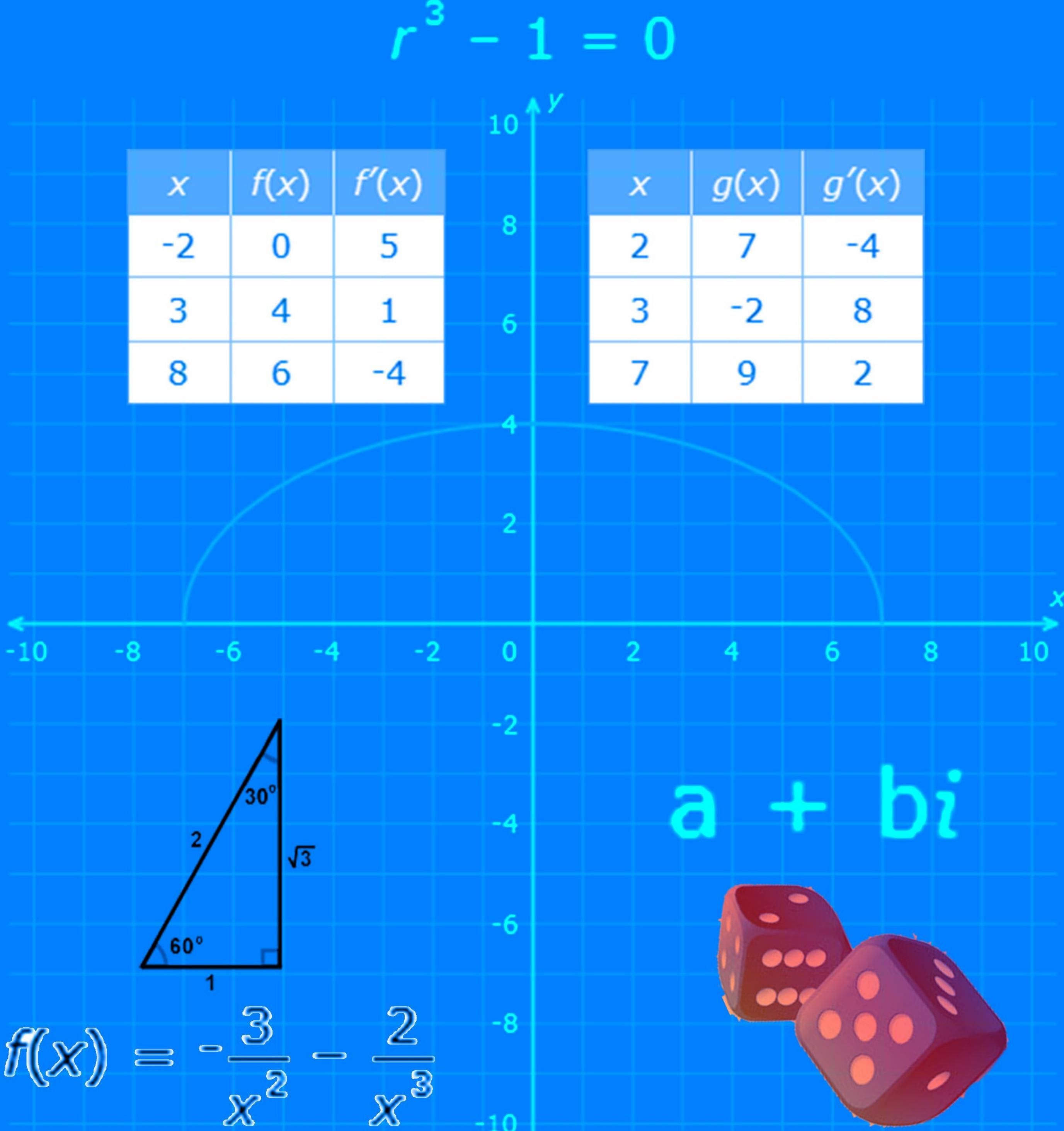
MINISTRY OF EDUCATION

YEAR 13 MATHEMATICS



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PREFACE

The development of this Textbook was entirely based on the **Year 13 Syllabus.**

It has a total of **nine** strands: Complex Numbers, Vectors, Algebra, Trigonometry, Probability & Inferential Statistics, Functions, Limits Continuity & Differentiability, Differentiation and Integration.

The contents of this book have been simplified so that it can be used by all students with different capabilities. It contains very useful materials to help students and teachers alike to prepare for the Year 13 external examination.

It is confidently believed that it will furnish Year 13 students with the necessary number and variety of exercises essential to successful instructions in mathematics.

The step – by – step instructions in the methods and examples will make it suitable for both direct one – to – one tutoring and as well as regular classroom use. Moreover, there are inclusions of external examination [Fiji Seventh Form Certificate or Fiji Year 13 Certificate Examination] questions and illustrations that will help students a great deal.

All examples that have been introduced can even be attempted by an average pupil without assistance. They have been carefully graded to suit the slow learners as well, while there are some problems that are provided for advance learners.

Teachers and students are also **advised to use other resources** for enhancing of teaching and learning. This textbook is **just a guide** to accomplish the learning outcomes.

ACKNOWLEDGEMENTS

Throughout the process in writing this textbook, a number of people have sacrificed their valuable time to assist the Ministry of Education. They are acknowledged for their active participation and without their insights, guidance and continued support; this book may not have been possible.

The Ministry of Education, therefore, hereby acknowledges the following people for their valuable contributions to this book:

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INTRODUCTION

This textbook takes students and teachers into pleasant journey ahead in Mathematics lessons. But before we begin, let's just look at the brief history of Mathematics.

The history of mathematics is nearly as old as humanity itself. Since the ancient times, mathematics has been fundamental to advances in science, engineering, and philosophy. It has evolved from simple counting, measurement and calculation, and the systematic study of the shapes and motions of physical objects, through the application of abstraction, imagination and logic, to the broad, complex and often abstract discipline we know today. (http://www.storyofmathematics.com/)

A high-quality mathematics education therefore provides a foundation for understanding the world, the ability to reason mathematically, an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject. (https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study/national-curriculum-in-england-mathematics-programmes-of-study#key-stage-4)

However, the teacher is to take the role of a facilitator rather than the teacher – centered teaching and incorporate more of technology driven lessons. Yet, it's harder to encourage students to learn and study due to the technology impact, then why not use technology to help students get motivated in Mathematics education.

In the book there are a variety of style types being used to help the teacher's guide students to a thorough understanding of the concepts. The style type is indicated by an icon as shown below:


Note for the teachers and students to elaborate a concept.


Pay attention to the examples. Listen to your teacher for any key ideas.


The answers given to each example. Understand and follow through the steps. If finding difficulty to follow, do not hesitate to ask your teacher.


Exercises based on the related concept learnt.

Did you know? These are some of the facts on the related concept or general ideas involving in Mathematics.

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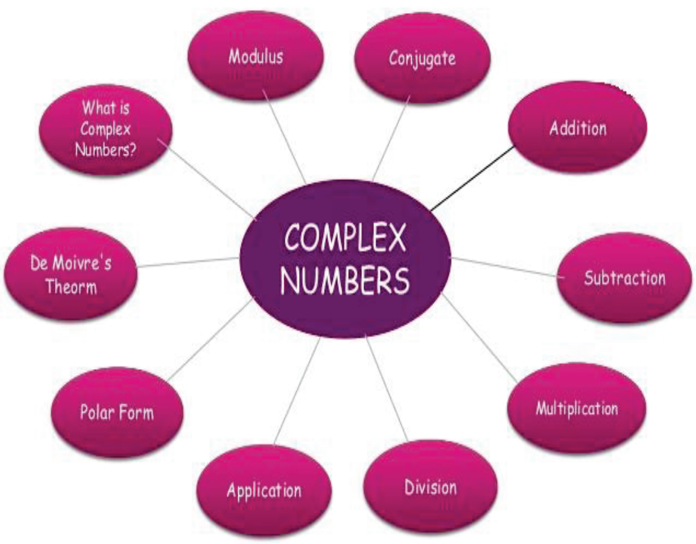

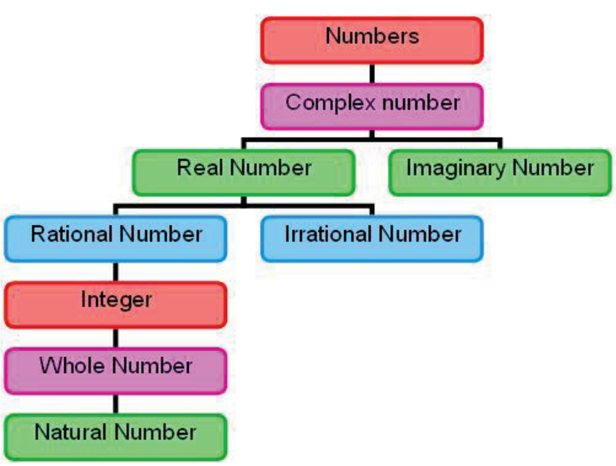

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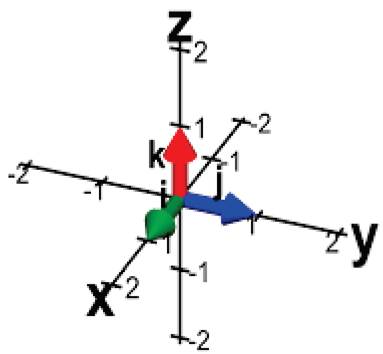

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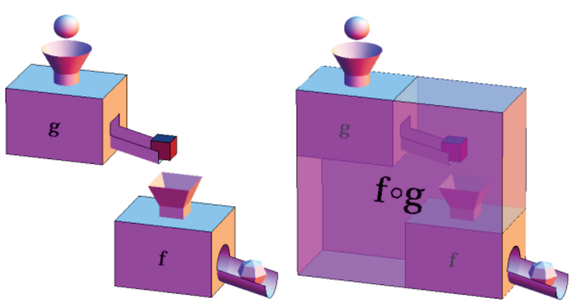

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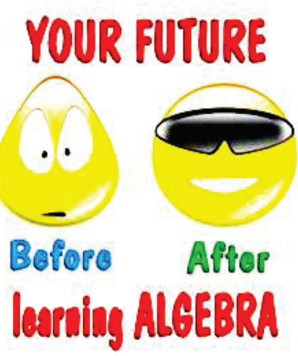

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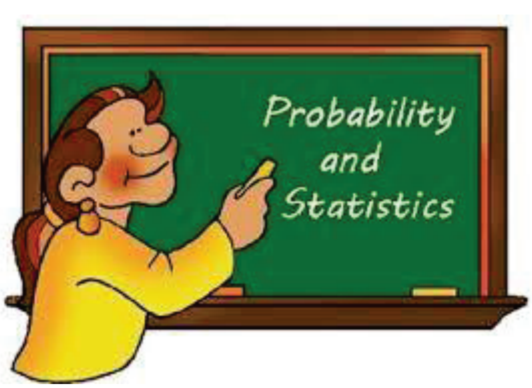

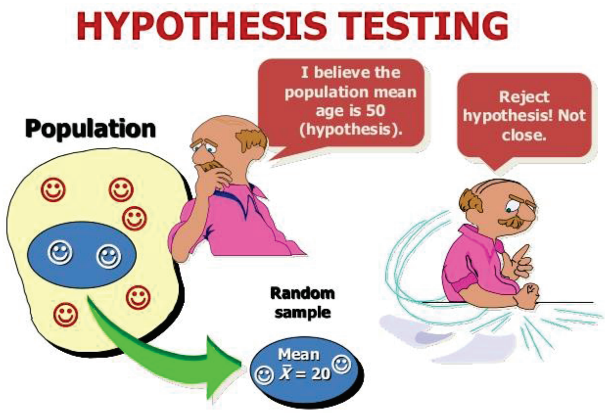

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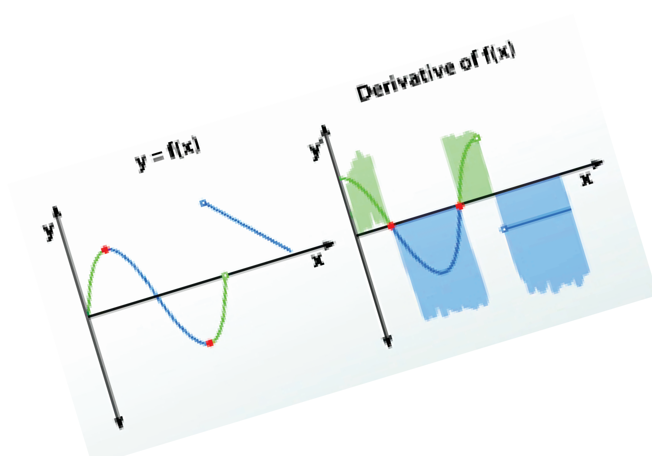

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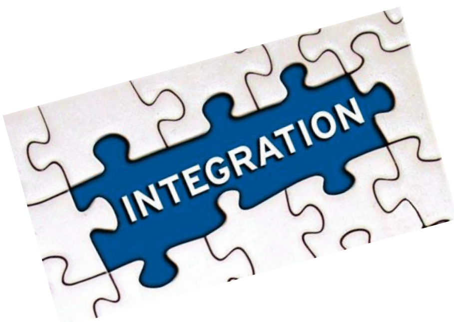
INTEGRATION

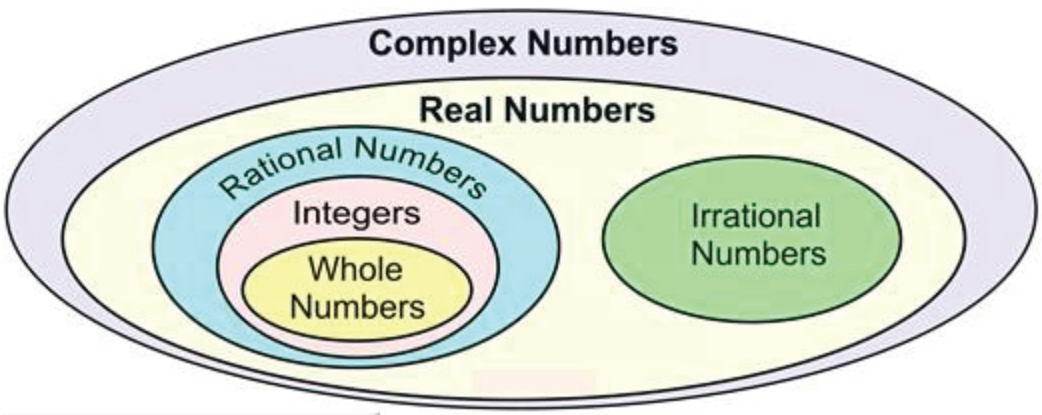


STRAND ONE

COMPLEX NUMBERS

SUB – STRAND 1.1

Introduction to Complex Numbers



LEARNING OUTCOMES

1.1.1 Define imaginary number i
Find square roots of negative numbers.

1.1.2 Solve quadratic equations with discriminant < 0.

1.1.3 Give parts of complex numbers.

1.1.4 Solve equations using equality rule.

1.1.1 The number i

- The inadequacy of real numbers in solving quadratic equation of the type $x^2 + 1 = 0$ gave rise to a new type of numbers called the **complex numbers**.

