centre and any radius, draw an arc to intersect the rays BA and BC, say at E and D respectively [see Fig.6.1(i)].
2. Next, taking D and E as centres and with the radius more than $\frac{1}{2}$ DE, draw arcs to intersect each other, say at F.
3. Draw the ray BF [see Fig.6.1(ii)]. This ray BF is the required bisector of the angle ABC.

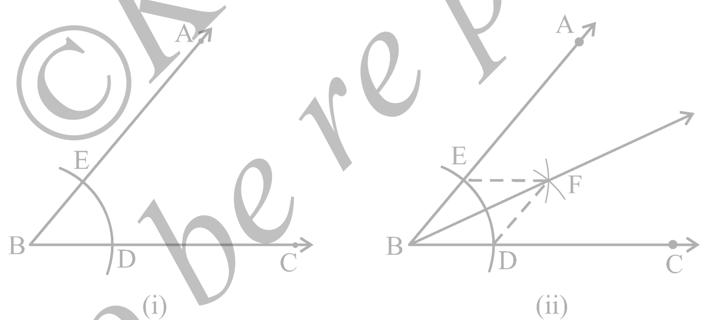


Fig. 6.1

Let us see how this method gives us the required angle bisector.

Join DF and EF.

In triangles BEF and BDF,

$$BE = BD \quad \text{(Radii of the same arc)}$$
$$EF = DF \quad \text{(Arcs of equal radii)}$$
$$BF = BF \quad \text{(Common)}$$

Therefore, $\Delta BEF \cong \Delta BDF$ (SSS rule)

This gives $\angle EBF = \angle DBF$ (CPCT)

Construction 6.2 : *To construct the perpendicular bisector of a given line segment.*

Given a line segment AB, we want to construct its perpendicular bisector.

Steps of Construction :

1. Taking A and B as centres and radius more than $\frac{1}{2}$ AB, draw arcs on both sides of the line segment AB (to intersect each other).
2. Let these arcs intersect each other at P and Q. Join PQ (see Fig.6.2).
3. Let PQ intersect AB at the point M. Then line PMQ is the required perpendicular bisector of AB.

Let us see how this method gives us the perpendicular bisector of AB.

Join A and B to both P and Q to form AP, AQ, BP and BQ.

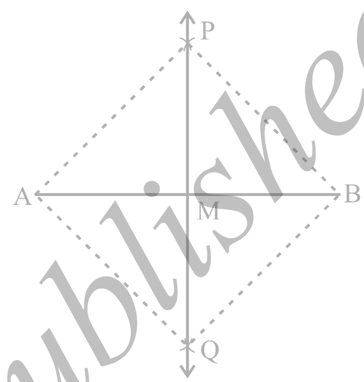


Fig. 6.2

In triangles PAQ and PBQ,

	AP = BP	(Arcs of equal radii)
	AQ = BQ	(Arcs of equal radii)
	PQ = PQ	(Common)
Therefore,	Δ PAQ $\cong$ Δ PBQ	(SSS rule)
So,	$\angle$ APM = $\angle$ BPM	(CPCT)

Now in triangles PMA and PMB,

	AP = BP	(As before)
	PM = PM	(Common)
	$\angle$ APM = $\angle$ BPM	(Proved above)
Therefore,	Δ PMA $\cong$ Δ PMB	(SAS rule)
So,	AM = BM and $\angle$ PMA = $\angle$ PMB	(CPCT)
As	$\angle$ PMA + $\angle$ PMB = 180°	(Linear pair axiom),

we get

$$\angle \text{PMA} = \angle \text{PMB} = 90°.$$

Therefore, PM, that is, PMQ is the perpendicular bisector of AB.

Construction 6.3 : *To construct an angle of 60^0 at the initial point of a given ray.*

Let us take a ray AB with initial point A [see Fig. 6.3(i)]. We want to construct a ray AC such that $\angle$ CAB = 60°. One way of doing so is given below.

Steps of Construction :

1. Taking A as centre and some radius, draw an arc of a circle, which intersects AB, say at a point D.
2. Taking D as centre and with the same radius as before, draw an arc intersecting the previously drawn arc, say at a point E.
3. Draw the ray AC passing through E [see Fig 6.3 (ii)].

Then $\angle$ CAB is the required angle of 60°. Now, let us see how this method gives us the required angle of 60°.

Join DE.

Then, AE = AD = DE (By construction)

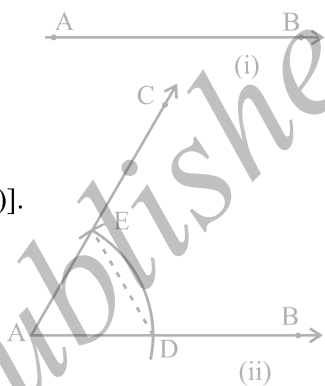


Fig. 6.3

Therefore, Δ EAD is an equilateral triangle and the $\angle$ EAD, which is the same as $\angle$ CAB is equal to 60°.

EXERCISE 6.1

1. Construct an angle of 90^0 at the initial point of a given ray and justify the construction.
2. Construct an angle of 45^0 at the initial point of a given ray and justify the construction.
3. Construct the angles of the following measurements:

   (i) 30° (ii) $22\frac{1}{2}°$ (iii) 15°

4. Construct the following angles and verify by measuring them by a protractor:

   (i) 75° (ii) 105° (iii) 135°

5. Construct an equilateral triangle, given its side and justify the construction.

6.3 Some Constructions of Triangles

So far, some basic constructions have been considered. Next, some constructions of triangles will be done by using the constructions given in earlier classes and given above. Recall from the Chapter 5 that SAS, SSS, ASA and RHS rules give the congruency of two triangles. Therefore, a triangle is unique if : (i) two sides and the included angle is given, (ii) three sides are given, (iii) two angles and the included side is given and, (iv) in a right triangle, hypotenuse and one side is given. You have already learnt how to construct such triangles in Class VII. Now, let us consider some more constructions of triangles. You may have noted that at least three parts of a triangle have to be given for constructing it but not all combinations of three parts are sufficient for the purpose. For example, if two sides and an angle (not the included angle) are given, then it is not always possible to construct such a triangle uniquely.

Construction 6.4 : *To construct a triangle, given its base, a base angle and sum of other two sides.*

Given the base BC, a base angle, say ∠B and the sum AB + AC of the other two sides of a triangle ABC, you are required to construct it.

Steps of Construction :

1. Draw the base BC and at the point B make an angle, say XBC equal to the given angle.
2. Cut a line segment BD equal to AB + AC from the ray BX.
3. Join DC and make an angle DCY equal to ∠BDC.
4. Let CY intersect BX at A (see Fig. 6.4).

Then, ABC is the required triangle.

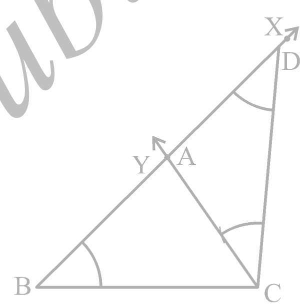


Fig. 6.4

Let us see how you get the required triangle.

Base BC and ∠B are drawn as given. Next in triangle ACD,

$$\angle ACD = \angle ADC \qquad \text{(By construction)}$$

Therefore, AC = AD and then

$$AB = BD - AD = BD - AC$$

$$AB + AC = BD$$

Alternative method :

Follow the first two steps as above. Then draw perpendicular bisector PQ of CD to intersect BD at a point A (see Fig 6.5). Join AC. Then ABC is the required triangle. Note that A lies on the perpendicular bisector of CD, therefore AD = AC.

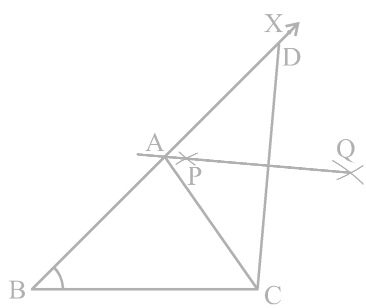


Fig. 6.5

Remark : The construction of the triangle is not possible if the sum AB + AC ≤ BC.

Construction 6.5 : *To construct a triangle given its base, a base angle and the difference of the other two sides.*

Given the base BC, a base angle, say ∠B and the difference of other two sides AB – AC or AC – AB, you have to construct the triangle ABC. Clearly there are following two cases:

Case (i) : Let AB > AC that is AB – AC is given.

Steps of Construction :

1. Draw the base BC and at point B make an angle say XBC equal to the given angle.
2. Cut the line segment BD equal to AB – AC from ray BX.
3. Join DC and draw the perpendicular bisector, say PQ of DC.
4. Let it intersect BX at a point A. Join AC (see Fig. 6.6).

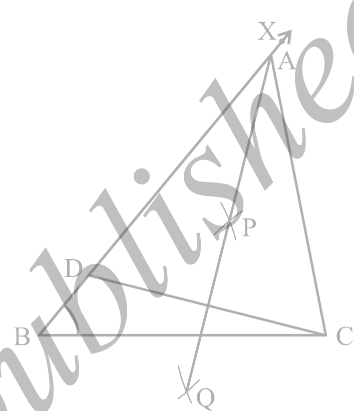


Fig. 6.6

Then ABC is the required triangle.

Let us now see how you have obtained the required triangle ABC.

Base BC and ∠B are drawn as given. The point A lies on the perpendicular bisector of DC. Therefore,

$$AD = AC$$

So, $$BD = AB – AD = AB – AC.$$

Case (ii) : Let AB < AC that is AC – AB is given.

Steps of Construction :

1. Same as in case (i).
2. Cut line segment BD equal to AC – AB from the line BX extended on opposite side of line segment BC.
3. Join DC and draw the perpendicular bisector, say PQ of DC.
4. Let PQ intersect BX at A. Join AC (see Fig. 6.7).

Then, ABC is the required triangle.

You can justify the construction as in case (i).

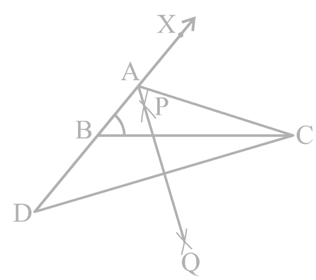


Fig. 6.7

Construction 6.6 : *To construct a triangle, given its perimeter and its two base angles.*

Given the base angles, say $\angle$ B and $\angle$ C and BC + CA + AB, you have to construct the triangle ABC.

Steps of Construction :

1. Draw a line segment, say XY equal to BC + CA + AB.
2. Make angles LXY equal to $\angle$B and MYX equal to $\angle$C.
3. Bisect $\angle$ LXY and $\angle$ MYX. Let these bisectors intersect at a point A [see Fig. 6.8(i)].

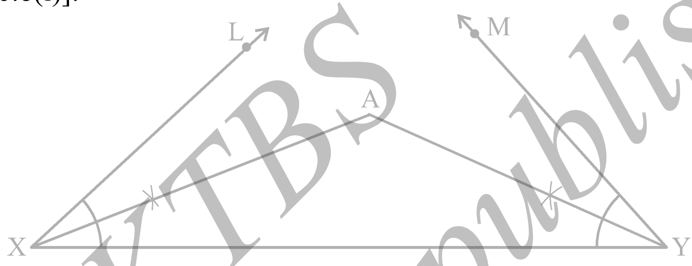


Fig. 6.8 (i)

4. Draw perpendicular bisectors PQ of AX and RS of AY.
5. Let PQ intersect XY at B and RS intersect XY at C. Join AB and AC [see Fig 6.8(ii)].

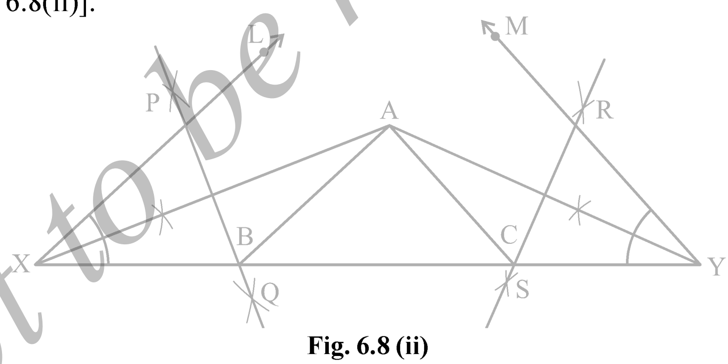


Fig. 6.8 (ii)

Then ABC is the required triangle. For the justification of the construction, you observe that, B lies on the perpendicular bisector PQ of AX.

Therefore, XB = AB and similarly, CY = AC.

This gives BC + CA + AB = BC + XB + CY = XY.