By letting $i^2 = -1$, in other words $i = \sqrt{-1}$, we are now able to work with a whole new range of equations. While i is called an imaginary unit, it is essential to some real world fields such as electrical engineering, quantum mechanics, cartography, and many others.

Recall one of the rules of surds: $\sqrt{ab} = \sqrt{a}\sqrt{b}$ which is handy in finding square root of negative numbers.

With the help of i, square roots of negative numbers can be interpreted in the following way:

a) $\sqrt{-4}$

$= \sqrt{-1 \times 4}$

$= \sqrt{-1} \times \sqrt{4}$

$= i \times 2$

$= 2i$

b) $\sqrt{-16}$

$= \sqrt{-1 \times 16}$

$= \sqrt{-1} \times \sqrt{16}$

$= i \times 4$

$= 4i$

c) $\sqrt{-9}$

$= \sqrt{-1 \times 9}$

$= \sqrt{-1} \times \sqrt{9}$

$= i \times 3$

$= 3i$

Exercise 1.1.1

Evaluate the following:

a) $\sqrt{-25}$

b) $\sqrt{-100}$

c) $\sqrt{-49}$

d) $\sqrt{-81}$

1.1.2 Quadratic Equations, where discriminant < 0

Quadratic equations have the form $ax^2 + bx + c = 0$.
Just identify and substitute the values of a, b **and** c in the **quadratic formula:**

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

You have to use imaginary numbers for square root of negative numbers.

Example 1 Solve $x^2+2x+5=0$

Answers Using the quadratic formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ where $a=1$, $b=2$ and $c=5$

$$x=\frac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)}=\frac{-2\pm\sqrt{-16}}{2}$$

$$x=\frac{-2\pm4i}{2}=-1+2i,\ -1-2i \qquad \therefore x\in\{-1+2i,\ -1-2i\}$$

Example 2 Solve the equation $4x^2+2x+1=0$.

Answers $a=4$, $b=2$ and $c=1$

$$x=\frac{-2\pm\sqrt{2^2-4(4)(1)}}{2(4)}$$

$$=\frac{-2\pm\sqrt{-12}}{8}$$

$$=\frac{-2\pm\sqrt{12}\,i}{8}$$

$$=\frac{-2\pm\sqrt{4}\sqrt{3}\,i}{8}$$

$$=\frac{-2\pm2\sqrt{3}\,i}{8}$$

$$=-\frac{1}{4}\pm\frac{\sqrt{3}}{4}i$$

Therefore $x=-\frac{1}{4}-\frac{\sqrt{3}}{4}i$ or $x=-\frac{1}{4}+\frac{\sqrt{3}}{4}i$.

Exercise 1.1.2

Solve the following equations and express the answers in rectangular form:

1. $4x^2 + 9 = 0$
2. $4x - 13 = 8x^2$
3. $3x^2 + 10 = 4x$
4. $2x^2 + 8 = 0$
5. $x^2 - 2x + 2 = 0$
6. $x^2 + 25 = 0$
7. $x^2 + x + 1 = 0$
8. $3x^2 - 4x + 2 = 0$
9. $x + \dfrac{5}{x} = 3$
10. $\dfrac{x^2}{2} = 5x - 17$

1.1.3 Parts of Complex Numbers

Complex number (z) can be written in the form $\boldsymbol{z = x + yi}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ are real numbers and i is the imaginary unit, satisfying the equation $i^2 = -1$. This is called **rectangular form.**

- The Complex number, z, is a single value number although it is made up of two parts 'x' and 'yi'.
- The '$\boldsymbol{x}$ – term' is called the real part, denoted by $Re(\boldsymbol{z})$.
- The '$\boldsymbol{y}$– term' is called the imaginary part (i.e. to the part attached to 'i'), denoted by $Im(\boldsymbol{z})$.
- This means that the Complex number can also be written as

  $\boldsymbol{z} = Re(z) + Im(\boldsymbol{z})\boldsymbol{i}$

For example, given the complex number $w = -3.5 + 2i$, the $Re(w)$ and $Im(w)$ are:

$Re(w) = -3.5$
$Im(w) = 2$

Exercise 1.1.3

1. State the real and imaginary parts of each of the complex numbers shown below:

   a) $z = 3 - 2i$ b) $z = -½ + 3i$ c) $z = \dfrac{\sqrt{3} - i}{2}$

   d) $z = -1 - \sqrt{3}\,i$ e) $z = -\sqrt{2}\,i$ f) $z = 6$

   g) $3z = 3 - 2i - z$ h) $2z + 2i = 3 - i$ i) $z = -3$

2. $Re(z)$ is $\sqrt{5}$ and $Im(z)$ is $\dfrac{1}{2}$. Write z in rectangular form.

3. $Re(w)$ is -1 and $Im(w)$ is $\dfrac{1}{\sqrt{3}}$. Write w in rectangular form.

1.1.4 Solving Equations using Equality rule

Two complex numbers are equal only if their real and imaginary parts are equal.
For example, consider $z_1 = a + bi$ and $z_2 = x + yi$.
If $z_1 = z_2$ then it can be said that $x = a$ **and** $y = b$.

To solve the equations, take into account the equality rule.

Example 1 Solve $3 - 4i = x + yi$

Answers

$$\begin{array}{ccc} 3 & + & -4i \\ \| & & \| \\ x & + & yi \\ Re & & Im \end{array}$$

$\therefore\ x = 3$ and $y = -4$

Example 2 Find the values of x and y if $2+i=x+(y-1)i$.

Answers $2+i=x+(y-1)i$

$$\begin{array}{cccc} 2 & + & 1 & i \\ & \| & & \\ x & + & (y-1) & i \\ Re & & Im & \end{array}$$

$x = 2$ and $y-1 = 1$, solve

$y = 1+1$

$\therefore$ $x = 2$ and $y = 2$

Example 3 Find the values of x and y such that $(2-i)x+(1+3i)y=7$.

Answers Expand and simplify

$$(2-i)x+(1+3i)y=7 \quad \Rightarrow 2x-xi+y+3yi=7 \quad \Rightarrow (2x+y)+(-x+3y)i=7+0i$$

Equate the imaginary and real parts.

$2x+y=7$ $\qquad (1)$

$-x+3y=0$ $\qquad (2)$

Solve Simultaneously:

Note: Solving Simultaneous Equations

Two common approaches are

- **Substitution Method**: Substitute one equation into another equation, then solve.
- **Elimination Method**: One of the variables to have same coefficient, then either add/subtract to solve.

Using the **Substitution method**:

Using equation (1), we have $(2x+y)=7 \qquad \Rightarrow y=7-2x \qquad (3)$

Substitute equation (3) into (2), we have

$(-x+3(7-2x))=0 \quad \Rightarrow -x+21-6x=0 \quad \Rightarrow -7x=-21 \quad \Rightarrow x=3$

Finally, substitute $x=3$ into equation (3), to get $y=1$.

Therefore, $x=3$ and $y=1$

Exercise 1.1.4

Find x and y such that

1. $x+2i=4-yi$
2. $x+2i=1+yi$
3. $(x+yi)(-i)=3$
4. $(x-y)+3i=4+yi$
5. $(x+2y)+(x-y)i=3+2i$
6. $(3+4i)(x+yi)=13+10i$

Most modern technology, from electric lighting to digital cameras could not have been invented without them." Imaginary numbers allow for complex analysis, which allows engineers to solve practical problems working in the plane.~ Stewart (http://www.slideshare.net/malikanaa/complex-number-45848431)

SUB – STRAND 1.2

Manipulation of Complex Numbers

(i) Addition:
$(a + ib) + (c + id) = (a + c) + i\,(b + d)$

(ii) Subtraction:
$(a + ib) - (c + id) = (a - c) + i\,(b - d)$

(iii) Multiplication:
$(a + ib)\,(c + id) = (ac - bd) + i\,(ad + bc)$

(iv) Division by a non- zero complex number:
$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2},\ (c+id) \neq 0.$$

LEARNING OUTCOMES

1.2.1 Simplify powers of *i*.
1.2.2 Add and subtract complex numbers.
1.2.3 Multiply complex numbers.
1.2.4 Divide complex numbers.

1.2.1 Working with 'i'

Since $i^2 = -1$ so $i = \sqrt{-1}$.

$$i^3 = i^2 \times i = -1 \times i = -i$$

$$i^5 = i^2 \times i^2 \times i = -1 \times -1 \times i = i$$

$$i^4 = i^2 \times i^2 = -1 \times -1 = 1$$

1.2.2 Addition and Subtraction

Given two complex numbers $z_1 = a + bi$ and $z_2 = x + yi$, the following holds true for **Addition and Subtraction**.

Recall in algebra, you can add or subtract like terms. Similarly you can add or subtract the real parts and the imaginary parts.

i.e. $z_1 \pm z_2$

$= (a + bi) + (x + yi)$

$= a + bi + x + yi$

Collecting *Real* and *Imaginary* parts yields:

$$\begin{array}{rl} & a + b\,i \\ + & x + y\,i \\ \hline & (a+x)+(b+y)i \end{array}$$

$$\therefore z_1 + z_2 = (a+x)+(b+y)i$$

Example 1 If $v = 1 - 5i$ and $w = -1 + 3i$. Find:

1. $v + w$
2. $w - v$

Answers

1. $v + w$

$= (1 - 5i) + (-1 + 3i)$

$$\begin{array}{rrr} = & 1 & -5\,i \\ + & -1 & +3\,i \\ \hline & 0 & -2i \end{array}$$

$\therefore v + w = -2i$

2. $w - v$

$= (-1 + 3i) - (1 - 5i)$

$$\begin{array}{rrr} = & -1 & +3\,i \\ - & 1 & -5\,i \\ \hline & -2 & +8\,i \end{array}$$

$\therefore w - v = -2 + 8i$

1.2.3 Multiplication

Given two complex numbers $z_1 = a + bi$ and $z_2 = x + yi$, the following holds true for **Multiplication**.

Recall in algebra, we use distributive law to expand the brackets. We will also use the fact that $i^2 = -1$ when simplifying the result.

i.e. $z_1 \times z_2$

$= (a + bi) \times (x + yi)$

$= (a + bi)\,(x + yi)$

$= a \times x + a \times yi + bi \times x + bi \times yi$

$= a\,x + a\,yi + bi\,x + bi\,yi$

$= a\,x + a\,yi + bi\,x + by\,i^2$

$= a\,x + a\,yi + bi\,x + by(-1)$ $i^2 = -1$

$= a\,x + a\,yi + bi\,x - by$ Collect real parts and imaginary parts

$\therefore z_1 \times z_2 = (ax - by) + (ay + bx)i$

Example 2   If $v = 1 - 5i$ and $w = -1 + 3i$. Find

1. $v \times w$ 2. w^2

Answers

1. $v \times w$

$= (1 - 5i) \times (-1 + 3i)$

$= \underbrace{1 \times -1} + \underbrace{1 \times 3i} \underbrace{- 5i \times -1} + \underbrace{- 5i \times 3i}$

$= -1 + 3i + 5i - 15\,i^2$

$= -1 + 3i + 5i - 15\,(-1)$

$= -1 + 3i + 5i + 15$ Add the real and imaginary parts

$= -1 + 15 + 3i + 5i$

$\therefore v \times w = 14 + 8\,i$

2. $w^2 = w \times w$

$= (-1 + 3i)^2$

$= (-1)^2 + 2 \times -1 \times 3\,i + (3i)^2$

$= 1 - 6\,i + 9\,i^2$

$= 1 - 6\,i + 9\,(-1)$

$= 1 - 6\,i - 9$

$= 1 - 9 - 6\,i$

$= -8 - 6\,i$

Shortcut
$(a + b)^2 = a^2 + 2ab + b^2$

1.2.4 Division

Conjugate

Conjugate is found by **changing the sign** of the coefficient of i. The complex conjugate is denoted by a line on top, i.e. $\bar{z}$

The complex roots of quadratics occur in **conjugate pairs**. This means if $a + bi$ is a root then $a - bi$ is also a root.