Again $\angle$BAX = $\angle$AXB (As in Δ AXB, AB = XB) and

$\angle$ABC = $\angle$BAX + $\angle$AXB = 2 $\angle$AXB = $\angle$LXY

Similarly, $\angle$ACB = $\angle$MYX as required.

Example 1 : Construct a triangle ABC, in which $\angle B = 60°$, $\angle C = 45°$ and AB + BC + CA = 11 cm.

Steps of Construction :

1. Draw a line segment PQ = 11 cm.(= AB + BC + CA).
2. At P construct an angle of 60° and at Q, an angle of 45°.

Fig. 6.9

3. Bisect these angles. Let the bisectors of these angles intersect at a point A.
4. Draw perpendicular bisectors DE of AP to intersect PQ at B and FG of AQ to intersect PQ at C.
5. Join AB and AC (see Fig. 6.9).

Then, ABC is the required triangle.

EXERCISE 6.2

1. Construct a triangle ABC in which BC = 7cm, $\angle B = 75°$ and AB + AC = 13 cm.
2. Construct a triangle ABC in which BC = 8cm, $\angle B = 45°$ and AB – AC = 3.5 cm.
3. Construct a triangle PQR in which QR = 6cm, $\angle Q = 60°$ and PR – PQ = 2cm.
4. Construct a triangle XYZ in which $\angle Y = 30°$, $\angle Z = 90°$ and XY + YZ + ZX = 11 cm.
5. Construct a right triangle whose base is 12cm and sum of its hypotenuse and other side is 18 cm.

6.4 Summary

In this chapter, you have done the following constructions using a ruler and a compass:

1. To bisect a given angle.
2. To draw the perpendicular bisector of a given line segment.
3. To construct an angle of 60° etc.
4. To construct a triangle given its base, a base angle and the sum of the other two sides.
5. To construct a triangle given its base, a base angle and the difference of the other two sides.
6. To construct a triangle given its perimeter and its two base angles.

CHAPTER 7

QUADRILATERALS

7.1 Introduction

You have studied many properties of a triangle in Chapters 3 and 5 and you know that on joining three non-collinear points in pairs, the figure so obtained is a triangle. Now, let us mark four points and see what we obtain on joining them in pairs in some order.

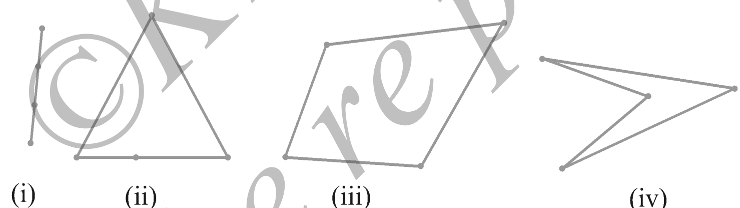


Fig. 7.1

Note that if all the points are collinear (in the same line), we obtain a line segment [see Fig. 7.1 (i)], if three out of four points are collinear, we get a triangle [see Fig. 7.1 (ii)], and if no three points out of four are collinear, we obtain a closed figure with four sides [see Fig. 7.1 (iii) and (iv)].

Such a figure formed by joining four points in an order is called a *quadrilateral*. In this book, we will consider only quadrilaterals of the type given in Fig. 7.1 (iii) but not as given in Fig. 7.1 (iv).

A quadrilateral has four sides, four angles and four vertices [see Fig. 7.2 (i)].

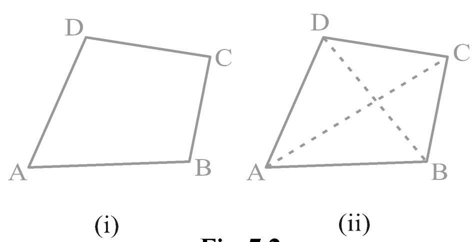


Fig. 7.2

In quadrilateral ABCD, AB, BC, CD and DA are the four sides; A, B, C and D are the four vertices and ∠ A, ∠ B, ∠ C and ∠ D are the four angles formed at the vertices.

Now join the opposite vertices A to C and B to D [see Fig. 7.2 (ii)].

AC and BD are the two diagonals of the quadrilateral ABCD.

In this chapter, we will study more about different types of quadrilaterals, their properties and especially those of parallelograms.

You may wonder why should we study about quadrilaterals (or parallelograms) Look around you and you will find so many objects which are of the shape of a quadrilateral - the floor, walls, ceiling, windows of your classroom, the blackboard, each face of the duster, each page of your book, the top of your study table etc. Some of these are given below (see Fig. 7.3).

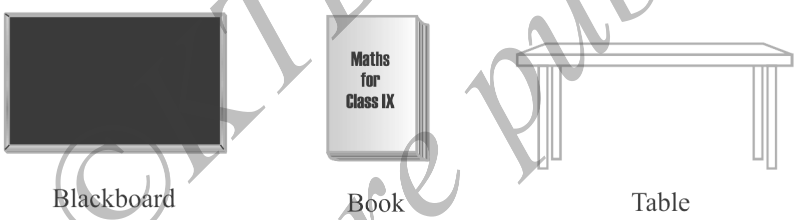


Fig. 7.3

Although most of the objects we see around are of the shape of special quadrilateral called rectangle, we shall study more about quadrilaterals and especially parallelograms because a rectangle is also a parallelogram and all properties of a parallelogram are true for a rectangle as well.

7.2 Angle Sum Property of a Quadrilateral

Let us now recall the angle sum property of a quadrilateral.

The sum of the angles of a quadrilateral is 360°. This can be verified by drawing a diagonal and dividing the quadrilateral into two triangles.

Let ABCD be a quadrilateral and AC be a diagonal (see Fig. 7.4).

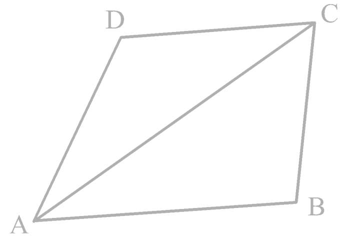


Fig. 7.4

What is the sum of angles in Δ ADC?

You know that

$$\angle DAC + \angle ACD + \angle D = 180° \qquad (1)$$

Similarly, in Δ ABC,

$$\angle CAB + \angle ACB + \angle B = 180° \qquad (2)$$

Adding (1) and (2), we get

$$\angle DAC + \angle ACD + \angle D + \angle CAB + \angle ACB + \angle B = 180° + 180° = 360°$$

Also, $\angle DAC + \angle CAB = \angle A$ and $\angle ACD + \angle ACB = \angle C$

So, $\angle A + \angle D + \angle B + \angle C = 360°$.

i.e., *the sum of the angles of a quadrilateral is 360°.*

7.3 Types of Quadrilaterals

Look at the different quadrilaterals drawn below:

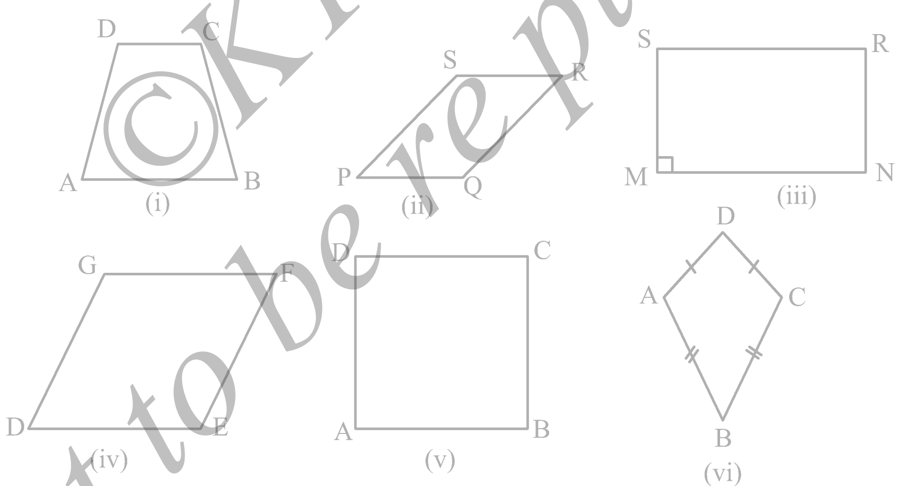


Fig. 7.5

Observe that :

- One pair of opposite sides of quadrilateral ABCD in Fig. 7.5 (i) namely, AB and CD are parallel. You know that it is called a *trapezium*.
- Both pairs of opposite sides of quadrilaterals given in Fig. 7.5 (ii), (iii) , (iv) and (v) are parallel. Recall that such quadrilaterals are called *parallelograms*.

  So, quadrilateral PQRS of Fig. 7.5 (ii) is a parallelogram.

Similarly, all quadrilaterals given in Fig. 7.5 (iii), (iv) and (v) are parallelograms.

- In parallelogram MNRS of Fig. 7.5 (iii), note that one of its angles namely $\angle$ M is a right angle. What is this special parallelogram called? Try to recall. It is called a *rectangle*.
- The parallelogram DEFG of Fig. 7.5 (iv) has all sides equal and we know that it is called a *rhombus*.
- The parallelogram ABCD of Fig. 7.5 (v) has $\angle$ A = 90° and all sides equal; it is called a *square*.
- In quadrilateral ABCD of Fig. 7.5 (vi), AD = CD and AB = CB i.e., two pairs of adjacent sides are equal. It is not a parallelogram. It is called a *kite*.

Note that a square, rectangle and rhombus are all parallelograms.

- A square is a rectangle and also a rhombus.
- A parallelogram is a trapezium.
- A kite is not a parallelogram.
- A trapezium is not a parallelogram (as only one pair of opposite sides is parallel in a trapezium and we require both pairs to be parallel in a parallelogram).
- A rectangle or a rhombus is not a square.

Look at the Fig. 7.6. We have a rectangle and a parallelogram with same perimeter 14 cm.

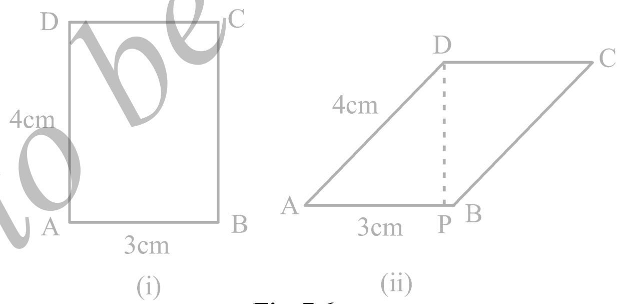


Fig. 7.6

Here the area of the parallelogram is DP × AB and this is less than the area of the rectangle, i.e., AB × AD as DP < AD. Generally sweet shopkeepers cut 'Burfis' in the shape of a parallelogram to accomodate more pieces in the same tray (see the shape of the Burfi before you eat it next time!).

Let us now review some properties of a parallelogram learnt in earlier classes.

7.4 Properties of a Parallelogram

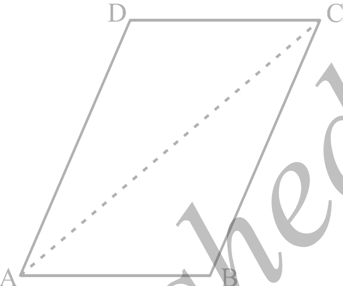


Fig. 7.7

Let us perform an activity.

Cut out a parallelogram from a sheet of paper and cut it along a diagonal (see Fig. 7.7). You obtain two triangles. What can you say about these triangles?

Place one triangle over the other. Turn one around, if necessary. What do you observe?

Observe that the two triangles are congruent to each other.

Repeat this activity with some more parallelograms. Each time you will observe that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem 7.1 : *A diagonal of a parallelogram divides it into two congruent triangles.*

Proof : Let ABCD be a parallelogram and AC be a diagonal (see Fig. 7.8). Observe that the diagonal AC divides parallelogram ABCD into two triangles, namely, Δ ABC and Δ CDA. We need to prove that these triangles are congruent.

In Δ ABC and Δ CDA, note that BC || AD and AC is a transversal.

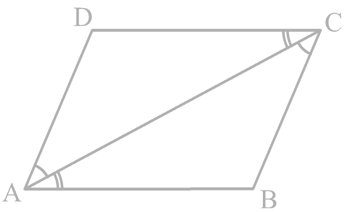


Fig. 7.8

So, $\angle$ BCA = $\angle$ DAC (Pair of alternate angles)

Also, AB || DC and AC is a transversal.

So, $\angle$ BAC = $\angle$ DCA (Pair of alternate angles)

and AC = CA (Common)

So, Δ ABC $\cong$ Δ CDA (ASA rule)

or, diagonal AC divides parallelogram ABCD into two congruent triangles ABC and CDA.