Given two complex numbers $z_1 = a + bi$ and $z_2 = x + yi$, then

$\overline{z_1} = a - bi$ and $\overline{z_2} = x - yi$

Complex Conjugate properties:

1) $\bar{\bar{z}} = z$

2) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

3) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Division

When dividing two complex numbers, multiply the numerator and denominator by the conjugate of the denominator.

i.e. $\frac{z_1}{z_2} = \frac{a+bi}{x+yi}$ multiply by conjugate of the denominator

$$= \frac{a+bi}{x+yi} \times \frac{x-yi}{x-yi} = \frac{(a+bi)(x-yi)}{(x+yi)(x-yi)}$$

$$\boxed{a^2 - b^2 = (a-b)(a+b)}$$

$$= \frac{ax - ayi + bxi - byi^2}{x^2 - (yi)^2}$$

$$= \frac{ax - ayi + bxi - by(-1)}{x^2 - y^2(-1)}$$

$$= \frac{ax + by + bxi - ayi}{x^2 + y^2}$$

$$= \frac{ax + by + (bx - ay)i}{x^2 + y^2}$$

In rectangular form:

$$\therefore \frac{z_1}{z_2} = \frac{ax+by}{x^2+y^2} + \frac{(bx-ay)}{x^2+y^2}i$$

Example 3 If $a = 1 - 5i$ and $b = -1 + 3i$, show that $\overline{ab} = \overline{a} \cdot \overline{b}$

Answer

$LHS = \overline{ab}$

$$= \overline{(1-5i)(-1+3i)}$$

$$= \overline{(1-5i)(-1+3i)}$$

$$= \overline{-1+3i+5i-15i^2}$$

$$= \overline{-1+3i+5i-15(-1)}$$

$$= \overline{-1+3i+5i+15}$$

$$= \overline{14+8i}$$

$$= 14-8i$$

$RHS = \overline{a} \cdot \overline{b}$

$$= \overline{(1-5i)} \cdot \overline{(-1+3i)}$$

$$= (1+5i)(-1-3i)$$

$$= -1-3i-5i-15i^2$$

$$= -1-3i-5i-15(-1)$$

$$= 14-8i$$

$$\therefore LHS = RHS$$

Example 4 Simplify and express $\dfrac{4-3i}{3+2i} + 3$ in the form $a+bi$.

Answer

$$\frac{4-3i}{3+2i} + \frac{3}{1} = \frac{4-3i}{3+2i} + \frac{3(3+2i)}{1(3+2i)}$$

$$= \frac{4-3i+9+6i}{3+2i}$$

$$= \frac{13+3i}{3+2i}$$

Next, multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{13+3i}{3+2i} = \frac{13+3i}{3+2i} \times \frac{3-2i}{3-2i}$$

$$= \frac{(13+3i)(3-2i)}{9-4i^2}$$

$$\boxed{a^2 - b^2 = (a-b)(a+b)}$$

$$= \frac{39-26i+9i-6i^2}{9+4}$$

$$= \frac{45-17i}{13}$$

$$= 3.46 - 1.31i$$

Exercise 1.2

1. Evaluate

   a) i^5 b) i^7 c) $\dfrac{1}{i^6}$ d) $3i^2 \times 4i^3$

2. Given that $z = 3 + i$ and $w = -1 + 2i$, evaluate the following:

   a) $z + w$ b) $z - w$ c) $z + 2w$

   d) $2z - 3w$ e) $z + z$ f) $w - w$

3. Simplify

   a) $(-1 - \sqrt{3}\,i)(1 + i)$ b) $(1 - i)^2$

   c) $-8i^4(9i^9 - 3i^6)$ d) $(1 + \sqrt{2}\,i)^2$

4. Two complex numbers, α and β are given as: $\alpha = 2 - i$ and $\beta = 3 + 2i$.

   a) Show that $\bar{\alpha} + \bar{\beta} = \overline{\alpha + \beta}$.

   b) Find α/β in the form $a + bi$.

5. Simplify

   a) $\dfrac{1}{1+i}$ b) $\dfrac{1+\sqrt{2}i}{i}$ c) $\dfrac{2-i}{3} - 1$ d) $\dfrac{3-2i}{1+i}$

6. Two complex numbers are given as: $z = 1 - \sqrt{3}i$ and $w = 1 + i$.

   a) Show that $z + \bar{z} = 2\mathrm{Re}(z)$

   b) Show that $\bar{\bar{w}} = w$

   c) Find $\dfrac{w}{z}$

   d) Simplify and express $\dfrac{\bar{w}}{\bar{z}}$ in the form $a + bi$

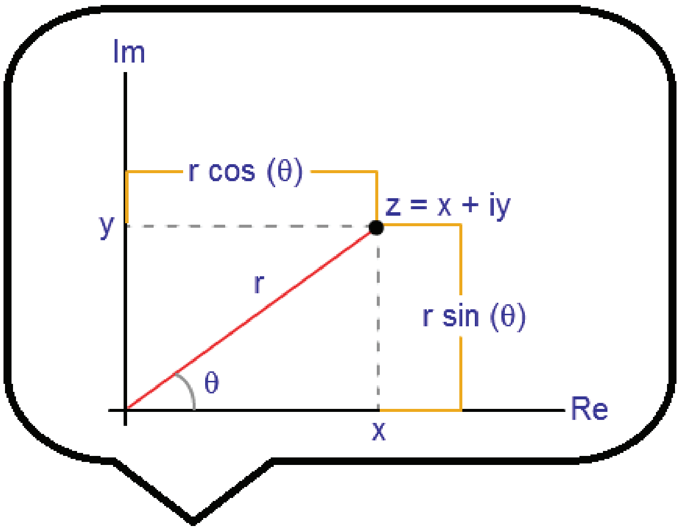
   e) Show that $\overline{\left(\dfrac{w}{z}\right)} = \dfrac{\bar{w}}{\bar{z}}$

DID YOU KNOW?

Imaginary numbers were originally posited by famed gambler/mathematician Girolamo Cardano and then expanded by Rafael Bombelli and John Wallis. They still existed as a peculiar, but essential problem in math until William Hamilton described the definition ~ Andy Kiersz (http://www.slideshare.net/malikanaa/complex-number-45848431)

SUB – STRAND 1.3

Geometrical Representation and Properties



LEARNING OUTCOMES

1.3.1 Represent rectangular form on the complex plane.
Find modulus and argument.

1.3.2 Convert rectangular to polar form.

1.3.3 Convert polar to rectangular form.

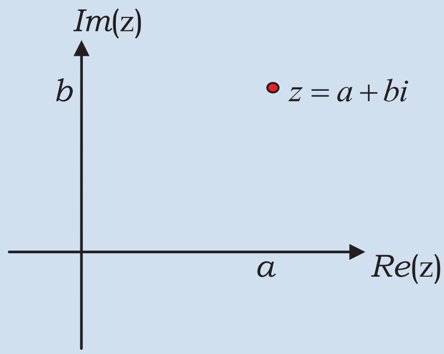
1.3.3 Represent polar form on an argand diagram.

1.3.4 Multiply and divide in polar form.

1.3.1 Geometric Representation of $z = a + bi$

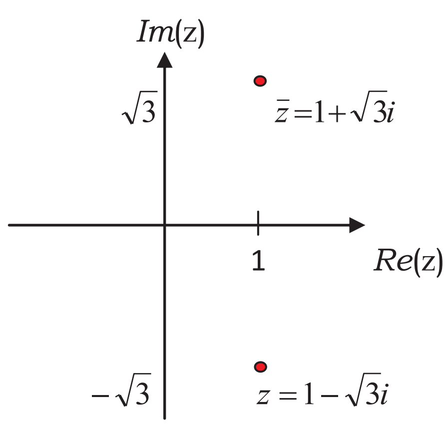
1.3.1.1 Argand Diagram

A complex number z is viewed as **point** in a two – dimensional Cartesian coordinate system called the complex plane or Argand diagram.
The x – **axis** shows the **real part** **(a)** and y – **axis** shows the **imaginary part** **(b)**.
Graphically, $z = a + bi$ is drawn below:



Example If $z = 1 - \sqrt{3}i$, plot the points z and $\bar{z}$ on an Argand diagram.

Answer



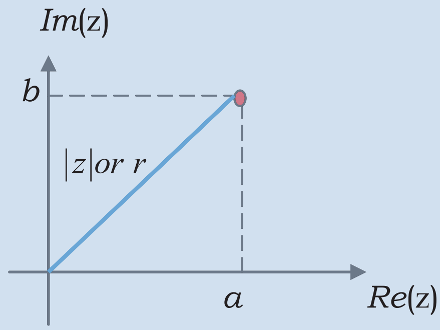
1.3.1.2 Modulus of z

The Modulus (magnitude or Absolute value) of a complex number is the **length** of the vector from the origin to the position of the complex number in the Argand diagram.

If $z = a + bi$, the modulus (using Pythagoras theorem) is defined as

$$\boxed{|z| = \sqrt{a^2 + b^2}.}$$

Graphically, modulus is shown below:



Properties of modulus:

1. $|\bar{z}| = |z|$
2. $\bar{z}z = |z|^2$
3. $|z^n| = |z|^n$
4. $|v \times w| = |v| \times |w|$
5. $\left|\frac{v}{w}\right| = \frac{|v|}{|w|}$

Example If $z = 1 - \sqrt{3}i$

Show that $\bar{z}z = |z|^2$

Answer

$\bar{z}z = |z|^2$: $Let \quad LHS = \bar{z}z, \quad RHS = |z|^2$

$$\begin{aligned} LHS &= \bar{z}z \\ &= (1 - \sqrt{3}i)(1 + \sqrt{3}i) \\ &= 1^2 - (\sqrt{3}i)^2 \\ &= 1 - 3i^2 \\ &= 1 - 3(-1) \\ &= 4 \end{aligned}$$

$$\begin{aligned} RHS &= |z|^2 \\ &= \left(\sqrt{1^2 + (-\sqrt{3})^2}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

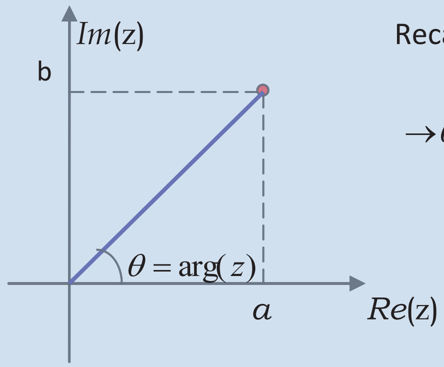
$$LHS = RHS$$

1.3.1.3 Argument of z [$arg(z)$]

The **angle from the positive x – axis** to the line segment joining the origin is called the **argument** of z . If $z = x + yi$ lies in the **first** or the **fourth** quadrant then

$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Graphically:



Recall, to find the angle use TOA, i.e.

$$\rightarrow \theta = \tan^{-1}\left(\frac{O}{A}\right)$$

Arguments have **positive values** if measured **anticlockwise** from the **positive x – axis** and **negative if measured clockwise.**

Principle argument – Arg(z) lies in the interval $-\pi < \theta \leq \pi$ or $-180° < \theta \leq 180°$
Note capital A for principle argument.

Properties of argument:

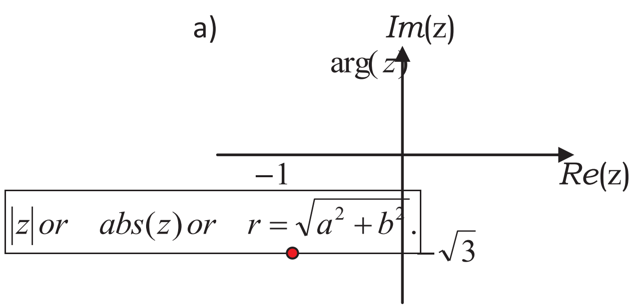
1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$
3. $\arg(z^n) = n \arg(z)$

Example A complex number is given as $z = -1 - \sqrt{3}i$.

a) Plot the point on an argand diagram.
b) Find $|z|$.
c) Find $\arg(z)$.

Answers

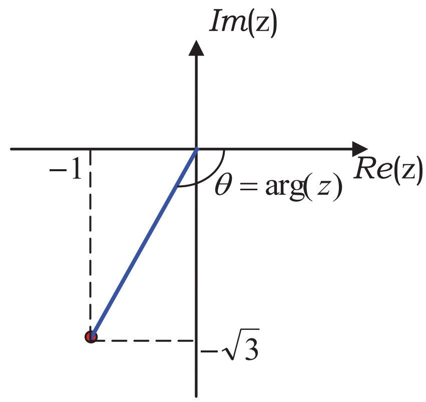
a)



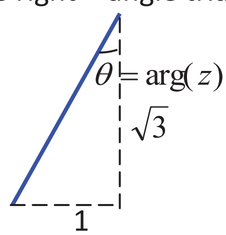
b)

$$|z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-1)^2 + (-\sqrt{3})^2}$$
$$= \sqrt{1+3}$$
$$= 2$$

c)



Consider the right – angle triangle



$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^o$$

$$\therefore Arg(z) = -(90^o + 30^o) = -120^o$$

This is the **principle argument**.

We could also say arg(z) = 270 – 30

$= 240°$

Exercise 1.3.1

Use the complex numbers shown below to answer questions 1 – 4.

i. $z_1 = -1 + i$ ii. $z_2 = -1 - \sqrt{2}i$ iii. $z_3 = \dfrac{1-\sqrt{3}i}{2}$

iv. $z_4 = -4i$ v. $z_5 = 3$ vi. $z_6 = \dfrac{3}{4} + \dfrac{1}{\sqrt{3}}i$

1. Represent the complex numbers on complex planes.
2. Find the magnitude and argument of each of the above complex numbers.
3. Verify the following properties:

   $i.\ |\overline{z_6}| = |z_6|$ $ii.\ |z_1 \times z_2| = |z_1| \times |z_2|$ $iii.\ \left|\dfrac{z_2}{z_1}\right| = \dfrac{|z_2|}{|z_1|}$

4. Verify the following properties:

   $i.\ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $ii.\ \arg(\dfrac{z_4}{z_2}) = \arg(z_4) - \arg(z_2)$

1.3.2 Polar or Trigonometric Form

Recall the **rectangular form** is written as $z = a + bi$.

- The polar coordinates is (r, θ)
  where r is modulus (**position of a point** in terms of its distance from the origin) and **θ** is the argument (**angle θ** from the positive x - axis).

- Using Casio fx-82 MS calculator (to be in degree mode) we can find r and θ

  - To find r: Press

Pol(	a	,	b	)	=

  - To find θ: Press

RCL	tan

- **Writing as polar form**
  The polar representation of z is $z = r\,[\cos\theta + i\sin\theta]$

  Short cut form to remember is $z = r \text{ cis } \theta$

Example A complex number is given as $z = -1 - \sqrt{3}i$. Write z in polar form.

Answers

$$z = a + bi$$
$$z = -1 - \sqrt{3}i$$

$$\therefore a = -1 \;\&\; b = -\sqrt{3}$$

Press

Pol(	-1	,	$-\sqrt{3}$	)	=

$\therefore r = 2$

RCL	tan

$\therefore \theta = -120^o$

In polar form

$$z = 2\ \left[\cos\left(-120^o\right) + i\sin\left(-120^o\right)\right]$$

$$= 2\,cis\,(-120^o)$$

1.3.3 Converting polar to rectangular form and vice versa

Recall that we have already converted rectangular form to polar form.

The **polar form** is written as $z = r\ cis\ \theta$ or $z = r\ [\cos\theta + i\sin\theta]$.

To convert polar form to **rectangular form** $z = x + yi$, you will use **distributive law** to expand.

Using Casio fx-82 MS calculator (to be in degree mode) find x and y

- To find x: Press | Shift | Pol(| r | , θ |) | = |
- To find y: Press | RCL | tan |

Example 1 A polar form of a complex number is given as

$$z = 3\ cis\ 60^o \quad .$$

Write z in rectangular form.

Answer Using distributive law

$$z = 3(\cos 60^o + i\sin 60^o)$$

$$= 3\cos 60^o + i \times 3\sin 60^o$$

$$= 3\times\frac{1}{2} + i\times 3\times\frac{\sqrt{3}}{2}$$

$$= \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

Press | Shift | Pol(| 3 | , 60 |) | = | $\therefore x = \frac{3}{2}$

Press | RCL | tan | $\therefore y = 2.60$

Example 2 The polar form of a complex number is given as

$$z = 2(\cos\ 45^o + i\sin\ 45^o)$$

Write z in rectangular form.

Answer

$$z = 2\times(\cos 45^o + i\sin 45^o)$$

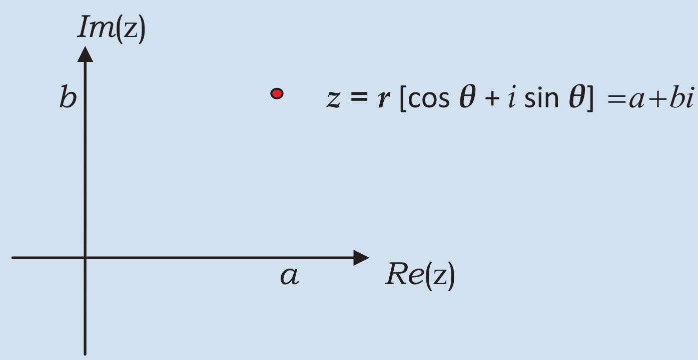
$$z = 2\times\cos 45^o + 2\times i\sin 45^o$$

$$= 2\cos 45^o + 2\sin 45^o i$$

$$= \sqrt{2} + \sqrt{2}i$$

1.3.4 Argand Diagram representing polar form

Recall Argand diagram of a complex number $z = a + bi$. The same point can be plotted using polar form $z = r\,[\cos\theta + i\sin\theta]$.
Convert $z = r\,[\cos\theta + i\sin\theta]$ to rectangular form $z = a + bi$ and plot.



Also note that the rectangular and polar forms of any given complex number represent the same point.

Example 3 A complex number is given as $z = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

a) Express z in rectangular form
b) Graph the complex number z on an Argand diagram.

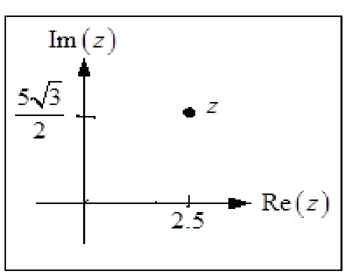
Answers

a)

$$z = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= \frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

b)



Exercise 1.3.4

1. For each of the complex numbers given below, write z in polar form:

   i. $z_1 = -1 + i$ ii. $z_2 = -1 - \sqrt{2}i$ iii. $z_3 = \dfrac{1 - \sqrt{3}i}{2}$

   iv. $z_4 = -4i$ v. $z_5 = 3$ vi. $z_6 = \dfrac{3}{4} + \dfrac{1}{\sqrt{3}}i$

2. Represent the following complex numbers on complex planes:

   i. $v=2\left(\cos 60^o + i\sin 60^o\right)$ ii. $w=3\left(\cos 30^o + i\sin 30^o\right)$

   iii. $z = 5\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ iv. $z = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

1.3.5 Multiplication and Division in Polar Form

If $z_1 = r_1$ $[\cos\theta_1 + i\sin\theta_1]$ and $z_2 = r_2$ $[\cos\theta_2 + i\sin\theta_2]$ then

Multiplication: $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

Example 4 Given two complex numbers: $v = 2\left(\cos 60^o + i\sin 60^o\right)$, $w = 3\left(\cos 30^o + i\sin 30^o\right)$. Find in rectangular form

a) $v \times w$

b) $\dfrac{v}{w}$

✍ **Answers**

a) $v \times w$

Using the property $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

$$v \cdot w = 2 \times 3 \quad [\cos(60^o + 30^o) + i\sin(60^o + 30^o)]$$
$$= 6(\cos 90^o + i\sin 90^o)$$

Use Distributive Law

$$v \bullet w = 6(\cos 90^o + i\sin 90^o)$$
$$= 6 \times \cos 90^o + 6 \times i\sin 90^o$$
$$= 6\cos 90^o + 6\sin 90^o \ i$$
$$= 0 + 6i$$
$$= 6i$$

b) $\frac{v}{w}$

Using the property $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$$\frac{v}{w} = \frac{2}{3} \quad [\cos(60^o - 30^o) + i\sin(60^o - 30^o)]$$
$$= \frac{2}{3}(\cos 30^o + i\sin 30^o)$$
$$= \frac{2}{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= \frac{\sqrt{3}}{3} + \frac{1}{3}i$$

Exercise 1.3

1. Write each complex number in rectangular form:

   a) $z = 2\left(\cos 300^\circ + i\sin 300^\circ\right)$

   b) $z = \sqrt{7}\left(\cos 40.9^\circ + i\sin 40.9^\circ\right)$

   c) $z = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

2. Given $p = 5\left(\cos 90^\circ + i\sin 90^\circ\right)$ and $q = 2\left(\cos -30^\circ + i\sin -30^\circ\right)$

   a) Represent p on an Argand diagram.
   b) Find $\overline{p}$
   c) Find $\overline{q}$
   d) Find pq and express your answer in rectangular form.
   e) Find $\dfrac{p}{q}$ and express your answer in rectangular form.

3. Two complex numbers are given as $\alpha = 3\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$ and $\beta = 2\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$

   Find:

   a) $\alpha\beta$ in the form $a + bi$

   b) $|\alpha\beta|$

   c) $\arg\left(\dfrac{\alpha}{\beta}\right)$

4. Given $p = 2\,cis\dfrac{\pi}{2}$ and $q = \sqrt{2}\ cis\,\pi$

   a) Represent q on an Argand diagram.
   b) Find $\overline{p}$
   c) Find pq in rectangular form.
   d) Find $\dfrac{p}{q}$ in rectangular form.

DID YOU KNOW?

Imaginary numbers are a fine and wonderful refuge of the divine spirit almost an amphibian between being and non-being. (1702) The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and not-being, which we call the imaginary root of negative unity. ~ Gottfried Wilhelm L

SUB – TOPIC 1.4

Powers and Roots Of Complex Numbers

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

where $n \geq 1$ is a positive integer.

Finding Roots

If $z^n = x + iy$

$z^n = r cis\theta$

$z = \sqrt[n]{r} cis\left[\frac{2\pi k + \theta}{n}\right] \quad k = 0,1,\ldots,n-1$

$e.g.(i)\ z^2 = 4i$

$z^2 = 4cis\frac{\pi}{2}$

$$z = 2cis\left[\frac{2\pi k + \frac{\pi}{2}}{2}\right] \quad k = 0,1$$

LEARNING OUTCOMES

1.4.1 Find any integer power of complex numbers using De Moivre's Theorem.

1.4.2 Find nth roots.

1.4.1 Raising Complex Numbers to Powers Using De Moivre's theorem

De Moivre's theorem, named after Abraham de Moivre, states that

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

Example A rectangular form of a complex number z is given as $z = 1 - \sqrt{3}\,i$.

a) Express z in polar form.

b) Use De Moivre's Theorem to find z^5.
[Give your answer in rectangular form.]

c) Show that $\left|z^5\right| = \left|z\right|^5$

Answers

a)
$$z = a + b\,i$$
$$z = 1 - \sqrt{3}\,i$$

$\therefore a = 1 \;\&\; b = -\sqrt{3}$

Press

Pol(	1	,	$-\sqrt{3}$	)	=

$\therefore r = 2$

RCL	tan

$\therefore \theta = -60^o$

Thus in polar form: $z = 2\left[\cos(-60^o) + i\sin(-60^o)\right]$

b) Using formula $\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$

$$z^5 = r^5 \quad (\cos 5\theta + i\sin 5\theta)$$
$$= 2^5 \quad \left[\cos(5\times -60^o) + i\sin(5\times -60^o)\right]$$
$$= 32 \quad \left[\cos(-300^o) + i\sin(-300^o)\right]$$

Distributive law

$$= 32\cos(-300^o) + i\times 32\sin(-300^o)$$
$$= 16 + i\times 32\times\frac{\sqrt{3}}{2}$$
$$= 16 + 16\sqrt{3}i$$

Calculator

Press

Shift	Pol(	32	, -300	)	=

$\therefore x = 16$

Press

RCL	tan

$\therefore y = 27.71$

c)

$$LHS = \left|z^5\right|$$

$$= \left|16 + 16\sqrt{3}i\right|$$

$$= \sqrt{16^2 + (16\sqrt{3})^2}$$

$$= 32$$

$$RHS = |z|^5$$

$$= \left|1 - \sqrt{3}i\right|^5$$

$$= \left(\sqrt{1^2 + (-\sqrt{3})^2}\right)^5$$

$$= 2^5$$

$$= 32$$

$$\therefore LHS = RHS$$

Exercise 1.4.1

1. A complex number is given as $z = 2 + \sqrt{3}i$.
   a) Express z in polar form.
   b) Find z^6 using De Moivre's theorem.

2. A complex number is given by $z = 2\sqrt{3} - i$.
   a) Write z in polar form.
   b) Find z^4 using De Moivre's theorem and express the answer in rectangular form.

3. A complex number z is given as $z = -3 + i$.
   a) Express z in polar form.
   b) Use De Moivre's Theorem to find $(-3+i)^3$ and express in rectangular form.

DID YOU? KNOW

Electrical engineers use complex numbers frequently in their careers. Calculating AC circuits, instead of direct current circuits, is a more complex process made simpler by utilizing complex numbers. Sales analysts may utilize complex numbers in order to make predictions and better understand the sales process. Economists use complex numbers in order to make profit predictions. When analyzing business cycles, complex numbers can also come into play
~ Stephanie Dube Dwilson

1.4.2 Finding nth roots

De Moivre's theorem can be used to find roots, $\boxed{w = z^{1/n} = \sqrt[n]{z}}$.

Let $z = [r, \theta]$

Thus,

$$\boxed{\begin{array}{c} z^{1/n} = r^{1/n}\left[\cos\left(\frac{2\pi k+\theta}{n}\right)+i\sin\left(\frac{2\pi k+\theta}{n}\right)\right], \textit{ if } \theta \textit{ is in radians} \\ or \\ z^{1/n} = r^{1/n}\left[\cos\left(\frac{360^o k+\theta}{n}\right)+i\sin\left(\frac{360^o k+\theta}{n}\right)\right], \textit{ if } \theta \textit{ is in } \deg \end{array}}$$

where k = 0, 1, 2, ... till the number of solution

Spacing of n-th roots

- For square roots there are two solutions 180^o apart.
- For cube root there are three solutions 120^o apart.
- For fourth root there are four solutions 90^o apart.
- Therefore, for nth root there are n solutions $\frac{360^o}{n}$ or $\frac{2\pi}{n}$ apart.

Example Solve $z^3 = 8\left(\cos 270^\circ + i\sin 270^\circ\right)$.[Leave the answers in polar form.]

Answers

Method 1

Given $z^3 = 8\left(\cos 270^\circ + i\sin 270^\circ\right)$, the roots are given by

$$W_k = \sqrt[3]{8}\left[\cos\left(\frac{270^\circ+360^\circ k}{3}\right)+i\sin\left(\frac{270^\circ+360^\circ k}{3}\right)\right],$$

where $k = \{0,1,2\}$, $n = 3$ and $r = \left|z^3\right| = 8$.

Hence, the three distinct roots (in polar form) are:

$$W_0 = \sqrt[3]{8}\left[\cos\left(\frac{270^\circ + 360^\circ(0)}{3}\right) + i\sin\left(\frac{270^\circ + 360^\circ(0)}{3}\right)\right] = 2\left(\cos 90^\circ + i\sin 90^\circ\right),$$

$$W_1 = \sqrt[3]{8}\left[\cos\left(\frac{270^\circ + 360^\circ(1)}{3}\right) + i\sin\left(\frac{270^\circ + 360^\circ(1)}{3}\right)\right] = 2\left(\cos 210^\circ + i\sin 210^\circ\right),$$

$$W_2 = \sqrt[3]{8}\left[\cos\left(\frac{270^\circ + 360^\circ(2)}{3}\right) + i\sin\left(\frac{270^\circ + 360^\circ(2)}{3}\right)\right] = 2\left(\cos 330^\circ + i\sin 330^\circ\right)$$

Method 2

Note: There are 3 solutions. $\frac{360}{3} = 120°$ **so add** $120°$ to the smallest angle $90°$ which is obtained from $\frac{270°}{3}$ **where** $270°$ is the argument of z and 3 refers to the number of roots.