Now, measure the opposite sides of parallelogram ABCD. What do you observe?

You will find that AB = DC and AD = BC.

This is another property of a parallelogram stated below:

Theorem 7.2 : *In a parallelogram, opposite sides are equal.*

You have already proved that a diagonal divides the parallelogram into two congruent triangles; so what can you say about the corresponding parts say, the corresponding sides? They are equal.

So, AB = DC and AD = BC

Now what is the converse of this result? You already know that whatever is given in a theorem, the same is to be proved in the converse and whatever is proved in the theorem it is given in the converse. Thus, Theorem 7.2 can be stated as given below :

If a quadrilateral is a parallelogram, then each pair of its opposite sides is equal. So its converse is :

Theorem 7.3 : *If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.*

Can you reason out why?

Let sides AB and CD of the quadrilateral ABCD be equal and also AD = BC (see Fig. 8.9). Draw diagonal AC.

Clearly, Δ ABC $\cong$ Δ CDA (Why?)

So, $\angle$ BAC = $\angle$ DCA

and $\angle$ BCA = $\angle$ DAC (Why?)

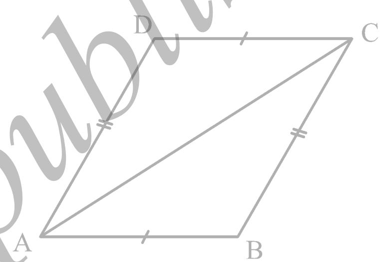


Fig. 7.9

Can you now say that ABCD is a parallelogram? Why?

You have just seen that in a parallelogram each pair of opposite sides is equal and conversely if each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram. Can we conclude the same result for the pairs of opposite angles?

Draw a parallelogram and measure its angles. What do you observe?

Each pair of opposite angles is equal.

Repeat this with some more parallelograms. We arrive at yet another result as given below.

Theorem 7.4 : *In a parallelogram, opposite angles are equal.*

Now, is the converse of this result also true? Yes. Using the angle sum property of a quadrilateral and the results of parallel lines intersected by a transversal, we can see that the converse is also true. So, we have the following theorem :

Theorem 7.5 : *If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.*

There is yet another property of a parallelogram. Let us study the same. Draw a parallelogram ABCD and draw both its diagonals intersecting at the point O (see Fig. 7.10).

Measure the lengths of OA, OB, OC and OD.

What do you observe? You will observe that

$$OA = OC \quad \text{and} \quad OB = OD.$$

or, O is the mid-point of both the diagonals.

Repeat this activity with some more parallelograms.

Each time you will find that O is the mid-point of both the diagonals.

So, we have the following theorem :

Theorem 7.6 : *The diagonals of a parallelogram bisect each other.*

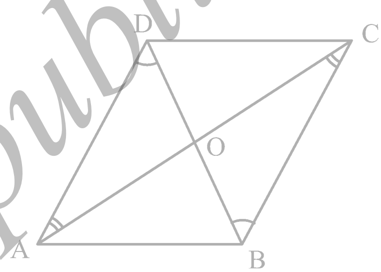


Fig. 7.10

Now, what would happen, if in a quadrilateral the diagonals bisect each other? Will it be a parallelogram? Indeed this is true.

This result is the converse of the result of Theorem 7.6. It is given below:

Theorem 7.7 : *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

You can reason out this result as follows:

Note that in Fig. 7.11, it is given that OA = OC and OB = OD.

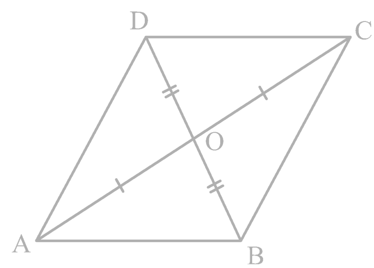


Fig. 7.11

So, Δ AOB $\cong$ Δ COD (Why?)

Therefore, $\angle$ ABO = $\angle$ CDO (Why?)

From this, we get AB || CD

Similarly, BC || AD

Therefore ABCD is a parallelogram.

Let us now take some examples.

Example 1 : Show that each angle of a rectangle is a right angle.

Solution : Let us recall what a rectangle is.

A rectangle is a parallelogram in which one angle is a right angle.

Fig. 7.12

Let ABCD be a rectangle in which $\angle$ A = 90°.

We have to show that $\angle$ B = $\angle$ C = $\angle$ D = 90°

We have, AD || BC and AB is a transversal (see Fig. 7.12).

So, $\angle$ A + $\angle$ B = 180° (Interior angles on the same side of the transversal)

But, $\angle$ A = 90°

So, $\angle$ B = 180° – $\angle$ A = 180° – 90° = 90°

Now, $\angle$ C = $\angle$ A and $\angle$ D = $\angle$ B

(Opposite angles of the parallellogram)

So, $\angle$ C = 90° and $\angle$ D = 90°.

Therefore, each of the angles of a rectangle is a right angle.

Example 2 : Show that the diagonals of a rhombus are perpendicular to each other.

Solution : Consider the rhombus ABCD (see Fig. 7.13).

You know that AB = BC = CD = DA (Why?)

Now, in Δ AOD and Δ COD,

OA = OC (Diagonals of a parallelogram bisect each other)

OD = OD (Common)

AD = CD

Therefore, Δ AOD ≅ Δ COD

(SSS congruence rule)

This gives, $\angle$ AOD = $\angle$ COD (CPCT)

But, $\angle$ AOD + $\angle$ COD = 180° (Linear pair)

So, 2$\angle$ AOD = 180°

or, $\angle$ AOD = 90°

So, the diagonals of a rhombus are perpendicular to each other.

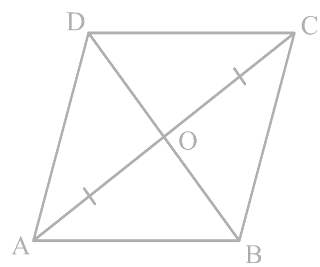


Fig. 7.13

Example 3 : ABC is an isosceles triangle in which AB = AC. AD bisects exterior angle PAC and CD || AB (see Fig. 7.14). Show that

(i) $\angle$ DAC = $\angle$ BCA and (ii) ABCD is a parallelogram.

Solution : (i) Δ ABC is isosceles in which AB = AC (Given)

So, $\angle$ ABC = $\angle$ ACB (Angles opposite to equal sides)

Also, $\angle$ PAC = $\angle$ ABC + $\angle$ ACB

(Exterior angle of a triangle)

or, $\angle$ PAC = 2$\angle$ ACB (1)

Now, AD bisects $\angle$ PAC.

So, $\angle$ PAC = 2$\angle$ DAC (2)

Therefore,

2$\angle$ DAC = 2$\angle$ ACB [From (1) and (2)]

or, $\angle$ DAC = $\angle$ ACB

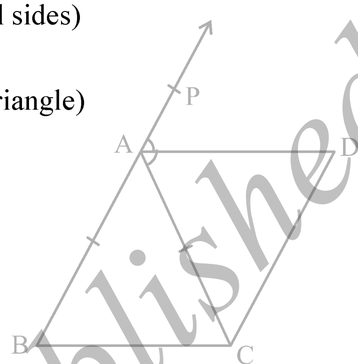


Fig. 7.14

(ii) Now, these equal angles form a pair of alternate angles when line segments BC and AD are intersected by a transversal AC.

So, BC || AD

Also, BA || CD (Given)

Now, both pairs of opposite sides of quadrilateral ABCD are parallel.

So, ABCD is a parallelogram.

Example 4 : Two parallel lines *l* and *m* are intersected by a transversal *p* (see Fig. 7.15). Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

Solution : It is given that PS || QR and transversal *p* intersects them at points A and C respectively.

The bisectors of $\angle$ PAC and $\angle$ ACQ intersect at B and bisectors of $\angle$ ACR and $\angle$ SAC intersect at D.

We are to show that quadrilateral ABCD is a rectangle.

Now, $\angle$ PAC = $\angle$ ACR

(Alternate angles as *l* || *m* and *p* is a transversal)

So, $\frac{1}{2}\angle$ PAC = $\frac{1}{2}\angle$ ACR

i.e., $\angle$ BAC = $\angle$ ACD

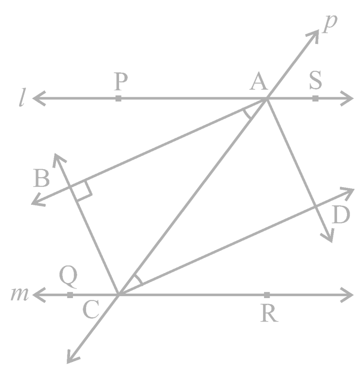


Fig. 7.15

These form a pair of alternate angles for lines AB and DC with AC as transversal and they are equal also.

So, $AB \parallel DC$

Similarly, $BC \parallel AD$ (Considering $\angle$ ACB and $\angle$ CAD)

Therefore, quadrilateral ABCD is a parallelogram.

Also, $\angle PAC + \angle CAS = 180°$ (Linear pair)

So, $\frac{1}{2}\angle PAC + \frac{1}{2}\angle CAS = \frac{1}{2} \times 180° = 90°$

or, $\angle BAC + \angle CAD = 90°$

or, $\angle BAD = 90°$

So, ABCD is a parallelogram in which one angle is 90°.

Therefore, ABCD is a rectangle.

Example 5 : Show that the bisectors of angles of a parallelogram form a rectangle.

Solution : Let P, Q, R and S be the points of intersection of the bisectors of $\angle$ A and $\angle$ B, $\angle$ B and $\angle$ C, $\angle$ C and $\angle$ D, and $\angle$ D and $\angle$ A respectively of parallelogram ABCD (see Fig. 7.16).

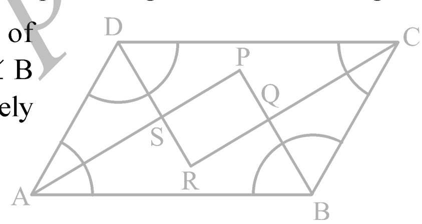


Fig. 7.16

In Δ ASD, what do you observe?

Since DS bisects $\angle$ D and AS bisects $\angle$ A, therefore,

$$\angle DAS + \angle ADS = \frac{1}{2}\angle A + \frac{1}{2}\angle D$$

$$= \frac{1}{2}(\angle A + \angle D)$$

$$= \frac{1}{2} \times 180°$$ ($\angle$ A and $\angle$ D are interior angles on the same side of the transversal)

$$= 90°$$

Also, $\angle DAS + \angle ADS + \angle DSA = 180°$ (Angle sum property of a triangle)

or, $90° + \angle DSA = 180°$

or, $\angle DSA = 90°$

So, $\angle PSR = 90°$ (Being vertically opposite to $\angle$ DSA)

Similarly, it can be shown that $\angle$ APB = 90° or $\angle$ SPQ = 90° (as it was shown for $\angle$ DSA). Similarly, $\angle$ PQR = 90° and $\angle$ SRQ = 90°.

So, PQRS is a quadrilateral in which all angles are right angles.

Can we conclude that it is a rectangle? Let us examine. We have shown that $\angle$ PSR = $\angle$ PQR = 90° and $\angle$ SPQ = $\angle$ SRQ = 90°. So both pairs of opposite angles are equal.

Therefore, PQRS is a parallelogram in which one angle (in fact all angles) is 90° and so, PQRS is a rectangle.

7.5 Another Condition for a Quadrilateral to be a Parallelogram

You have studied many properties of a parallelogram in this chapter and you have also verified that if in a quadrilateral any one of those properties is satisfied, then it becomes a parallelogram.

We now study yet another condition which is the least required condition for a quadrilateral to be a parallelogram.

It is stated in the form of a theorem as given below:

Theorem 7.8 : *A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.*

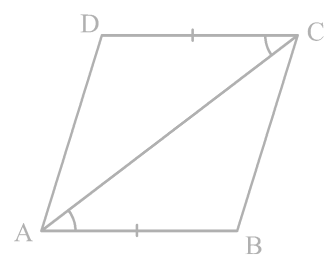


Fig. 7.17

Look at Fig 7.17 in which AB = CD and AB || CD. Let us draw a diagonal AC. You can show that Δ ABC ≅ Δ CDA by SAS congruence rule.

So, BC || AD (Why?)

Let us now take an example to apply this property of a parallelogram.

Example 6 : ABCD is a parallelogram in which P and Q are mid-points of opposite sides AB and CD (see Fig. 7.18). If AQ intersects DP at S and BQ intersects CP at R, show that:

(i) APCQ is a parallelogram.