So the angles are :

$90°$,

$90° + 120° = 210°$ [this confirms with above angles]

$210° + 120° = 330°$

Roots in polar form are $\{2\,cis\,90°,\ 2\,cis\,210°,\ 2\,cis\,330°\}$

Example 2 The polar form of a complex number is given as $z = 16\,cis\,\pi$
Find the fourth roots of z.

Answers

Using Method 2
First Root

$$z_1 = 16^{1/4}(cis\,\pi)^{1/4}$$
$$= 16^{1/4}\,cis\frac{\pi}{4}$$
$$= 2\,cis\frac{\pi}{4}$$

For fourth root we expect four solutions which are $\frac{2\pi}{4} = \frac{1}{2}\pi$ radians apart.

Second root

Add $\frac{1}{2}\pi$ to the first root.

$$z_2 = 2\ cis\,(\frac{\pi}{4} + \frac{\pi}{2})$$
$$= 2\ cis\,(\frac{3\pi}{4})$$

Third root

Add $\frac{1}{2}\pi$ to the second root.

$$z_2 = 2\ cis\,(\frac{3\pi}{4} + \frac{\pi}{2})$$
$$= 2\ cis\,(\frac{5\pi}{4})$$

4th root

Add $\frac{1}{2}\pi$ to the third root.

$$z_2 = 2\ cis\,(\frac{5\pi}{4} + \frac{\pi}{2})$$
$$= 2\ cis\,(\frac{7\pi}{4})$$

Roots in polar form are $\{2\ cis\frac{\pi}{4},\ 2\ cis\frac{3\pi}{4},\ 2\ cis\frac{5\pi}{4},\ 2\ cis\frac{7\pi}{4}\}$

Roots in rectangular form are: $\left\{\sqrt{2}+\sqrt{2}i,\ -\sqrt{2}+\sqrt{2}i, -\sqrt{2}-\sqrt{2}i,\ \sqrt{2}-\sqrt{2}i\right\}$

Exercise 1.4.2

1. A complex number z given in polar form is $\sqrt{8}\left(\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right)$. Find the fourth roots of z. (Give your answer in polar form.)

2. A complex number z is given as $z = \sqrt{3}+i$.

   a) Express z in polar form.
   b) Use De Moivre's Theorem to find z^4.
   c) Find all the cube roots of $z = \sqrt{3}+i$.

3. A complex number z is given as $z = 1+i$.

   a) Plot z on an argand diagram.
   b) Find the modulus and argument.
   c) Express z in polar form.
   d) Use De Moivre's Theorem to find z^4.
   e) Find the square roots of $z = 1+i$.

4. Solve

   a) $z^3 = 8i$ b) $z^4 = i+1$ c) $z^3 + i = 0$ d) $z^4 = 81$

5. Solve

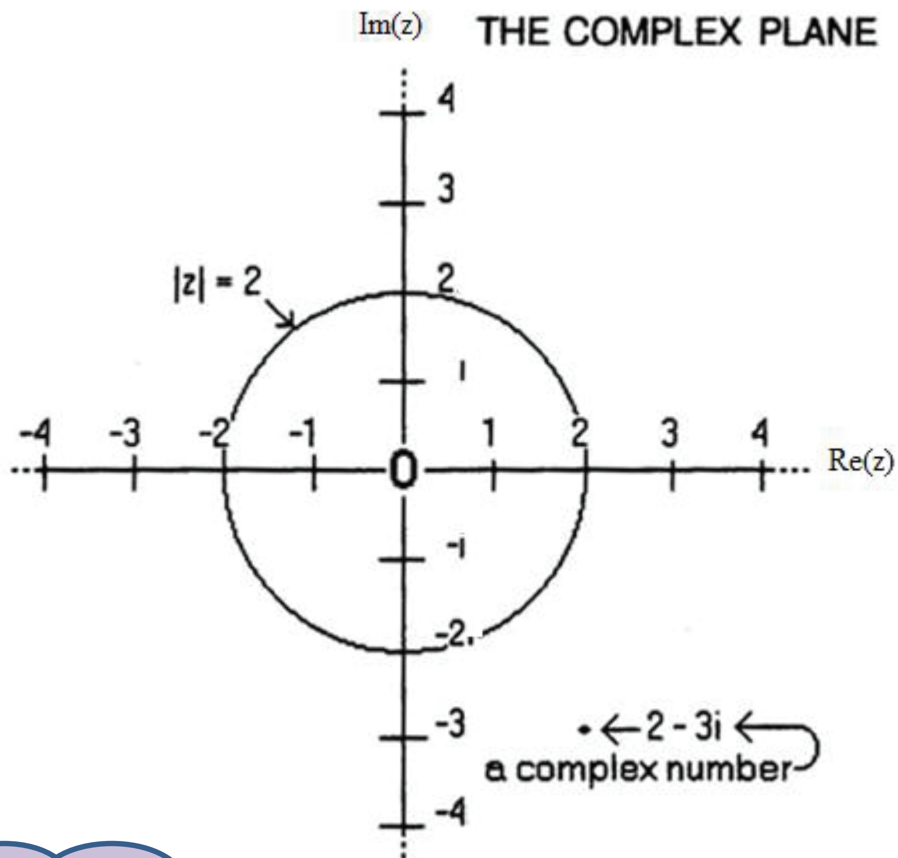
   a) $z^3 = 27(\cos 90° + i\sin 90°)$ b) $z^2 = 4(\cos 60° + i\sin 60°)$

DID YOU? KNOW

Adam and Eve are like an imaginary number, the square root of minus one. You can never see any concrete proof that it exists, but if you include it in your equations, you can calculate all manner of things that couldn't be imagined without it ~ — Philip Pullman

SUB – TOPIC 1.5

Curves and regions



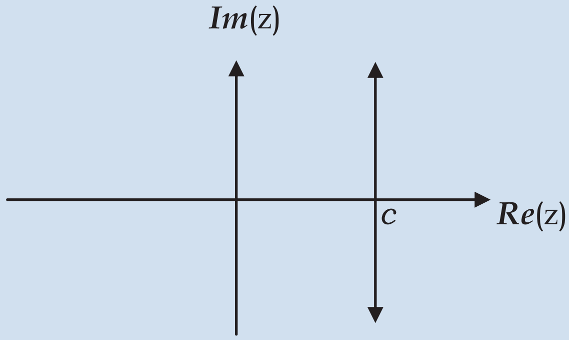
CONTENT LEARNING OUTCOMES

1.5.1 Sketch lines parallel to *x* and *y* axis.
1.5.2 Sketch rays.
1.5.3 Sketch circles given the equation $|z| = k$
1.5.4 Sketch regions associated with above lines and curves.

1.5.1 Horizontal and vertical lines

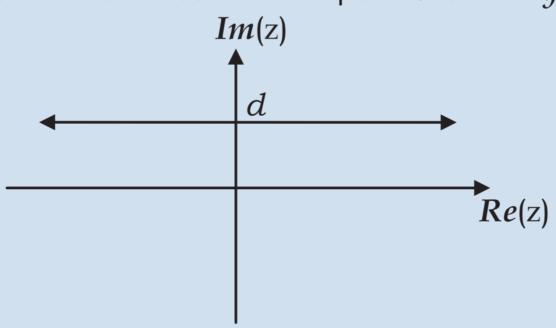
Re(z) refers to the real part which is shown as **vertical line** on the $x-$ **axis.**
For instance, sketch ***Re(z)*** = c

This means the set of all complex numbers with a real part of c, thus a vertical line will be drawn at point c on the $x-$ axis



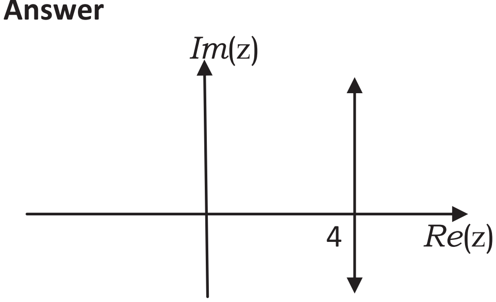
Im(z)refers to the imaginary part which is shown as the **horizontal line** on the $y-$ **axis.**
For instance, sketch ***Im(z)*** = d

This means the set of all complex numbers with a real part of d, thus a horizontal line will be drawn at point d on the $y-$ axis



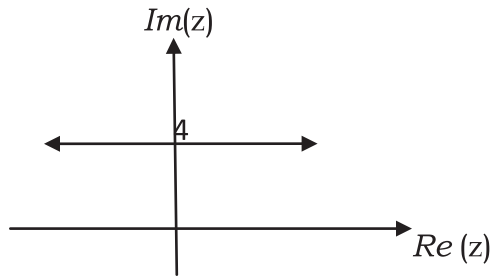
Example 1 Sketch ***Re(z)*** = 4

Answer



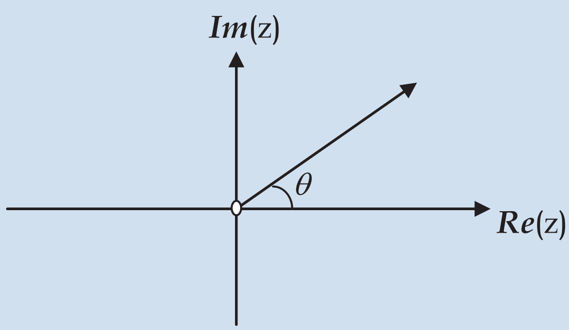
Example 2 Sketch $Im(z) = 4$

Answer



1.5.2 Rays

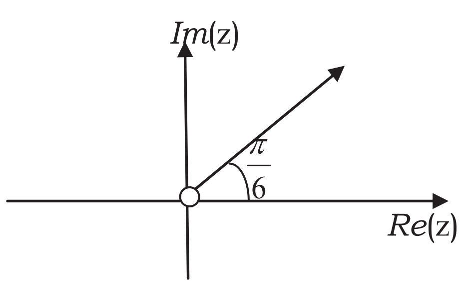
The $Arg(z) = \theta$ represents a ray from the origin making an angle θ with the positive x – **axis.** Note that the origin is excluded.



Example 3 Sketch $Arg(z) = \frac{\pi}{6}$

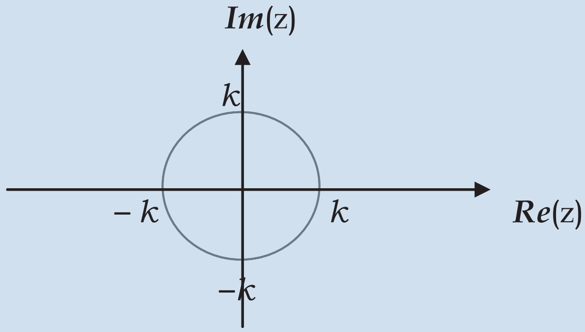
Answer

This is a ray from the origin making an angle of $\frac{\pi}{6}$ radians with the positive x-axis.



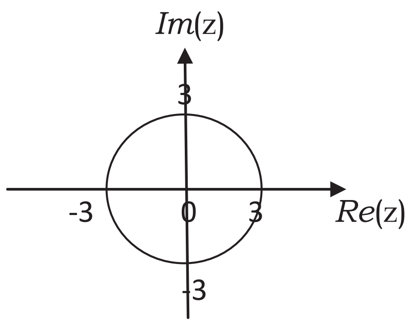
1.5.3 Circles

In general, the Graphs of Modulus Function, $|z| = k$ corresponds to the points on a circle with radius k and centre (0,0).



Example 4 Sketch $|z| = 3$

Answer



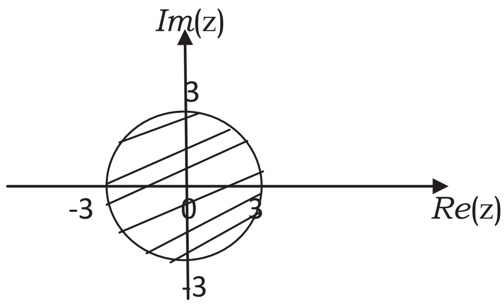
1.5.4 Regions given as inequality

Recall the sketching of graphs using the inequality signs like **<, >, ≤, or ≥** .
A solid line for **≤** or **≥**, and a dashed line for **<** or **>**.
Shade above or right of the line or outside the circle for **>** or **≥**
Shade below or left of the line or inside the circle for **<** or **≤**.

Example 5 Sketch $|z| \le 3$

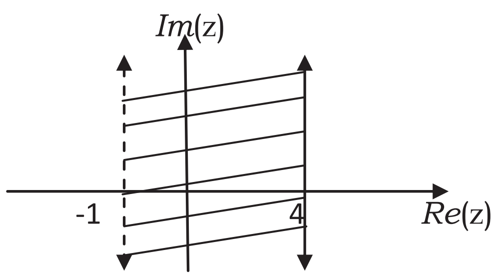
Answer

This contains all the points on and inside the circle with radius = 3



Example 6 Sketch -1< $Re(z) \le 4$

Answer



Exercise 1.5.1

On argand diagrams, sketch the following graphs

1. $|z| = 3.$
2. $|z| < 2.$
3. $|z| \ge 1.$
4. $Re(z) = 4$
5. $-3 < \mathrm{Re}(z) \le 2$
6. $0 \le \mathrm{Re}(z) < 1$
7. $Im(z) = -5$
8. $-1 < Im(z) \le 2$
9. $1 \le Im(z) < 5$
10. $Arg(z) = \frac{\pi}{3}$
11. $Arg(z) = 30^{\circ}$
12. $Arg(z) = -\frac{\pi}{2}$

Review Exercise 1

1. A complex number has $\mathrm{Re}(z)=-2$ and $\mathrm{Im}(z)=\sqrt{3}$. Find:
   a) $\overline{z}$, the conjugate of z, in rectangular form.
   b) $z\times\overline{z}$
   c) $\dfrac{1}{z}$ in the form $a+bi.$

2. Solve for x and y in $(-2+\sqrt{3}i)(x+yi)=3-2i$

3. Solve the equation $x^2-2x+4=0$.

4. A complex number z is given as $z=1-\sqrt{2}i$.
   a) Express z in polar form.
   b) Use De Moivre's Theorem to find z^4.
   c) Find all the fourth roots of $z=1-\sqrt{2}i$.