(ii) DPBQ is a parallelogram.

(iii) PSQR is a parallelogram.

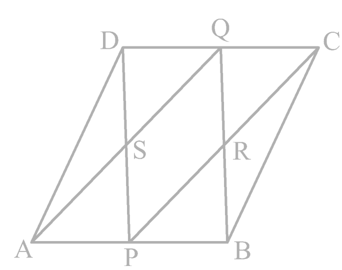


Fig. 7.18

Solution : (i) In quadrilateral APCQ,

$AP \parallel QC$ (Since $AB \parallel CD$) (1)

$AP = \frac{1}{2} AB, \quad CQ = \frac{1}{2} CD$ (Given)

Also, $AB = CD$ (Why?)

So, $AP = QC$ (2)

Therefore, APCQ is a parallelogram [From (1) and (2) and Theorem 7.8]

(ii) Similarly, quadrilateral DPBQ is a parallelogram, because

$DQ \parallel PB$ and $DQ = PB$

(iii) In quadrilateral PSQR,

$SP \parallel QR$ (SP is a part of DP and QR is a part of QB)

Similarly, $SQ \parallel PR$

So, PSQR is a parallelogram.

EXERCISE 7.1

1. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.
2. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
3. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
4. Show that the diagonals of a square are equal and bisect each other at right angles.
5. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then it is a square.
6. Diagonal AC of a parallelogram ABCD bisects $\angle$ A (see Fig. 7.19). Show that
   (i) it bisects $\angle$ C also,
   (ii) ABCD is a rhombus.

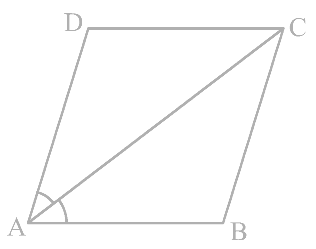


Fig. 7.19

7. ABCD is a rhombus. Show that diagonal AC bisects $\angle$ A as well as $\angle$ C and diagonal BD bisects $\angle$ B as well as $\angle$ D.
8. ABCD is a rectangle in which diagonal AC bisects $\angle$ A as well as $\angle$ C. Show that: (i) ABCD is a square (ii) diagonal BD bisects $\angle$ B as well as $\angle$ D.

9. In parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 7.20). Show that:

(i) Δ APD $\cong \Delta$ CQB

(ii) AP = CQ

(iii) Δ AQB $\cong \Delta$ CPD

(iv) AQ = CP

(v) APCQ is a parallelogram

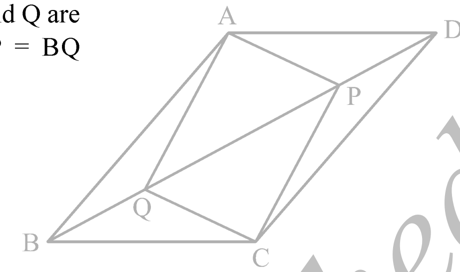


Fig. 7.20

10. ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD (see Fig. 7.21). Show that

(i) Δ APB $\cong \Delta$ CQD

(ii) AP = CQ

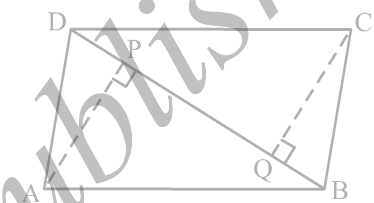


Fig. 7.21

11. In Δ ABC and Δ DEF, AB = DE, AB || DE, BC = EF and BC || EF. Vertices A, B and C are joined to vertices D, E and F respectively (see Fig. 7.22). Show that

(i) quadrilateral ABED is a parallelogram

(ii) quadrilateral BEFC is a parallelogram

(iii) AD || CF and AD = CF

(iv) quadrilateral ACFD is a parallelogram

(v) AC = DF

(vi) Δ ABC $\cong \Delta$ DEF.

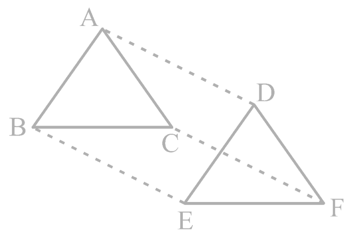


Fig. 7.22

12. ABCD is a trapezium in which AB || CD and AD = BC (see Fig. 7.23). Show that

(i) $\angle$ A = $\angle$ B

(ii) $\angle$ C = $\angle$ D

(iii) Δ ABC $\cong \Delta$ BAD

(iv) diagonal AC = diagonal BD

[*Hint*: Extend AB and draw a line through C parallel to DA intersecting AB produced at E.]

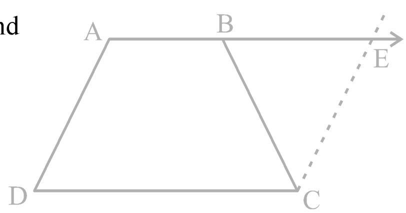


Fig. 7.23

7.6 The Mid-point Theorem

You have studied many properties of a triangle as well as a quadrilateral. Now let us study yet another result which is related to the mid-point of sides of a triangle. Perform the following activity.

Draw a triangle and mark the mid-points E and F of two sides of the triangle. Join the points E and F (see Fig. 7.24).

Measure EF and BC. Measure $\angle$ AEF and $\angle$ ABC.

What do you observe? You will find that :

$$EF = \frac{1}{2} BC \text{ and } \angle AEF = \angle ABC$$

so, EF || BC

Repeat this activity with some more triangles.

So, you arrive at the following theorem:

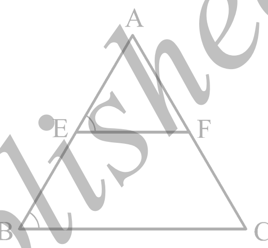


Fig. 7.24

Theorem 7.9 : *The line segment joining the mid-points of two sides of a triangle is parallel to the third side.*

You can prove this theorem using the following clue:

Observe Fig 7.25 in which E and F are mid-points of AB and AC respectively and CD || BA.

Δ AEF $\cong$ Δ CDF (ASA Rule)

So, EF = DF and BE = AE = DC (Why?)

Therefore, BCDE is a parallelogram. (Why?)

This gives EF || BC.

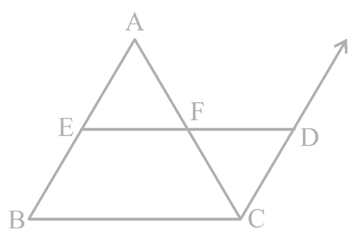


Fig. 7.25

In this case, also note that EF = $\frac{1}{2}$ ED = $\frac{1}{2}$ BC.

Can you state the converse of Theorem 7.9? Is the converse true?

You will see that converse of the above theorem is also true which is stated as below:

Theorem 7.10 : *The line drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.*

In Fig 7.26, observe that E is the mid-point of AB, line l is passsing through E and is parallel to BC and CM || BA.

Prove that AF = CF by using the congruence of Δ AEF and Δ CDF.

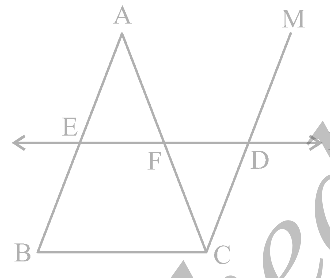


Fig. 7.26

Example 7 : In Δ ABC, D, E and F are respectively the mid-points of sides AB, BC and CA (see Fig. 7.27). Show that Δ ABC is divided into four congruent triangles by joining D, E and F.

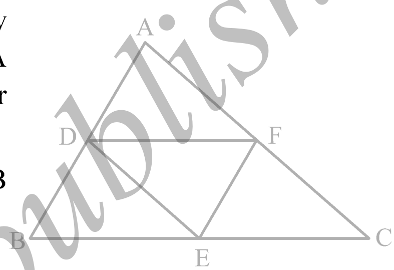


Fig. 7.27

Solution : As D and E are mid-points of sides AB and BC of the triangle ABC, by Theorem 7.9,

DE || AC

Similarly, DF || BC and EF || AB

Therefore ADEF, BDFE and DFCE are all parallelograms.

Now DE is a diagonal of the parallelogram BDFE,

therefore, Δ BDE $\cong$ Δ FED

Similarly Δ DAF $\cong$ Δ FED

and Δ EFC $\cong$ Δ FED

So, all the four triangles are congruent.

Example 8 : l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p (see Fig. 7.28). Show that l, m and n cut off equal intercepts DE and EF on q also.

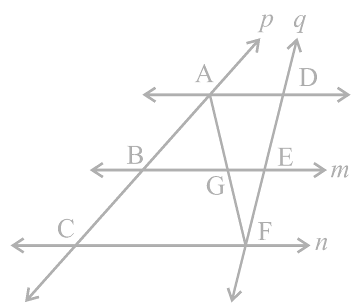


Fig. 7.28

Solution : We are given that AB = BC and have to prove that DE = EF.

Let us join A to F intersecting m at G..

The trapezium ACFD is divided into two triangles;

namely Δ ACF and Δ AFD.

In Δ ACF, it is given that B is the mid-point of AC (AB = BC)

and BG || CF (since *m* || *n*).

So, G is the mid-point of AF (by using Theorem 7.10)

Now, in Δ AFD, we can apply the same argument as G is the mid-point of AF, GE || AD and so by Theorem 7.10, E is the mid-point of DF,

i.e., DE = EF.

In other words, *l*, *m* and *n* cut off equal intercepts on *q* also.

EXERCISE 7.2

1. ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA (see Fig 7.29). AC is a diagonal. Show that :

(i) SR || AC and SR = $\frac{1}{2}$ AC

(ii) PQ = SR

(iii) PQRS is a parallelogram.

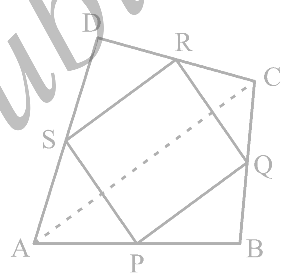


Fig. 7.29

2. ABCD is a rhombus and P, Q, R and S are ©wthe mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rectangle.

3. ABCD is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rhombus.

4. ABCD is a trapezium in which AB || DC, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (see Fig. 7.30). Show that F is the mid-point of BC.

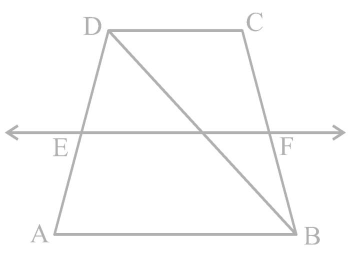


Fig. 7.30

5. In a parallelogram ABCD, E and F are the mid-points of sides AB and CD respectively (see Fig. 7.31). Show that the line segments AF and EC trisect the diagonal BD.

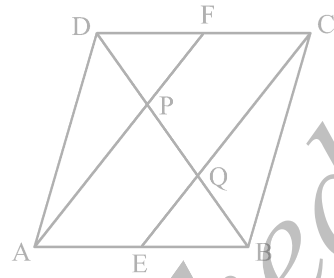


Fig. 7.31

6. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.
7. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that
   (i) D is the mid-point of AC (ii) $MD \perp AC$
   (iii) $CM = MA = \frac{1}{2} AB$

7.7 Summary

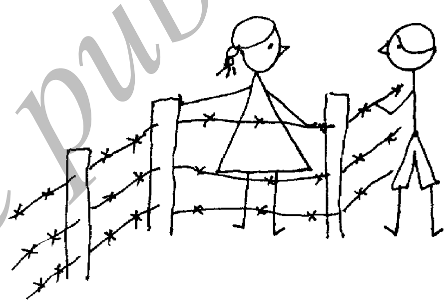
In this chapter, you have studied the following points :

1. Sum of the angles of a quadrilateral is 360°.
2. A diagonal of a parallelogram divides it into two congruent triangles.
3. In a parallelogram,
   (i) opposite sides are equal (ii) opposite angles are equal
   (iii) diagonals bisect each other
4. A quadrilateral is a parallelogram, if
   (i) opposite sides are equal or (ii) opposite angles are equal
   or (iii) diagonals bisect each other
   or (iv)a pair of opposite sides is equal and parallel
5. Diagonals of a rectangle bisect each other and are equal and vice-versa.
6. Diagonals of a rhombus bisect each other at right angles and vice-versa.
7. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
8. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
9. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.
10. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

APPENDIX 1

PROOFS IN MATHEMATICS

A1.1 Introduction



Suppose your family owns a plot of land and there is no fencing around it. Your neighbour decides one day to fence off his land. After he has fenced his land, you discover that a part of your family's land has been enclosed by his fence. How will you prove to your neighbour that he has tried to encroach on your land? Your first step may be to seek the help of the village elders to sort out the difference in boundaries. But, suppose opinion is divided among the elders. Some feel you are right and others feel your neighbour is right. What can you do? Your only option is to find a way of establishing your claim for the boundaries of your land that is acceptable to all. For example, a government approved survey map of your village can be used, if necessary in a court of law, to prove (claim) that you are correct and your neighbour is wrong.