5. Sketch the following graphs:
   a) *Re(z)*= -3
   b) $|z|=4$
   c) *Arg(z)*= $-\dfrac{\pi}{3}$
   d) $-1\leq$ *Re(z)* ≤ 1

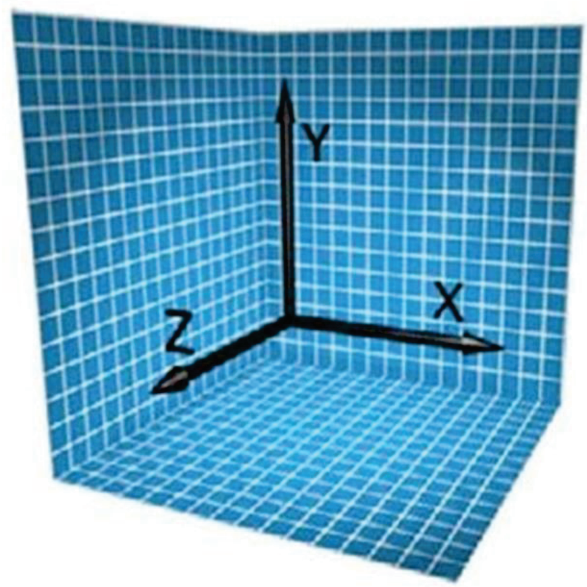
DID YOU? KNOW — *The complex numbers are used broadly in electrical engineering, quantum mechanics and complex mathematical theory* ~ anonymous

STRAND TWO

VECTORS

SUB – STRAND 2.1

Arithmetic Operations on vectors



LEARNING OUTCOME

- Apply arithmetic operations on vectors.

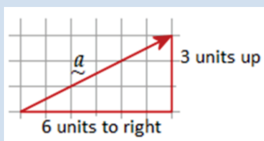
2.1.1 Three – Dimensional Vectors

A **vector** has both **magnitude** (length, or size) and **direction**. Recall Vectors in 2-dimensions. It is denoted by a skew symbol (~) at the bottom of the letter that is $\underset{\sim}{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. It can be drawn on the $x - y$ plane where the first element on the top represent the x – value and the second element at the bottom represents the y – value.

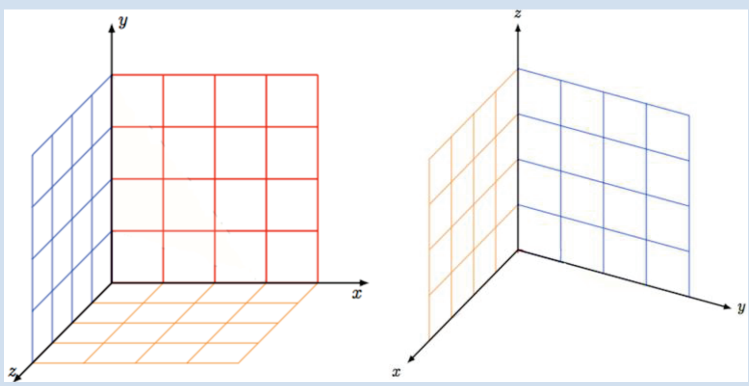
+ x: shift x units to the right side	– x: shift x units to the left side
+ y: shift y units upwards	– y: shift y units downwards

Example Draw vector $\underset{\sim}{a} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

Answers

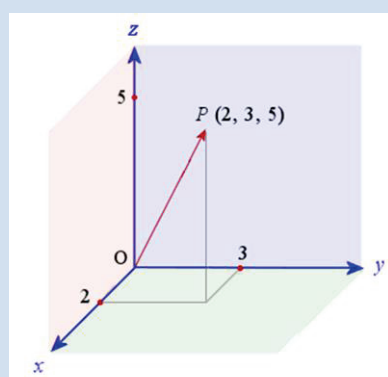


Extend this idea to represent 3-dimensional vectors. Three dimensional vectors comprises of x, y and z components, which are right—angles to each other.



Example

Draw vector $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$



2.1.2 Arithmetic Operations on Vector

2.1.2.1 Scalar Multiplication

The scalar is multiplied with each element of the vector.

Given that the vector $\underset{\sim}{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and k is a scalar, then $k\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$

Example 1 Evaluate $-2\begin{pmatrix} 3 \\ -2 \\ \frac{1}{2} \end{pmatrix}$

Answer

$$-2\begin{pmatrix} 3 \\ -2 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix}$$

Example 2 Given vector $\mathbf{p} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and vector $\mathbf{q} = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$.

Find the constant *k* such that $5\mathbf{p} + k\mathbf{q} = \begin{pmatrix} 20 \\ -14 \\ 7 \end{pmatrix}$

Answer

$$5\mathbf{p} + k\mathbf{q} = \begin{pmatrix} 20 \\ -14 \\ 7 \end{pmatrix} \Rightarrow 5\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + k\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 20 \\ -14 \\ 7 \end{pmatrix}$$

Multiply the scalar by each element of the vector.

$10 - 5k = 20 \quad \Rightarrow k = -2$

$-10 + 2k = -14 \quad \Rightarrow k = -2$

$15 + 4k = 7 \quad \Rightarrow k = -2$

$\therefore k = -2$

Solve any one of the equations to obtain the value of k.

2.1.2.2 Vector Addition and Subtraction

While adding or subtracting vectors, just add or subtract the respective components

Given Vector $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

$$\underset{\sim}{a} \pm \underset{\sim}{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}$$

Example 3

$\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, find

1. $\underset{\sim}{a} + \underset{\sim}{b}$
2. $\underset{\sim}{b} - \underset{\sim}{a}$

Answers

$\underset{\sim}{a} + \underset{\sim}{b}$

$$= \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

$\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{b} + (-\underset{\sim}{a})$

$$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$$

2.1.3 Finding Vector given two points

If two points P_1 and P_2 are known, the vector from P_1 to P_2 , $\overrightarrow{P_1P_2}$ is found by

$$\overrightarrow{P_1P_2} = P_2 - P_1$$

Example 4 Point P_1 = (1, -3, 4) and P_2 = (1, 3, -1)

Find the vector $\overrightarrow{P_1P_2}$.

Answer $\overrightarrow{P_1P_2} = P_2 - P_1$

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 3--3 \\ -1-4 \end{pmatrix}$$

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 0 \\ 6 \\ -5 \end{pmatrix}$$

Exercise 2.1

1. Vectors are given as $\underset{\sim}{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\underset{\sim}{y} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ and $\underset{\sim}{z} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.

   Evaluate the following:

   a) $\underset{\sim}{x} + \underset{\sim}{y} - \underset{\sim}{z}$ b) $2\underset{\sim}{x} - \underset{\sim}{z}$ c) $\underset{\sim}{x} + (-3)\underset{\sim}{y} - \underset{\sim}{z}$

2. Given vectors $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ and $\underset{\sim}{d} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$, find the constant k such that

   $$3\underset{\sim}{c} + k\underset{\sim}{d} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

3. Point P_1 = (0, -3, 4) and P_2 = (-1, 2, -1)

   Find the vector $\overrightarrow{P_1P_2}$.

4. Point P_1 = (0, -3, 4) and P_2 = (-1, 2, -1)

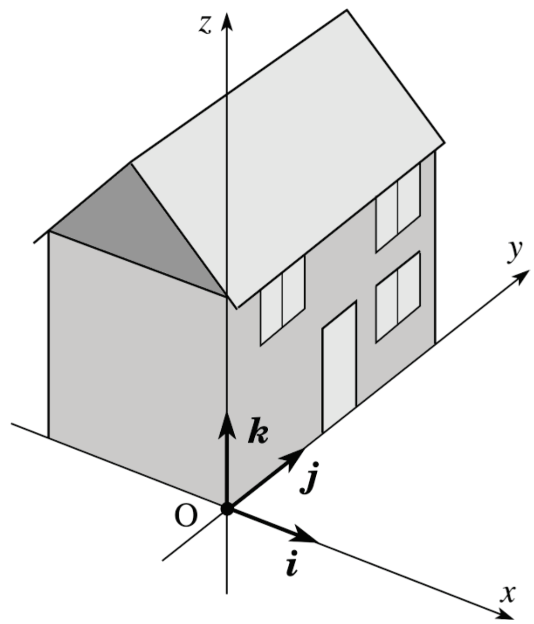
   Find the vector $\overrightarrow{P_1P_2}$.

DID YOU KNOW?

Vectors are used in everyday life to locate individuals and objects. They are also used to describe objects acting under the influence of an external force ~ Reference An IAC Publishing Labs Company

SUB – STRAND 2.2

Norm of a Vector and Unit vectors



LEARNING OUTCOMES

2.2.1 Find norm and unit vectors.

2.2.1 Norm and Unit Vectors

2.2.1.1 Norm of a vector

Norm of a vector $\underset{\sim}{a}$ is the length of the vector. It is also known as modulus or magnitude. The symbol is $|a|$ and is found using Pythagoras theorem:

$$\boxed{|a| = \sqrt{x^2 + y^2 + z^2}}$$

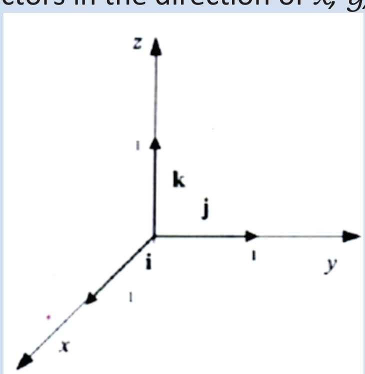
Example 1 Find the magnitude of $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

Answer

$$|\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26} \text{ or } 5.10$$

2.2.1.2 Unit Vectors

A unit vector is any vector which is of 1 unit length.
Unit vectors in the direction of x, y, and $z-$ axis are denoted by **i**, **j** and **k**.



- **i** , **j** and **k** in column vector can be represented as $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
- Normalizing a vector which is finding a unit vector in the same direction

$$\boxed{\frac{d}{|d|} \rightarrow \textit{unit vector in the direction of vector } d}$$

Example 2 Point P_1 = (3, 0, -1) and P_2 = (-2, 1, 4)

Find the vector $\overrightarrow{P_1P_2}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Answers

$$\overrightarrow{P_1P_2} = P_2 - P_1 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 5 \end{pmatrix}.$$

$$= -5\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

Example 2 The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are defined by:

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \underset{\sim}{b} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

a) Express $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

b) Find $\underset{\sim}{a} - 2\underset{\sim}{b}$ and express in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Answers

a) $\underset{\sim}{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\underset{\sim}{b} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

b)

$$\underset{\sim}{a} - 2\underset{\sim}{b}$$

$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} - 2(\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} - 2\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

$$= -\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$$

***Example* 4** A vector is given as $\mathbf{p} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$

a) Find the modulus of $\mathbf{p} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$

b) Find the unit vector in the direction of $\mathbf{p} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

✍ **Answers**

a) $|\mathbf{p}| = \sqrt{4^2 + (-4)^2 + 7^2} = \sqrt{81} = 9.$

b) Normalize

$$\frac{\mathbf{p}}{|\mathbf{p}|} = \frac{1}{9}(4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}) = \frac{4}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{7}{9}\mathbf{k}$$

Exercise 2.2.1

1. A vector is given as $\underset{\sim}{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$.

   a) Express in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

   b) Find the modulus of $\underset{\sim}{v}$.

   c) Find the unit vector in the direction of $\underset{\sim}{v}$.

2. Vectors are given as $\underset{\sim}{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\underset{\sim}{y} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ and $\underset{\sim}{z} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.

   a) Find $2\underset{\sim}{x} - 3\underset{\sim}{y}$

   b) Find the $|\underset{\sim}{x}|$, $|\underset{\sim}{y}|$ and $|\underset{\sim}{z}|$.

   c) Express the three vectors in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

3. Point P_1 = (-2, 0, -1) and P_2 = (-2, -5, 4). Find the vector $\overrightarrow{P_1P_2}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

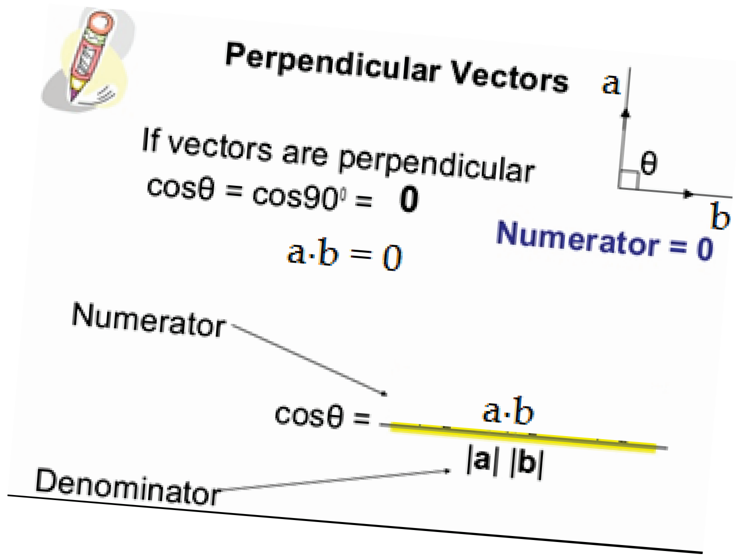
4. Vectors are given as $\underset{\sim}{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\underset{\sim}{y} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ and $\underset{\sim}{z} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.

   a) Express $\underset{\sim}{x}, \underset{\sim}{y}$ and $\underset{\sim}{z}$ in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

   b) Find the unit vector in the direction of $\underset{\sim}{z}$.