Let us look at another situation. Suppose your mother has paid the electricity bill of your house for the month of August, 2005. The bill for September, 2005, however, claims that the bill for August has not been paid. How will you disprove the claim made by the electricity department? You will have to produce a receipt proving that your August bill has been paid.

You have just seen some examples that show that in our daily life we are often called upon to prove that a certain statement or claim is true or false. However, we also accept many statements without bothering to prove them. But, in mathematics we only accept a statement as true or false (except for some axioms) if it has been proved to be so, according to the logic of mathematics.

In fact, proofs in mathematics have been in existence for thousands of years, and they are central to any branch of mathematics. The first known proof is believed to have been given by the Greek philosopher and mathematician Thales. While mathematics was central to many ancient civilisations like Mesopotamia, Egypt, China and India, there is no clear evidence that they used proofs the way we do today.

In this chapter, we will look at what a statement is, what kind of reasoning is involved in mathematics, and what a mathematical proof consists of.

A1.2 Mathematically Acceptable Statements

In this section, we shall try to explain the meaning of a mathematically acceptable statement. A 'statement' is a sentence which is not an order or an exclamatory sentence. And, of course, a statement is not a question! For example,

"What is the colour of your hair?" is not a statement, it is a question.

"Please go and bring me some water." is a request or an order, not a statement.

"What a marvellous sunset!" is an exclamatory remark, not a statement.

However, "The colour of your hair is black" is a statement.

In general, statements can be one of the following:

- *always true*
- *always false*
- *ambiguous*

The word 'ambiguous' needs some explanation. There are two situations which make a statement ambiguous. The first situation is when we cannot decide if the statement is always true or always false. For example, "Tomorrow is Thursday" is ambiguous, since enough of a context is not given to us to decide if the statement is true or false.

The second situation leading to ambiguity is when the statement is subjective, that is, it is true for some people and not true for others. For example, "Dogs are intelligent" is ambiguous because some people believe this is true and others do not.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 8 days in a week.

(ii) It is raining here.

(iii) The sun sets in the west.

(iv) Gauri is a kind girl.

(v) The product of two odd integers is even.

(vi) The product of two even natural numbers is even.

Solution :

(i) This statement is always false, since there are 7 days in a week.

(ii) This statement is ambiguous, since it is not clear where 'here' is.

(iii) This statement is always true. The sun sets in the west no matter where we live.

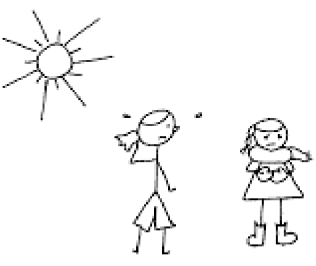
(iv) This statement is ambiguous, since it is subjective–Gauri may be kind to some and not to others.

(v) This statement is always false. The product of two odd integers is always odd.

(vi) This statement is always true. However, to justify that it is true we need to do some work. It will be proved in Section A1.4.

As mentioned before, in our daily life, we are not so careful about the validity of statements. For example, suppose your friend tells you that in July it rains everyday in Manantavadi, Kerala. In all probability, you will believe her, even though it may not have rained for a day or two in July. Unless you are a lawyer, you will not argue with her!

As another example, consider statements we often make to each other like "it is very hot today". We easily accept such statements because we know the context even though these statements are ambiguous. 'It is very hot today' can mean different things to different people because what is very hot for a person from Kumaon may not be hot for a person from Chennai.



But a mathematical statement cannot be ambiguous. *In mathematics, a statement is only acceptable or valid, if it is either true or false.* We say that a statement is true, if it is always true otherwise it is called a false statement.

For example, $5 + 2 = 7$ is always true, so '$5 + 2 = 7$' is a true statement and $5 + 3 = 7$ is a false statement.

Example 2 : State whether the following statements are true or false:

(i) The sum of the interior angles of a triangle is 180°.

(ii) Every odd number greater than 1 is prime.

(iii) For any real number x, $4x + x = 5x$.

(iv) For every real number x, $2x > x$.

(v) For every real number x, $x^2 \geq x$.

(vi) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) This statement is true. You have already proved this in Chapter 6.

(ii) This statement is false; for example, 9 is not a prime number.

(iii) This statement is true.

(iv) This statement is false; for example, $2 \times (-1) = -2$, and -2 is not greater than -1.

(v) This statement is false; for example, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{4}$ is not greater than $\frac{1}{2}$.

(vi) This statement is false, since a rhombus has equal sides but need not be a square.

You might have noticed that to establish that a statement is not true according to mathematics, all we need to do is to find one case or example where it breaks down. So in (ii), since 9 is not a prime, it is an example that shows that the statement "Every odd number greater than 1 is prime" is not true. Such an example, that counters a statement, is called a *counter-example.* We shall discuss counter-examples in greater detail in Section A1.5.

You might have also noticed that while Statements (iv), (v) and (vi) are false, they can be restated with some conditions in order to make them true.

Example 3 : Restate the following statements with appropriate conditions, so that they become true statements.

(i) For every real number x, $2x > x$.

(ii) For every real number x, $x^2 \geq x$.

(iii) If you divide a number by itself, you will always get 1.

(iv) The angle subtended by a chord of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) If $x > 0$, then $2x > x$.

(ii) If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

(iii) If you divide a number except zero by itself, you will always get 1.

(iv) The angle subtended by a diameter of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides and interior angles equal, then it is a square.

EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.
   (i) There are 13 months in a year.
   (ii) Diwali falls on a Friday.
   (iii) The temperature in Magadi is 26^0 C.
   (iv) The earth has one moon.
   (v) Dogs can fly.
   (vi) February has only 28 days.
2. State whether the following statements are true or false. Give reasons for your answers.
   (i) The sum of the interior angles of a quadrilateral is 350°.
   (ii) For any real number x, $x^2 \geq 0$.
   (iii) A rhombus is a parallelogram.
   (iv) The sum of two even numbers is even.
   (v) The sum of two odd numbers is odd.
3. Restate the following statements with appropriate conditions, so that they become true statements.
   (i) All prime numbers are odd.
   (ii) Two times a real number is always even.
   (iii) For any x, $3x + 1 > 4$.
   (iv) For any x, $x^3 \geq 0$.
   (v) In every triangle, a median is also an angle bisector.

A1.3 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning.* To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.

You are given four cards. Each card has a number printed on one side and a letter on the other side.

Suppose you are told that these cards follow the rule:

"If a card has an even number on one side, then it has a vowel on the other side."

What is the **smallest number** of cards you need to turn over to check if the rule is true?

Of course, you have the option of turning over all the cards and checking. But can you manage with turning over a fewer number of cards?

Notice that the statement mentions that a card with an even number on one side has a vowel on the other. It does not state that a card with a vowel on one side must have an even number on the other side. That may or may not be so. The rule also does not state that a card with an odd number on one side must have a consonant on the other side. It may or may not.

So, do we need to turn over 'A'? No! Whether there is an even number or an odd number on the other side, the rule still holds.

What about '5'? Again we do not need to turn it over, because whether there is a vowel or a consonant on the other side, the rule still holds.

But you do need to turn over V and 6. If V has an even number on the other side, then the rule has been broken. Similarly, if 6 has a consonant on the other side, then the rule has been broken.

The kind of reasoning we have used to solve this puzzle is called **deductive reasoning**. It is called 'deductive' because we arrive at (i.e., deduce or infer) a result or a statement from a previously established statement using logic. For example, in the puzzle above, by a series of logical arguments we deduced that we need to turn over only V and 6.

Deductive reasoning also helps us to conclude that a particular statement is true, because it is a special case of a more general statement that is known to be true. For example, once we prove that the product of two odd numbers is always odd, we can immediately conclude (without computation) that 70001×134563 is odd simply because 70001 and 134563 are odd.

Deductive reasoning has been a part of human thinking for centuries, and is used all the time in our daily life. For example, suppose the statements "The flower Solaris blooms, only if the maximum temperature is above 28° C on the previous day" and "Solaris bloomed in Imaginary Valley on 15th September, 2005" are true. Then using deductive reasoning, we can conclude that the maximum temperature in Imaginary Valley on 14th September, 2005 was more than 28° C.

Unfortunately we do not always use correct reasoning in our daily life! We often come to many conclusions based on faulty reasoning. For example, if your friend does not smile at you one day, then you may conclude that she is angry with you. While it may be true that "if she is angry with me, she will not smile at me", it may also be true that "if she has a bad headache, she will not smile at me". Why don't you examine some conclusions that you have arrived at in your day-to-day existence, and see if they are based on valid or faulty reasoning?

EXERCISE A1.2

1. Use deductive reasoning to answer the following:

(i) Humans are mammals. All mammals are vertebrates. Based on these two statements, what can you conclude about humans?

(ii) Anthony is a barber. Dinesh had his hair cut. Can you conclude that Antony cut Dinesh's hair?

(iii) Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?

(iv) If it rains for more than four hours on a particular day, the gutters will have to be cleaned the next day. It has rained for 6 hours today. What can we conclude about the condition of the gutters tomorrow?

(v) What is the fallacy in the cow's reasoning in the cartoon below?

2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

   "If a card has a consonant on one side, then it has an odd number on the other side."

   B **3** **U** **8**

A1.4 Theorems, Conjectures and Axioms

So far we have discussed statements and how to check their validity. In this section, you will study how to distinguish between the three different kinds of statements mathematics is built up from, namely, a theorem, a conjecture and an axiom.

You have already come across many theorems before. So, what is a theorem? A mathematical statement whose truth has been established (proved) is called a *theorem*. For example, the following statements are theorems, as you will see in Section A1.5.

Theorem A1.1 : *The sum of the interior angles of a triangle is* $180°$.

Theorem A1.2 : *The product of two even natural numbers is even.*

Theorem A1.3 : *The product of any three consecutive even natural numbers is divisible by* 16.

A *conjecture* is a statement which we believe is true, based on our mathematical understanding and experience, that is, our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses. Let us look at some patterns and see what kind of intelligent guesses we can make.

Example 4 : Take any three consecutive even numbers and add them, say,

$2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$, $20 + 22 + 24 = 66$.

Is there any pattern you can guess in these sums? What can you conjecture about them?

Solution : One conjecture could be :

(i) the sum of three consecutive even numbers is even.

Another could be :

(ii) the sum of three consecutive even numbers is divisible by 6.

Example 5 : Consider the following pattern of numbers called the Pascal's Triangle:

Line												Sum of numbers
1						1						1
2					1		1					2
3				1		2		1				4
4			1		3		3		1			8
5		1		4		6		4		1		16
6	1		5		10		10		5		1	32
7			:						:			:
8			:						:			:

What can you conjecture about the sum of the numbers in Lines 7 and 8? What about the sum of the numbers in Line 21? Do you see a pattern? Make a guess about a formula for the sum of the numbers in line *n*.

Solution : Sum of the numbers in Line 7 = $2 \times 32 = 64 = 2^6$

Sum of the numbers in Line 8 = $2 \times 64 = 128 = 2^7$

Sum of the numbers in Line 21 = 2^{20}

Sum of the numbers in Line $n = 2^{n-1}$

Example 6 : Consider the so-called triangular numbers T_n:

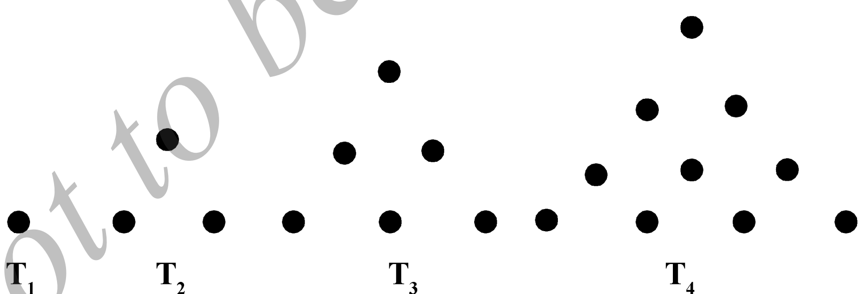
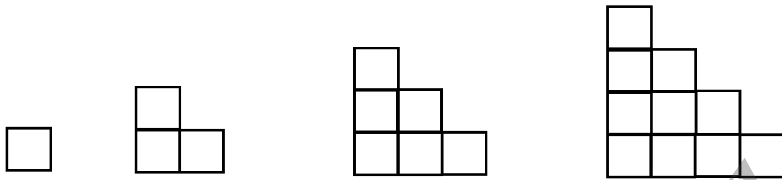


Fig. A1.1

The dots here are arranged in such a way that they form a triangle. Here $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, and so on. Can you guess what T_5 is? What about T_6? What about T_n?

Make a conjecture about T_n.

It might help if you redraw them in the following way.



T_1 T_2 T_3 T_4

Fig. A1.2

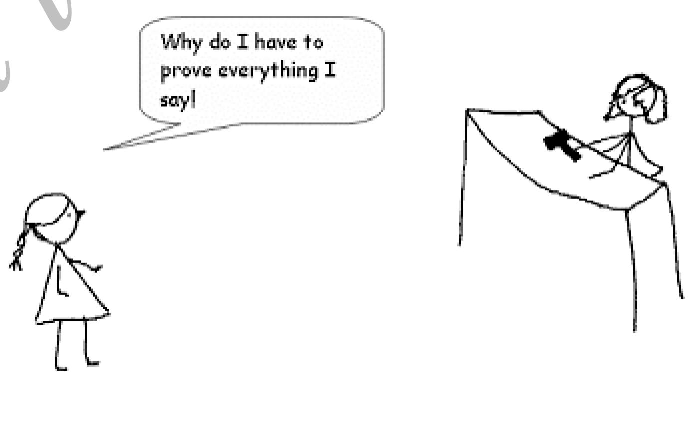
Solution : $T_5 = 1 + 2 + 3 + 4 + 5 = 15 = \dfrac{5 \times 6}{2}$

$$T_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \times 7}{2}$$

$$T_n = \frac{n \times (n+1)}{2}$$

A favourite example of a conjecture that has been open (that is, it has not been proved to be true or false) is the Goldbach conjecture named after the mathematician Christian Goldbach (1690 – 1764). This conjecture states that "*every even integer greater than* 4 *can be expressed as the sum of two odd primes*." Perhaps you will prove that this result is either true or false, and will become famous!

You might have wondered – do we need to prove everything we encounter in mathematics, and if not, why not?



The fact is that every area in mathematics is based on some statements which are assumed to be true and are not proved. These are 'self-evident truths' which we take to be true without proof. These statements are called *axioms*. In Chapter 5, you would have studied the axioms and postulates of Euclid. (We do not distinguish between axioms and postulates these days.)

For example, the first postulate of Euclid states:

A straight line may be drawn from any point to any other point.

And the third postulate states:

A circle may be drawn with any centre and any radius.

These statements appear to be perfectly true and Euclid assumed them to be true. Why? This is because we cannot prove everything and we need to start somewhere. We need some statements which we accept as true and then we can build up our knowledge using the rules of logic based on these axioms.

You might then wonder why we don't just accept all statements to be true when they appear self-evident. There are many reasons for this. Very often our intuition can be wrong, pictures or patterns can deceive and the only way to be sure that something is true is to prove it. For example, many of us believe that if a number is multiplied by another, the result will be larger than both the numbers. But we know that this is not always true: for example, $5 \times 0.2 = 1$, which is less than 5.

Also, look at the Fig. A1.3. Which line segment is longer, AB or CD?

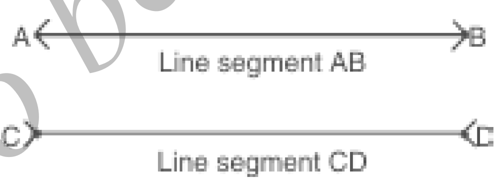


Fig. A1.3

It turns out that both are of exactly the same length, even though AB appears shorter!

You might wonder then, about the validity of axioms. Axioms have been chosen based on our intuition and what appears to be self-evident. Therefore, we expect them to be true. However, it is possible that later on we discover that a particular axiom is not true. What is a safeguard against this possibility? We take the following steps:

(i) Keep the axioms to the bare minimum. For instance, based on only axioms and five postulates of Euclid, we can derive hundreds of theorems.

(ii) Make sure the axioms are consistent.

We say a collection of axioms is *inconsistent,* if we can use one axiom to show that another axiom is not true. For example, consider the following two statements. We will show that they are inconsistent.

Statement1: No whole number is equal to its successor.

Statement 2: A whole number divided by zero is a whole number.

(Remember, **division by zero is not defined**. But just for the moment, we assume that it is possible, and see what happens.)

From Statement 2, we get $\frac{1}{0} = a$, where a is some whole number. This implies that, $1 = 0$. But this disproves Statement 1, which states that no whole number is equal to its successor.

(iii) A false axiom will, sooner or later, result in a contradiction. We say that *there is a contradiction, when we find a statement such that, both the statement and its negation are true.* For example, consider Statement 1 and Statement 2 above once again.

From Statement 1, we can derive the result that $2 \neq 1$.

Now look at $x^2 - x^2$. We will factorise it in two different ways as follows:

(i) $x^2 - x^2 = x(x - x)$ and

(ii) $x^2 - x^2 = (x + x)(x - x)$

So, $x(x - x) = (x + x)(x - x)$.

From Statement 2, we can cancel $(x - x)$ from both sides.

We get $x = 2x$, which in turn implies $2 = 1$.

So we have both the statement $2 \neq 1$ and its negation, $2 = 1$, true. This is a contradiction. The contradiction arose because of the false axiom, that a whole number divided by zero is a whole number.

So, the statements we choose as axioms require a lot of thought and insight. We must make sure they do not lead to inconsistencies or logical contradictions. Moreover, the choice of axioms themselves, sometimes leads us to new discoveries. From Chapter 5, you are familiar with Euclid's fifth postulate and the discoveries of non-Euclidean geometries. You saw that mathematicians believed that the fifth postulate need not be a postulate and is actually a theorem that can be proved using just the first four postulates. Amazingly these attempts led to the discovery of non-Euclidean geometries.

We end the section by recalling the differences between an axiom, a theorem and a conjecture. An **axiom** is a mathematical statement which is assumed to be true without proof; a **conjecture** is a mathematical statement whose truth or falsity is yet to be established; and a **theorem** is a mathematical statement whose truth has been logically established.

EXERCISE A1.3

1. Take any three consecutive even numbers and find their product; for example, $2 \times 4 \times 6 = 48$, $4 \times 6 \times 8 = 192$, and so on. Make three conjectures about these products.

2. Go back to Pascal's triangle.

Line 1 : $1 = 11^0$

Line 2 : $1 \ 1 = 11^1$

Line 3 : $1 \ 2 \ 1 = 11^2$

Make a conjecture about Line 4 and Line 5. Does your conjecture hold? Does your conjecture hold for Line 6 too?

3. Let us look at the triangular numbers (see Fig.A1.2) again. Add two consecutive triangular numbers. For example, $T_1 + T_2 = 4$, $T_2 + T_3 = 9$, $T_3 + T_4 = 16$.

What about $T_4 + T_5$? Make a conjecture about $T_{n-1} + T_n$.

4. Look at the following pattern:

$1^2 = 1$

$11^2 = 121$

$111^2 = 12321$

$1111^2 = 1234321$

$11111^2 = 123454321$

Make a conjecture about each of the following:

$111111^2 =$

$1111111^2 =$

Check if your conjecture is true.

5. List five axioms (postulates) used in this book.

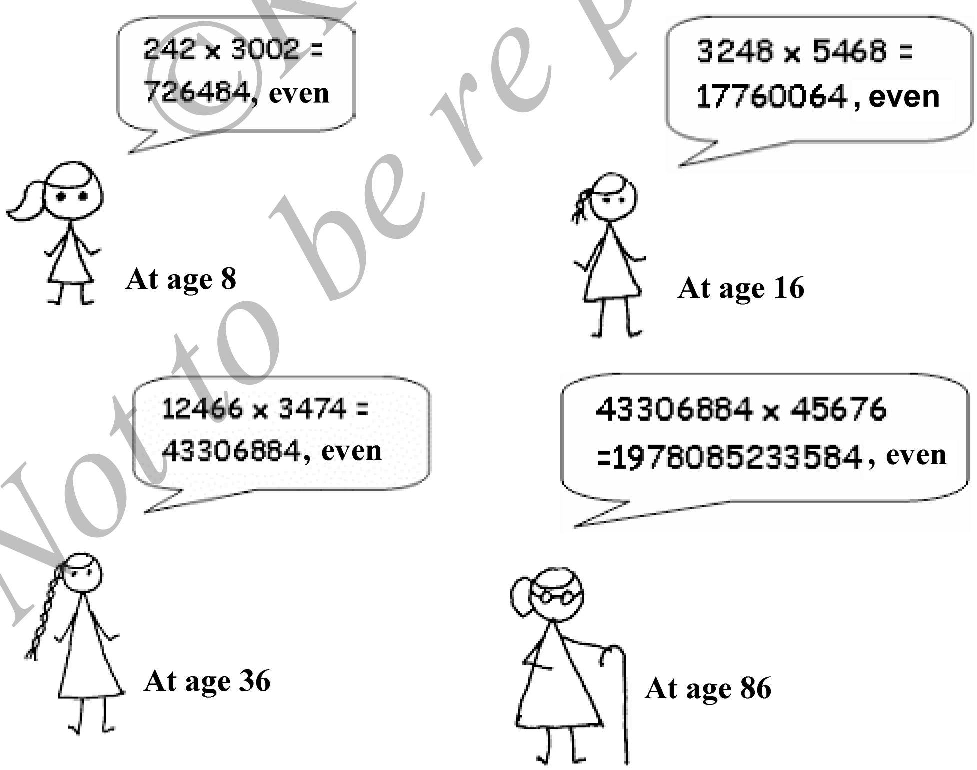
A1.5 What is a Mathematical Proof?

Let us now look at various aspects of proofs. We start with understanding the difference between verification and proof. Before you studied proofs in mathematics, you were mainly asked to verify statements.

For example, you might have been asked to verify with examples that "the product of two even numbers is even". So you might have picked up two random even numbers, say 24 and 2006, and checked that 24 × 2006 = 48144 is even. You might have done so for many more examples.

Also, you might have been asked as an activity to draw several triangles in the class and compute the sum of their interior angles. Apart from errors due to measurement, you would have found that the interior angles of a triangle add up to 180°.

What is the flaw in this method? There are several problems with the process of verification. While it may help you to make a statement you believe is true, you cannot be *sure* that it is true in *all* cases. For example, the multiplication of several pairs of even numbers may lead us to guess that the product of two even numbers is even. However, it does not ensure that the product of all pairs of even numbers is even. You cannot physically check the products of all possible pairs of even numbers. If you did, then like the girl in the cartoon, you will be calculating the products of even numbers for the rest of your life. Similarly, there may be some triangles which you have not yet drawn whose interior angles do not add up to 180°. We cannot measure the interior angles of all possible triangles.



Moreover, verification can often be misleading. For example, we might be tempted to conclude from Pascal's triangle (Q.2 of Exercise A1.3), based on earlier verifications, that $11^5 = 15101051$. But in fact $11^5 = 161051$.

So, you need another approach that does not depend upon verification for some cases only. There is another approach, namely 'proving a statement'. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a *mathematical proof.*

In Example 2 of Section A1.2, you saw that to establish that a mathematical statement is false, it is enough to produce a single counter-example. So while it is not enough to establish the validity of a mathematical statement by checking or verifying it for thousands of cases, it is enough to produce one counter-example to *disprove* a statement (i.e., to show that something is false). This point is worth emphasising.

To show that a mathematical statement is false, it is enough to find a single counter-example.

So, 7 + 5 = 12 is a counter-example to the statement that the sum of two odd numbers is odd.

Let us now look at the list of basic ingredients in a proof:

(i) To prove a theorem, we should have a rough idea as to how to proceed.

(ii) The information already given to us in a theorem (i.e., the hypothesis) has to be clearly understood and used.

For example, in Theorem A1.2, which states that the product of two even numbers is even, we are given two even natural numbers. So, we should use their properties. In the Factor Theorem (in Chapter 2), you are given a polynomial $p(x)$ and are told that $p(a) = 0$. You have to use this to show that $(x - a)$ is a factor of $p(x)$. Similarly, for the converse of the Factor Theorem, you are given that $(x - a)$ is a factor of $p(x)$, and you have to use this hypothesis to prove that $p(a) =0$.