SUB – STRAND 2.3

Dot product and its application



LEARNING OUTCOMES

2.3.1 Compute scalar or dot product.

2.3.2 Calculate the angle between two vectors.

2.3.3 Identify parallel and perpendicular vectors.

2.3.1 Scalar product or Dot product

The result obtained will always be scalar.
Given $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$, then '$a$ dot b' is defined as

$$\text{Scalar (Dot) Product}: \quad \underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 1 Points A and B have position vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

Determine the scalar product of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Answer

$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 1 \times -2 + -2 \times 1 + -2 \times 3 \\ &= -2 - 2 - 6 \\ &= -10 \end{aligned}$$

2.3.2 Angle between two vectors

To find the angle between the two vectors, use formula

$$a \cdot b = |a||b| \cos\theta$$

Rearrange to get the angle:

$$a.b = |a||b|\cos\theta$$

$$\frac{a.b}{|a||b|} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{a.b}{|a||b|}\right)$$

Example Points A and B have position vectors $\underset{\sim}{a}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}-2\\1\\3\end{pmatrix}$.

i. Determine the scalar product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

ii. Find $|a|$ and $|b|$.

iii. Find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Answers

i. $\underset{\sim}{a}.\underset{\sim}{b}=a_1b_1+a_2b_2+a_3b_3$

$=1\times-2+-2\times1+-2\times3$

$=-2-2-6$

$=-10$

ii. $|a|=\sqrt{x^2+y^2+z^2}$

$=\sqrt{1^2+(-2)^2+(-2)^2}$

$=\sqrt{1+4+4}$

$=\sqrt{9}$

$=3$

$|b|=\sqrt{x^2+y^2+z^2}$

$=\sqrt{(-2)^2+1^2+3^2}$

$=\sqrt{4+1+9}$

$=\sqrt{14}$

iii. $\theta=\cos^{-1}\left(\frac{a.b}{|a||b|}\right)$

$=\cos^{-1}\left(\frac{-10}{3\times\sqrt{14}}\right)$

$=152.98^o$

2.3.3 Properties of Angle

Properties of Angle between two vectors

1. If two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, then the angle between them is either zero or 180^0, that is

$$\boxed{\theta = 0^0, 180^0}$$

One vector will be **scalar multiple** of the other.

2. If two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular to each other, the angle will be 90^o. This means that their dot product is zero. This is also known as Orthogonal Vectors.

$$\boxed{\begin{array}{l}\theta = 90^0 \\ \underset{\sim}{a} . \underset{\sim}{b} = 0\end{array}}$$

Example If $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

Show that the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal.

Answer

$$\begin{aligned}\underset{\sim}{a} . \underset{\sim}{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 1 \times 1 + -3 \times 3 + 4 \times 2 \\ &= 1 - 9 + 8 \\ &= 0\end{aligned}$$

Hence, it is orthogonal.

Exercise 2.3

1. Given $\underset{\sim}{a} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -6 \\ 2 \\ 8 \end{pmatrix}$

   a) Find the scalar product.
   b) Find the angle between the vectors.
   c) Are the vectors parallel or orthogonal? Explain.

2. Vectors are given as $\underset{\sim}{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\underset{\sim}{y} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ and $\underset{\sim}{z} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$. Find the scalar product of:

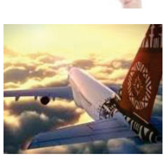
   a) $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

   b) $\underset{\sim}{x}$ and $\underset{\sim}{z}$.

3. Given vector $p = \begin{pmatrix} -2 \\ +4 \\ +3 \end{pmatrix}$ and $q = \begin{pmatrix} +1 \\ -3 \\ +2 \end{pmatrix}$. Find the angle between p and q.

4. Given vectors $\underset{\sim}{a} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$. Show that the two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

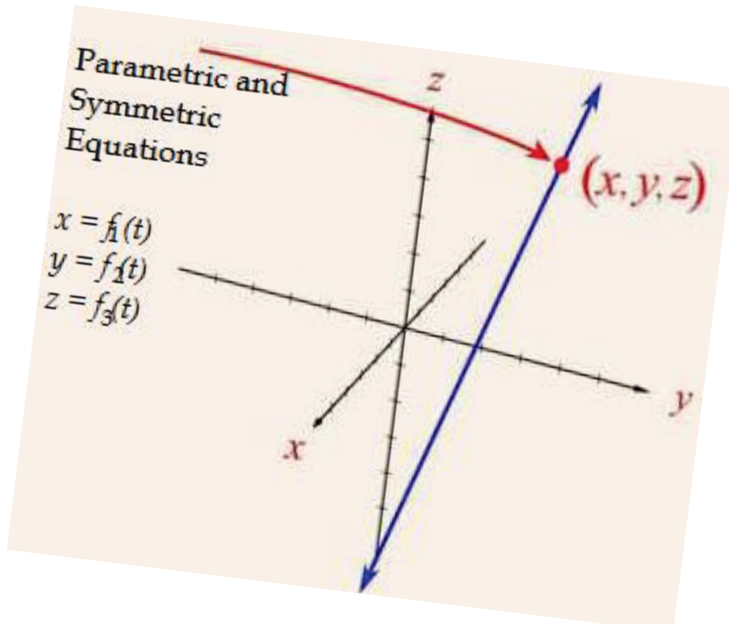
5. Given vectors $\mathbf{c} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ y \\ -2 \end{pmatrix}$, find the value of y if $\mathbf{c}$ and $\mathbf{d}$ are orthogonal.



A great example would be how airplane pilots receive instructions to land at airports. During a visual approach, the Air Traffic Control instructs pilots to fly a particular heading (direction) for a certain distance (magnitude). This is exactly what a vector quantity is- something that has a magnitude and direction. This is why the Air Traffic Controllers might sometimes use the phrase "expect vectors for the visual approach..." when the plane nears the airport.~ Krishnaprasad Bindumadhavan

SUB – TOPIC 2.4

Equation of lines



LEARNING OUTCOME

2.4.1 Determine equation of lines: vector, parametric and symmetric equations.

2.4.1 Equation of Lines

A line which passes through a point (x_0, y_0, z_0) and is parallel to the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

has Vector equation: $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}}$

Parametric equation: $\boxed{x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct}$

Symmetric Equation:

Symmetric Equation is also known as Cartesian Equation of line. This form of equation does not include parameter, t.

<u>Finding Symmetric Equation from Parametric Equation</u> :

$$x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$$

Making t the subject of each equation: $t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Hence, **Symmetric or Cartesian Form o**f an equation of a line which passes through a point (x_0, y_0, z_0) and is parallel to the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Example 1 Find the equation of the line passing through (0, 3, -1) and parallel to the vector $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

Answers

Vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

Parametric Equation $x = t,\quad y = 3 - t,\quad z = -1 + 3t$

Make t the subject for each equation: $\frac{x-0}{1} = \frac{y-3}{-1} = \frac{z+1}{3}$

Simplifying gives: $x = 3 - y = \frac{z+1}{3}$

Thus, the equation of the line in **symmetric form** is $x = 3 - y = \frac{z+1}{3}$

Example 2 Find the equation of a line passing through the points P_1 = (3, 0, -1) and P_2 = (-2, 1, 4) .

Answers

Find the vector from P_1 to P_2.

$$\overrightarrow{P_1P_2} = P_2 - P_1 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 5 \end{pmatrix}.$$

P_1 = (3, 0, -1) is the point on the line so the **vector equation** is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -5 \\ 1 \\ 5 \end{pmatrix}$$

The **parametric equation** is $x = 3 - 5t, \quad y = t, \quad z = -1+5t$

[Note: We can also use point P_2]

Example 3: The equation of a line is given as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

a) Give the coordinates of a point that lies on the line.

b) Give a vector which is parallel to the line.

Answers

a) (0, 3, -1)

b) $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

Example 3 The **symmetric equation** of a line is given as $\frac{3-x}{-2}=y+1=\frac{4z-8}{-4}$

Write the **parametric** and **vector** equation of this line.

Answers

The general form of symmetric equation of the line is

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

The given equation can be rewritten as

$$\frac{-(x-3)}{-2}=\frac{y+1}{1}=\frac{4(z-2)}{-4} \Rightarrow \frac{x-3}{2}=\frac{y+1}{1}=\frac{z-2}{-1} \Rightarrow \frac{x-3}{2}=\frac{y-(-1)}{1}=\frac{z-2}{-1}$$

The point on the line is (3, -1, 2) and the vector is $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

Thus, the **vector equation** of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}+t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Make x, y and z the subject to get the **parametric equation**

$$x=3+2t$$
$$y=-1+t$$
$$z=2+(-t)$$

Exercise 2.4.1

1. Find the vector, parametric and symmetric equations of the line passing through P_1 = (-2, 3, 5) and P_2 = (3, -1, -2).

2. Write down the vector ,parametric and symmetric equations of the line passing through (2, -1, 3) in the direction of $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

3. For the line $\frac{x+2}{3}=\frac{4-y}{2}=2z+4$
   a) State the coordinates of a point that lies on this line.
   b) State the direction vector.
   c) Write the vector and parametric equation of the line.

4. Write down the vector equation of the line parallel to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$, and passing through the point (1, 4, -2)

5. Find the parametric equation of a line through (-2, 0, 5) that is parallel to the line $x = 3 + 2t, \quad y = 4 - t, \quad z = 6 + 2t$

6. For the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$,

   a) State the coordinates of a point that lies on this line.
   b) State the direction vector.
   c) Write the equation of the line in parametric form.

7. The vector equations of two lines $\mathbf{L1}$ and $\mathbf{L2}$ are given as

$\mathbf{L1}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$\mathbf{L2}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

   a) Give the direction vector for $\mathbf{L1}$.
   b) Write the coordinates of a point on $\mathbf{L2}$.
   c) Express $\mathbf{L1}$ in symmetric form.

8. The parametric equation of a line is x = 3 – 5t, y = 2– t, z = t

   a) State the coordinates of a point that lies on this line.
   b) State the direction vector.
   c) Write the equation of the line in vector form.

DID YOU KNOW?

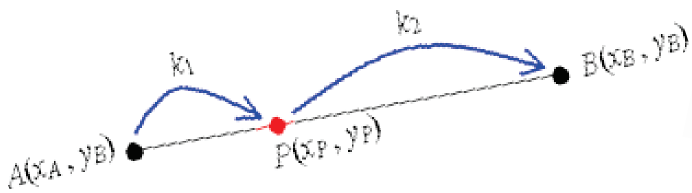
Vectors are important in modeling and solving real – life problems. For example in an airplane, a dot product would give the combined effect of the coordinates in different dimensions on each other. It would actually be the product with cosine of their mutual angle. ~ Ahmad Bokhari

SUB – TOPIC 2.5

Division of a Line Segment

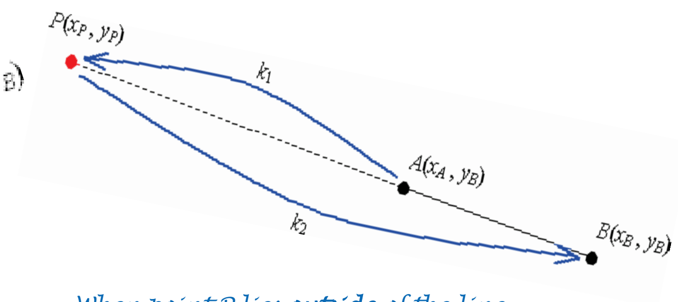
Theory:

(a) Internal division of a line segment:



When point P lies **between** points A and B, it divides AB **internally**.

(b) External division of a line segment:



When point P lies **outside** of the line segment AB, it can divide AB **externally**.

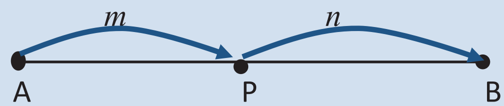
LEARNING OUTCOME

2.5.1 Find the coordinates of the point which divides a line segment in a given ratio.

2.5.1 Ratio Formula

2.5.1.1 Dividing a line segment in the ratio $m : n$

- **Finding a point which divides the interval internally** in the ratio of $m : n$



Let's divide interval AB in $(m + n)$ parts and denote point P as the point where it is m sub-intervals away from point A and n sub-intervals from point B.

Consider first point A = (a_1, a_2, a_3) and the second point B = (b_1, b_2, b_3)

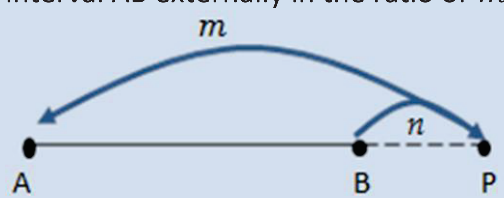
The **position vectors** of points A and B are A = $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ B = $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

If point P divides the line segment in the ratio $m : n$, then

$$\boxed{\mathrm{P} = \frac{m\mathrm{B} + n\mathrm{A}}{m+n}}$$

- **Finding the point which divides the line segment externally**

The division of the interval AB externally in the ratio of $m : n$ would be given as

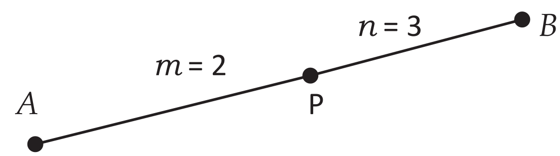


Note that point P is not on interval AB but an extension of interval AB. Point P is formed as it is moved back from B by a ratio of n sections.

In this case n would be negative.