You can also use constructions during the process of proving a theorem. For example, to prove that the sum of the angles of a triangle is 180°, we draw a line parallel to one of the sides through the vertex opposite to the side, and use properties of parallel lines.

(iii) A proof is made up of a successive sequence of mathematical statements. Each statement in a proof is logically deduced from a previous statement in the proof, or from a theorem proved earlier, or an axiom, or our hypothesis.

(iv) The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

To understand these ingredients, we will analyse Theorem A1.1 and its proof. You have already studied this theorem in Chapter 6. But first, a few comments on proofs in geometry. We often resort to diagrams to help us prove theorems, and this is very important. However, each statement in the proof has to be established **using only logic**. Very often, we hear students make statements like "Those two angles are equal because in the drawing they look equal" or "that angle must be 90^0, because the two lines look as if they are perpendicular to each other". Beware of being deceived by what you see (remember Fig A1.3)! .

So now let us go to Theorem A1.1.

Theorem A1.1 : *The sum of the interior angles of a triangle is* 180°.

Proof : Consider a triangle ABC (see Fig. A1.4).

We have to prove that $\angle$ ABC + $\angle$ BCA + $\angle$ CAB = 180° (1)

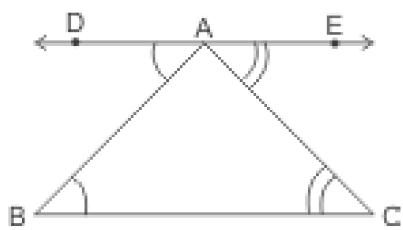


Fig A 1.4

Construct a line DE parallel to BC passing through A. (2)

DE is parallel to BC and AB is a transversal.

So, $\angle$ DAB and $\angle$ ABC are alternate angles. Therefore, by Theorem 6.2, Chapter 6, they are equal, i.e. $\angle$ DAB = $\angle$ ABC (3)

Similarly, $\angle$ CAE = $\angle$ ACB (4)

Therefore, $\angle$ ABC + $\angle$ BAC+ $\angle$ ACB = $\angle$ DAB +$\angle$ BAC + $\angle$ CAE (5)

But $\angle$ DAB +$\angle$ BAC + $\angle$ CAE = 180°, since they form a straight angle. (6)

Hence, $\angle$ ABC + $\angle$ BAC+ $\angle$ ACB = 180°. ■ (7)

Now, we comment on each step of the proof.

Step 1 : Our theorem is concerned with a property of triangles, so we begin with a triangle.

Step 2 : This is the key idea – the intuitive leap or understanding of how to proceed so as to be able to prove the theorem. Very often geometric proofs require a construction.

Steps 3 and 4 : Here we conclude that $\angle$ DAE = $\angle$ ABC and $\angle$ CAE = $\angle$ ACB, by using the fact that DE is parallel to BC (our construction), and the previously proved Theorem 6.2, which states that if two parallel lines are intersected by a transversal, then the alternate angles are equal.

Step 5 : Here we use Euclid's axiom (see Chapter 5) which states that: "If equals are added to equals, the wholes are equal" to deduce

$\angle$ ABC + $\angle$ BAC+ $\angle$ ACB = $\angle$ DAB +$\angle$ BAC + $\angle$ CAE.

That is, the sum of the interior angles of the triangle are equal to the sum of the angles on a straight line.

Step 6 : Here we use the Linear pair axiom of Chapter 6, which states that the angles on a straight line add up to 180°, to show that $\angle$ DAB +$\angle$ BAC + $\angle$ CAE = 180°.

Step 7 : We use Euclid's axiom which states that "things which are equal to the same thing are equal to each other" to conclude that $\angle$ ABC + $\angle$ BAC + $\angle$ACB = $\angle$ DAB +$\angle$ BAC + $\angle$ CAE = 180°. Notice that Step 7 is the claim made in the theorem we set out to prove.

We now prove Theorems A1.2 and A1.3 without analysing them.

Theorem A1.2 : *The product of two even natural numbers is even.*

Proof : Let x and y be any two even natural numbers.

We want to prove that xy is even.

Since x and y are even, they are divisible by 2 and can be expressed in the form

$x = 2m$, for some natural number m and $y = 2n$, for some natural number n.

Then $xy = 4\,mn$. Since $4\,mn$ is divisible by 2, so is xy.

Therefore, xy is even. ■

Theorem A1.3 : *The product of any three consecutive even natural numbers is divisible by* 16.

Proof : Any three consecutive even numbers will be of the form $2n$, $2n + 2$ and $2n + 4$, for some natural number n. We need to prove that their product $2n(2n + 2)(2n + 4)$ is divisible by 16.

Now, $2n(2n + 2)(2n + 4) = 2n \times 2(n +1) \times 2(n + 2)$

$= 2 \times 2 \times 2n(n + 1)(n + 2) = 8n(n + 1)(n + 2)$.

Now we have two cases. Either n is even or odd. Let us examine each case.

Suppose n is even : Then we can write $n = 2m$, for some natural number m.

And, then $2n(2n + 2)(2n + 4) = 8n(n + 1)(n + 2) = 16m(2m + 1)(2m + 2)$.

Therefore, $2n(2n + 2)(2n + 4)$ is divisible by 16.

Next, suppose n is odd. Then $n + 1$ is even and we can write $n + 1 = 2r$, for some natural number r.

We then have :

$$\begin{aligned} 2n(2n + 2)(2n + 4) &= 8n(n + 1)(n + 2) \\ &= 8(2r - 1) \times 2r \times (2r + 1) \\ &= 16r(2r - 1)(2r + 1) \end{aligned}$$

Therefore, $2n(2n + 2)(2n + 4)$ is divisible by 16.

So, in both cases we have shown that the product of any three consecutive even numbers is divisible by 16. ■

We conclude this chapter with a few remarks on the difference between how mathematicians discover results and how formal rigorous proofs are written down. As mentioned above, each proof has a key intuitive idea (sometimes more than one). Intuition is central to a mathematician's way of thinking and discovering results. Very often the proof of a theorem comes to a mathematician all jumbled up. A mathematician will often experiment with several routes of thought, and logic, and examples, before she/he can hit upon the correct solution or proof. It is only after the creative phase subsides that all the arguments are gathered together to form a proper proof.

It is worth mentioning here that the great Indian mathematician Srinivasa Ramanujan used very high levels of intuition to arrive at many of his statements, which he claimed were true. Many of these have turned out be true and are well known theorems. However, even to this day mathematicians all over the world are struggling to prove (or disprove) some of his claims (conjectures).

Srinivasa Ramanujan (1887–1920)
Fig. A1.5

EXERCISE A1.4

1. Find counter-examples to disprove the following statements:

(i) If the corresponding angles in two triangles are equal, then the triangles are congruent.

(ii) A quadrilateral with all sides equal is a square.

(iii) A quadrilateral with all angles equal is a square.

(iv) For integers a and b, $\sqrt{a^2 + b^2} = a + b$

(v) $2n^2 + 11$ is a prime for all whole numbers n.

(vi) $n^2 - n + 41$ is a prime for all positive integers n.

2. Take your favourite proof and analyse it step-by-step along the lines discussed in Section A1.5 (what is given, what has been proved, what theorems and axioms have been used, and so on).

3. Prove that the sum of two odd numbers is even.

4. Prove that the product of two odd numbers is odd.

5. Prove that the sum of three consecutive even numbers is divisible by 6.

6. Prove that infinitely many points lie on the line whose equation is $y = 2x$.

(*Hint* : Consider the point (n, $2n$) for any integer n.)

7. You must have had a friend who must have told you to think of a number and do various things to it, and then without knowing your original number, telling you what number you ended up with. Here are two examples. Examine why they work.

(i) Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.

(ii) Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11 and 13.

A1.6 Summary

In this Appendix, you have studied the following points:

1. In mathematics, a statement is only acceptable if it is either always true or always false.
2. To show that a mathematical statement is false, it is enough to find a single counter-example.
3. Axioms are statements which are assumed to be true without proof.
4. A conjecture is a statement we believe is true based on our mathematical intuition, but which we are yet to prove.
5. A mathematical statement whose truth has been established (or proved) is called a theorem.
6. The main logical tool in proving mathematical statements is deductive reasoning.
7. A proof is made up of a successive sequence of mathematical statements. Each statement in a proof is logically deduced from a previouly known statement, or from a theorem proved earlier, or an axiom, or the hypothesis.

ANSWERS/HINTS

EXERCISE 1.1

1. Yes. $0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3}$ etc., denominator q can also be taken as negative integer.

2. There can be infinitely many rationals betwen numbers 3 and 4, one way is to take them

$3 = \frac{21}{6+1}, 4 = \frac{28}{6+1}$. Then the six numbers are $\frac{22}{7}, \frac{23}{7}, \frac{24}{7}, \frac{25}{7}, \frac{26}{7}, \frac{27}{7}$.

3. $\frac{3}{5} = \frac{30}{50}, \frac{4}{5} = \frac{40}{50}$. Therefore, five rationals are : $\frac{31}{50}, \frac{32}{50}, \frac{33}{50}, \frac{34}{50}, \frac{35}{50}$.

4. (i) True, since the collection of whole numbers contains all the natural numbers.

(ii) False, for example -2 is not a whole number.

(iii) False, for example $\frac{1}{2}$ is a rational number but not a whole number.

EXERCISE 1.2

1. (i) True, since collection of real numbers is made up of rational and irrational numbers.

(ii) False, 'm' not only be a natural number, it may be a possitive rational.

(iii) False, for example 2 is real but not irrational.

2. No. For example, $\sqrt{4} = 2$ is a rational number.

3. Repeat the procedure as in Fig. 1.8 several times. First obtain $\sqrt{4}$ and then $\sqrt{5}$.

EXERCISE 1.3

1. (i) 0.36, terminating. (ii) $0.\overline{09}$, non-terminating repeating.

   (iii) 4.125, terminating. (iv) $0.\overline{230769}$, non-terminating repeating.

   (v) $0.\overline{18}$ non-terminating repeating. (vi) 0.8225 terminating.

2. $\frac{2}{7} = 2 \times \frac{1}{7} = 0.\overline{285714}$, $\frac{3}{7} = 3 \times \frac{1}{7} = 0.\overline{428571}$,

   $\frac{4}{7} = 4 \times \frac{1}{7} = 0.\overline{571428}$, $\frac{5}{7} = 5 \times \frac{1}{7} = 0.\overline{714285}$,

   $\frac{6}{7} = 6 \times \frac{1}{7} = 0.\overline{857142}$

3. (i) $\frac{2}{3}$ [Let $x = 0.666\ldots$ So $10x = 6.666\ldots$ or, $10x = 6 + x$ or, $x = \frac{6}{9} = \frac{2}{3}$]

   (ii) $\frac{43}{90}$ (iii) $\frac{1}{999}$

4. 1 [Let $x = 0.9999\ldots$ So $10\,x = 9.999\ldots$ or, $10\,x = 9 + x$ or, $x = 1$]

5. $0.\overline{0588235294117647}$

6. The prime factorisation of q has only powers of 2 or powers of 5 or both.

7. 0.01001000100001…, 0.202002000200002…, 0.003000300003…

8. 0.75075007500075000075…, 0.767076700767000767…, 0.808008000800008…

9. (i) and (v) irrational; (ii), (iii) and (iv) rational.

EXERCISE 1.4

1. Proceed as in Section 1.4 for 2.665.
2. Proceed as in Example 11.

EXERCISE 1.5

1. (i) Irrational (ii) Rational

   (iii) Rational (iv) Irrational

   (v) Irrational

2. (i) $6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}$ (ii) 6

(iii) $7 + 2\sqrt{10}$ (iv) 3

3. There is no contradiction. Remember that when you measure a length with a scale or any other device, you only get an approximate rational value. So, you may not realise that either *c* or *d* is irrational.

4. Refer Fig. 1.17.

5. (i) $\frac{\sqrt{7}}{7}$ (ii) $\sqrt{7} + \sqrt{6}$

(iii) $\frac{\sqrt{5} - \sqrt{2}}{3}$ (iv) $\frac{\sqrt{7} + 2}{3}$

EXERCISE 1.6

1. (i) 8 (ii) 2 (iii) 5

2. (i) 27 (ii) 4 (iii) 8

(iv) $\frac{1}{5}\left[(125)^{-\frac{1}{3}} = \left(5^3\right)^{-\frac{1}{3}} = 5^{-1}\right]$

3. (i) $2^{\frac{13}{15}}$ (ii) 3^{-21} (iii) $11^{\frac{1}{4}}$ (iv) $56^{\frac{1}{2}}$

EXERCISE 2.1

1. (i) False. This can be seen visually by the student.

(ii) False. This contradicts Axiom 2.1.

(iii) True. (Postulate 2)

(iv) True. If you superimpose the region bounded by one circle on the other, then they coincide. So, their centres and boundaries coincide. Therefore, their radii will coincide.

(v) True. The first axiom of Euclid.

3. There are several undefined terms which the student should list. They are consistent, because they deal with two different situations — (i) says that given two points A and

B, there is a point C lying on the line in between them; (ii) says that given A and B, you can take C not lying on the line through A and B.

These 'postulates' do not follow from Euclid's postulates. However, they follow from Axiom 2.1.

4. $AC = BC$

So, $AC + AC = BC + AC$ (Equals are added to equals)

i.e., $2AC = AB$ (BC + AC coincides with AB)

Therefore, $AC = \frac{1}{2} AB$

5. Make a temporary assumption that different points C and D are two mid-points of AB. Now, you show that points C and D are not two different points.

6. $AC = BD$ (Given) (1)

$AC = AB + BC$ (Point B lies between A and C) (2)

$BD = BC + CD$ (Point C lies between B and D) (3)

Substituting (2) and (3) in (1), you get

$$AB + BC = BC + CD$$

So, $AB = CD$ (Subtracting equals from equals)

7. Since this is true for any thing in any part of the world, this is a universal truth.

EXERCISE 2.2

1. Any formulation the student gives should be discussed in the class for its validity.

2. If a straight line l falls on two straight lines m and n such that sum of the interior angles on one side of l is two right angles, then by Euclid's fifth postulate the line will not meet on this side of l. Next, you know that the sum of the interior angles on the other side of line l will also be two right angles. Therefore, they will not meet on the other side also. So, the lines m and n never meet and are, therefore, parallel.

EXERCISE 3.1

1. 30°, 250° **2.** 126° **4.** Sum of all the angles at a point = 360°

5. $\angle$ QOS = $\angle$ SOR + $\angle$ ROQ and $\angle$ POS = $\angle$ POR – $\angle$ SOR. **6.** 122°, 302°

EXERCISE 3.2

1. 130°, 130° **2.** 126° **3.** 126°, 36°, 54° **4.** 60° **5.** 50°, 77°

6. Angle of incidence = Angle of reflection. At point B, draw BE $\perp$ PQ and at point C, draw CF $\perp$ RS.

EXERCISE 3.3

1. 65° **2.** 32°, 121° **3.** 92° **4.** 60° **5.** 37°, 53°

6. Sum of the angles of Δ PQR = Sum of the angles of Δ QTR and $\angle$ PRS = $\angle$ QPR + $\angle$ PQR.

EXERCISE 4.1

1. (i) and (ii) are polynomials in one variable, (v) is a polynomial in three variables,

(iii), (iv) are not polynomials, because in each of these exponent of the variable is not a whole number.

2. (i) 1 (ii) –1 (iii) $\frac{\pi}{2}$ (iv) 0

3. $3x^{35} - 4$; $\sqrt{2}\,y^{100}$ (You can write some more polynomials with different coefficients.)

4. (i) 3 (ii) 2 (iii) 1 (iv) 0

5. (i) quadratic (ii) cubic (iii) quadratic (iv) linear

(v) linear (vi) quadratic (vii) cubic

EXERCISE 4.2

1. (i) 3 (ii) –6 (iii) –3

2. (i) 1, 1, 3 (ii) 2, 4, 4 (iii) 0, 1, 8 (iv) –1, 0, 3

3. (i) Yes (ii) No (iii) Yes (iv) Yes

(v) Yes (vi) Yes

(vii) $-\frac{1}{\sqrt{3}}$ is a zero, but $\frac{2}{\sqrt{3}}$ is not a zero of the polynomial (viii) No

4. (i) –5 (ii) 5 (iii) $\frac{-5}{2}$ (iv) $\frac{2}{3}$

(v) 0 (vi) 0 (vii) $-\frac{d}{c}$

EXERCISE 4.3

1. (i) 0 (ii) $\frac{27}{8}$

(iii) 1 (iv) $-\pi^3 + 3\pi^2 - 3\pi + 1$ (v) $-\frac{27}{8}$

2. $5a$ **3.** No, since remainder is not zero.

EXERCISE 4.4

1. $(x + 1)$ is a factor of (i), but not the factor of (ii), (iii) and (iv).

2. (i) Yes (ii) No (iii) Yes

3. (i) –2 (ii) $-\left(2 + \sqrt{2}\right)$ (iii) $\sqrt{2} - 1$ (iv) $\frac{3}{2}$

4. (i) $(3x - 1)(4x - 1)$ (ii) $(x + 3)(2x + 1)$ (iii) $(2x + 3)(3x - 2)$ (iv) $(x + 1)(3x - 4)$

5. (i) $(x - 2)(x - 1)(x + 1)$ (ii) $(x + 1)(x + 1)(x - 5)$

(iii) $(x + 1)(x + 2)(x + 10)$ (iv) $(y - 1)(y + 1)(2y + 1)$

EXERCISE 4.5

1. (i) $x^2 + 14x + 40$ (ii) $x^2 - 2x - 80$ (iii) $9x^2 - 3x - 20$

(iv) $y^4 - \frac{9}{4}$ (v) $9 - 4x^2$

2. (i) 11021 (ii) 9120 (iii) 9984

3. (i) $(3x + y)(3x + y)$ (ii) $(2y - 1)(2y - 1)$ (iii) $\left(x + \frac{y}{10}\right)\left(x - \frac{y}{10}\right)$

4. (i) $x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz$

(ii) $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$

(iii) $4x^2 + 9y^2 + 4z^2 - 12xy + 12yz - 8xz$

(iv) $9a^2 + 49b^2 + c^2 - 42ab + 14bc - 6ac$

(v) $4x^2 + 25y^2 + 9z^2 - 20xy - 30yz + 12xz$

(vi) $\frac{a^2}{16} + \frac{b^2}{4} + 1 - \frac{ab}{4} - b + \frac{a}{2}$

5. (i) $(2x + 3y - 4z)(2x + 3y - 4z)$ (ii) $\left(-\sqrt{2}\,x + y + 2\sqrt{2}\,z\right)\left(-\sqrt{2}\,x + y + 2\sqrt{2}\,z\right)$

6. (i) $8x^3 + 12x^2 + 6x + 1$ (ii) $8a^3 - 27b^3 - 36a^2b + 54ab^2$

(iii) $\frac{27}{8}x^3 + \frac{27}{4}x^2 + \frac{9}{2}x + 1$ (iv) $x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4xy^2}{3}$

7. (i) 970299 (ii) 1061208 (iii) 994011992

8. (i) $(2a + b)(2a + b)(2a + b)$ (ii) $(2a - b)(2a - b)(2a - b)$

(iii) $(3 - 5a)(3 - 5a)(3 - 5a)$ (iv) $(4a - 3b)(4a - 3b)(4a - 3b)$

(v) $\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)$

10. (i) $(3y + 5z)(9y^2 + 25z^2 - 15yz)$ (ii) $(4m - 7n)(16m^2 + 49n^2 + 28mn)$

11. $(3x + y + z)(9x^2 + y^2 + z^2 - 3xy - yz - 3xz)$

12. Simiplify RHS.

13. Put $x + y + z = 0$ in Identity VIII.

14. (i) -1260. Let $a = -12$, $b = 7$, $c = 5$. Here $a + b + c = 0$. Use the result given in Q13.

(ii) 16380

15. (i) One possible answer is : Length = $5a - 3$, Breadth = $5a - 4$

(ii) One possible answer is : Length = $7y - 3$, Breadth = $5y + 4$

16. (i) One possible answer is : 3, x and $x - 4$.

(ii) One possible answer is : $4k$, $3y + 5$ and $y - 1$.

EXERCISE 5.1

1. They are equal. **6.** $\angle$ BAC = $\angle$ DAE

EXERCISE 5.2

6. $\angle BCD = \angle BCA + \angle DCA = \angle B + \angle D$ **7.** each is of 45°

EXERCISE 5.3

3. **(ii)** From (i), $\angle ABM = \angle PQN$

EXERCISE 5.4

4. Join BD and show $\angle B > \angle D$. Join AC and show $\angle A > \angle C$.

5. $\angle Q + \angle QPS > \angle R + \angle RPS$, etc.

EXERCISE 7.1

1. 36°, 60°, 108° and 156°.

6. (i) From Δ DAC and Δ BCA, show $\angle DAC = \angle BCA$ and $\angle ACD = \angle CAB$, etc.

(ii) Show $\angle BAC = \angle BCA$, using Theorem 7.4.

EXERCISE 7.2

2. Show PQRS is a parallelogram. Also show PQ || AC and PS || BD. So, $\angle P = 90°$.

5. AECF is a parallelogram. So, AF || CE, etc.

EXERCISE A1.1

1. (i) False. There are 12 months in a year.

(ii) Ambiguous. In a given year, Diwali may or may not fall on a Friday.

(iii) Ambiguous. At some time in the year, the temperature in Magadi, may be 26° C.

(iv) Always true.

(v) False. Dogs cannot fly.

(vi) Ambiguous. In a leap year, February has 29 days.

2. (i) False. The sum of the interior angles of a quadrilateral is 360°.

(ii) True (iii) True (iv) True

(v) False, for example, 7 + 5 = 12, which is not an odd number.

3. (i) All prime numbers greater than 2 are odd. (ii) Two times a natural number is always even. (iii) For any $x > 1$, $3x + 1 > 4$. (iv) For any $x \geq 0$, $x^3 \geq 0$.

(v) In an equilateral triangle, a median is also an angle bisector.

EXERCISE A1.2

1. (i) Humans are vertebrates. (ii) No. Dinesh could have got his hair cut by anybody else. (iii) Gulag has a red tongue. (iv) We conclude that the gutters will have to be cleaned tomorrow. (v) All animals having tails need not be dogs. For example, animals such as buffaloes, monkeys, cats, etc. have tails but are not dogs.

2. You need to turn over B and 8. If B has an even number on the other side, then the rule has been broken. Similarly, if 8 has a consonant on the other side, then the rule has been broken.

EXERCISE A1.3

1. Three possible conjectures are:

(i) The product of any three consecutive even numbers is even. (ii) The product of any three consecutive even numbers is divisible by 4. (iii) The product of any three consecutive even numbers is divisible by 6.

2. Line 4: 1 3 3 1 $= 11^3$; Line 5: 1 4 6 4 1 $= 11^4$; the conjecture holds for Line 4 and Line 5; No, because $11^5 \neq 15101051$.

3. $T_4 + T_5 = 25 = 5^2$; $T_{n-1} + T_n = n^2$.

4. $111111^2 = 12345654321$; $1111111^2 = 1234567654321$

5. Student's own answer. For example, Euclid's postulates.

EXERCISE A1.4

1. (i) You can give any two triangles with the same angles but of different sides.

(ii) A rhombus has equal sides but may not be a square.

(iii) A rectangle has equal angles but may not be a square.

(iv) For $a = 3$ and $b = 4$, the statement is not true.

(v) For $n = 11$, $2n^2 + 11 = 253$ which is not a prime.

(vi) For $n = 41$, $n^2 - n + 41$ is not a prime.

2. Student's own answer.

3. Let x and y be two odd numbers. Then $x = 2m + 1$ for some natural number m and $y = 2n + 1$ for some natural number n.

$x + y = 2\,(m + n + 1)$. Therefore, $x + y$ is divisible by 2 and is even.

4. See Q.3. $xy = (2m + 1)(2n + 1) = 2\,(2mn + m + n) + 1$.

Therefore, $x\,y$ is not divisible by 2, and so it is odd.

5. Let $2n$, $2n + 2$ and $2n + 4$ be three consecutive even numbers. Then their sum is $6(n + 1)$, which is divisible by 6.

7. (i) Let your original number be n. Then we are doing the following operations:

$$n \to 2n \to 2\,n + 9 \to 2\,n + 9 + n = 3n + 9 \to \frac{3n + 9}{3} = n + 3 \to n + 3 + 4 = n + 7 \to n + 7 - n = 7.$$

(ii) Note that $7 \times 11 \times 13 = 1001$. Take any three digit number say, *abc*. Then $abc \times 1001 = abcabc$. Therefore, the six digit number *abcabc* is divisible by 7, 11 and 13.

****