Example 1 Point A has the coordinate (6, 1, 5) and point B has the coordinate (-1, 4, 3). Point P divides the line AB such that AP: AB = 2 : 5. Find the coordinates of point P.

Answer



Note the ratio carefully: AP= 2 , AB = 5 so PB =3

Finding the point:

$$\mathrm{P} = \frac{m\mathrm{B} + n\mathrm{A}}{m+n}$$

$$= \frac{2\begin{pmatrix}-1\\4\\3\end{pmatrix} + 3\begin{pmatrix}6\\1\\5\end{pmatrix}}{2+3}$$

$$= \frac{\begin{pmatrix}-2\\8\\6\end{pmatrix} + \begin{pmatrix}18\\3\\15\end{pmatrix}}{2+3}$$

$$= \begin{pmatrix}16\\11\\21\end{pmatrix} \div 5$$

$$= \frac{1}{5}\begin{pmatrix}16\\11\\21\end{pmatrix}$$

$$= \begin{pmatrix}3.2\\2.2\\4.2\end{pmatrix}$$

$\therefore P\ (3.2\ ,\ 2.2\ ,\ 4.2)$

Example 2 A and B are the points $(2,1,3)$ and $(-2,5,-4)$, respectively. Find the coordinates of point X on the line AB such that $AX = \frac{1}{3}XB$.

✍ **Answer**

We are given $\frac{AX}{XB}=\frac{m}{n}=\frac{1}{3}$, that is, $m=1$ and $n=3$. →

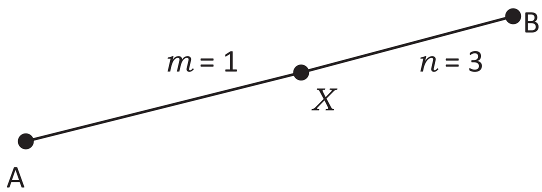
$$X=\frac{m\mathrm{B}+n\mathrm{A}}{m+n}$$

$$=\frac{1\begin{pmatrix}-2\\5\\-4\end{pmatrix}+3\begin{pmatrix}2\\1\\3\end{pmatrix}}{1+3}$$

$$=\frac{\begin{pmatrix}-2\\5\\-4\end{pmatrix}+\begin{pmatrix}6\\3\\9\end{pmatrix}}{4}$$

$$=\frac{1}{4}\begin{pmatrix}4\\8\\5\end{pmatrix}$$

$$=(1,2,{}^{5}\!/_{4})$$



Example 3 A and B are the points (-1, 2, 4) and (3, 0, -2), respectively. Find the coordinates of point P on the line AB such that $\frac{AP}{PB}=\frac{-1}{3}$.

✍ **Answer**

$$\mathrm{P}=\frac{m\mathrm{B}+n\mathrm{A}}{m+n}$$

$$=\frac{-1\begin{pmatrix}3\\0\\-2\end{pmatrix}+3\begin{pmatrix}-1\\2\\4\end{pmatrix}}{-1+3}$$

$$=\frac{\begin{pmatrix}-3\\0\\2\end{pmatrix}+\begin{pmatrix}-3\\6\\12\end{pmatrix}}{2}$$

$$=\frac{1}{2}\begin{pmatrix}-6\\6\\14\end{pmatrix}$$

$$=(-3,3,7)$$

Exercise 2.5.1

1. If P is the point (1, 1, 0) and R is the point (1, 6, -5) , find the coordinates of a point Q on the line PR given that $PQ: QR$ = 3:2.

2. If R and S are the points $(-3,4,1)$ and $(5,0,-1)$, respectively, find the coordinates of point P such that $RP:RS=1:4$.

3. A and B are the points (-3, 5, 10) and (12, -5, -15), respectively. Find the coordinates of point P on the line AB such that $\frac{AP}{AB}=\frac{2}{5}$.

4. If Q is the point (-2, 1, 1) and R is the point (-2, 4, -5) , find the coordinates of a point P on the line QR given that $QR = 2PR$.

5. Point A has the coordinate $(3,-5,2)$ and point B has the coordinate $(6,-5,-1)$. Point P divides the line AB such that $AP:PB=-2:5$. Find the coordinates of point P.

6. A is the point (1, 5, -4) and B is the point (7, -3, -1) on a line AB. Find the coordinates of P on this line such that AP : PB = 6 : −1.

7. A is the point (-3, 4, -2) and B is the point (1, 2, -3) on a line AB. Find the coordinates of P on this line such that AP : PB = 5 : −1.

Review Exercise 2

1. Given vectors $c = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ and $d = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, find the constant k such that

   $3c + k\,d = \begin{pmatrix} -7 \\ 14 \\ 0 \end{pmatrix}$.

2. Given vector $\mathbf{p} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$, find the scalar product of $\mathbf{p}$ and $\mathbf{q}$.

3. Are the vectors $c = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ and $d = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ Orthogonal ?

4. Find the unit vector in the direction of $\underset{\sim}{p} = 2\underset{\sim}{i} - \underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$.

5. Given $\underset{\sim}{a} = 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

   find
   a) $2a - 3b$
   b) $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$
   c) the scalar product
   d) the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

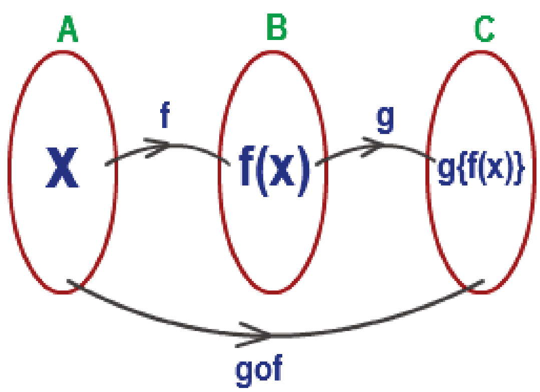
6. Find the parametric equation of a line through the origin that is parallel to the line $x = t$, $y = -1 + t$, $z = 6 + 2t$

STRAND THREE

FUNCTIONS

SUB – STRAND 3.1

Operations on Functions



The domain of the composite function f(g(x)) is the set of all x in the domain of g such that g(x) is in the domain of f.

The composition of the function f with the function g is defined by

$$(f \circ g)(x) = f[g(x)]$$

Two step process to find y = f(g(x)):

1. ***Find h = g(x).***
2. ***Find y = f(h) = f(g(x))***

LEARNING OUTCOMES

3.1.1 Apply arithmetic operations and operation of composition on various functions.

3.1.2 Give domain and range of combined functions.

3.1.1 New Functions from Old

➢ Combining functions

If $f(x)$ and $g(x)$ are the two functions, there are many ways we can combine them:

1. **Addition**

$$(f+g)(x) = f(x) + g(x)$$

2. **Subtraction**

$$(f-g)(x) = f(x) - g(x)$$

3. **Multiplication**

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

4. **Division**

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$$

➢ A composite function, denoted by ‘o’ , is defined by

$$f \circ g(x) = f(g(x))$$

Example If $f(x) = x^2$ and $g(x) = x + 4$, then

$f \circ g(x) = f(x+4) = (x+4)^2$

$g \circ f(x) = g(x^2) = x^2 + 4$

Example 1 Functions f and g are given as $f(x) = 3x^2 + 2$ and $g(x) = 5x - 2$.

Find the following:

a) $(f + g)(x)$

b) $(f - g)(x)$

c) $(f \cdot g)(x)$

d) $\left(\dfrac{f}{g}\right)(x)$

e) $f \circ g(x)$

f) $g \circ f(x)$

Answers

a) $(f+g)(x) = f(x) + g(x)$

$= (3x^2 + 2) + (5x - 2)$

$= 3x^2 + 5x + \cancel{2 - 2}$

$= 3x^2 + 5x$

b) $(f-g)(x) = f(x) - g(x)$

$= (3x^2 + 2) - (5x - 2)$

$= 3x^2 - 5x + 2 + 2$

$= 3x^2 - 5x + 4$

c)

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$= (3x^2 + 2) \cdot (5x - 2)$

$= 3x^2 . 5x - 2 . 3x^2 + 2 . 5x + 2 . -2$

$= 15x^3 - 6x^2 + 10x - 4$

d)

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{3x^2 + 2}{5x - 2}$$

e) $f \mathbf{o}\, g(x) = f\,[5x - 2]$

$= 3(5x - 2)^2 + 2$

$= 3(25x^2 - 20x + 4) + 2$

$= 75x^2 - 60x + 12 + 2$

$= 75x^2 - 60x + 14$

A Short-cut way

$(5x - 2)^2$

$= (5x)^2 - 2 . 2 . 5x + (-2)^2$

$= 25x^2 - 20x + 4$

f) $g\, \mathbf{o}\, f(x) = g[3x^2 + 2]$

$= 5(3x^2 + 2) - 2$

$= 15x^2 + 10 - 2$

$= 15x^2 + 8$

3.1.2 Domain and Range of Combined Functions

- The domain of addition, subtraction and multiplication of two functions $f(x)$ and $g(x)$ is the intersection of the domains of f and g.

$$d_{f+g} = d_{f-g} = d_{f \bullet g} = d_f \cap d_g$$

- The domain of $\frac{f}{g}$ has an additional restriction that $g(x) \neq 0$

$$d_{\frac{f}{g}} = d_f \cap d_g, g(x) \neq 0$$

- The domain of $f \mathbf{o}\, g(x)$ = **domain of the inside function**, $g(x)$
  $f \mathbf{o}\, g(x)$ exists if the **Range of** $g(x)$ is a **subset** of the Domain of $f(x)$.

Example 2 Functions f and g are as follows

$f(x)=\sqrt{x+3}$ and $g(x)=\sqrt{2-x}$.

Give the domain of $(f+g)(x)$

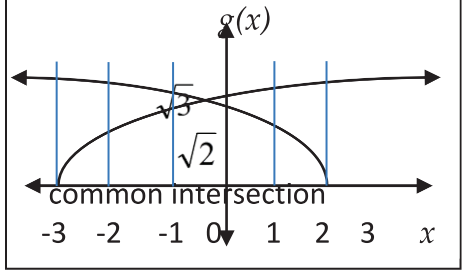
Answer

$$d_f=\{x:x\geq -3\} \qquad d_g=\{x:x\leq 2\}$$

$$d_{f+g}=d_f\cap d_g$$

$$=\{x:-3\leq x\leq 2\}$$

The diagram below shows the graph of $f(x)$ = $\sqrt{x+3}$ and $g(x)$ = $\sqrt{2-x}$



Example 3 If $f(x) = x + 4$ and $g(x) = \sqrt{x}$

a) Find $f \circ g(x)$.

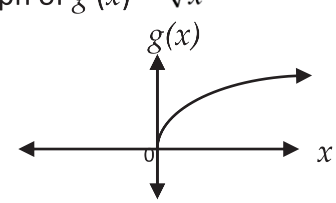
b) State the domain $f \circ g(x)$.

Answers

a) $f \circ g(x) = f(\sqrt{x}) = \sqrt{x} + 4$

b) The domain of $(f \circ g)(x)$ will be given by the domain of the inside function, g.

The diagram below shows the graph of $g(x)$ = $\sqrt{x}$



domain of g is $x \geq 0$

$\therefore$ domain of $(f \circ g)(x)$ is $\{x: x \geq 0, x \in \mathrm{R}\}$

Example 4 Functions f and g are defined by

$$f : x \rightarrow x^2 - 1$$
$$g : x \rightarrow \sqrt{x+2}$$

Find:

b) $f \mathbf{o}\, g\,(\,x\,)$.

c) the domain of $f \mathbf{o}\, g\,(\,x\,)$.

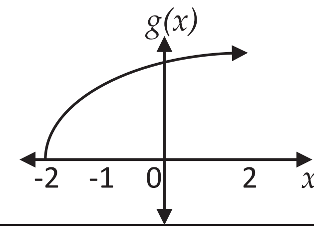
d) the range of $f \mathbf{o}\, g\,(\,x\,)$.

Answers

a)

$$(fog)(x) = f\left(g(x)\right) = \left(\sqrt{x+2}\right)^2 - 1 = (x+2) - 1 = x+1$$

The diagram below shows the graph of $g(x) = \sqrt{x+2}$



b) The domain of $f \circ g\,(\,x\,) =$ domain of g

domain of g is $x \geq -2$

Therefore, domain of $f \mathbf{o}\, g$ is $x \geq -2$

c) Since domain is $x \geq -2$, substitute x = -2 in the expression for $f \mathbf{o}\, g\,(\,x\,)$ which is the **smallest domain** of $f \circ g\,(\,x\,)$

$f \mathbf{o}\, g\,(\,x\,) = x + 1$

$= (-2) + 1$

$= -1.$

(If we substitute any number bigger than -2, we will get answer more than -1)

So the range is $y \geq -1$

Exercise 3.1

1. f and g are defined as $f : x \rightarrow x^2 + 1$ and $g : x \rightarrow \sqrt{x-5}$

Find and state its domain:

a) $(f + g)(x)$ b) $(f - g)(x)$

c) $(f \cdot g)(x)$ d) $\left(\dfrac{f}{g}\right)(x)$

2. f and g are defined as $f : x \to x^2 + 1$ and $g : x \to \dfrac{1}{x-4}$

State :

a) the domain of f.

b) the range of g.

c) $f \circ g\,(x)$.

3. f and g are defined $f : x \to x^2 + 1$ and $g : x \to \sqrt{x-2}$

a) State the domain of g.

b) Find $f \circ g\,(x)$.

c) State the domain of $f \circ g\,(x)$.

d) State the range of $f \circ g\,(x)$.

4. The function f and g are defined by $f : x \to x^2$ and $g : x \to x - 3$

Find:

(a) $g \circ f\,(x)$.

(b) the domain and range of $g \circ f\,(x)$.

5. The function f and g are defined by $f(x) = \dfrac{1}{1-x}$ and $g(x) = x^2 + 3$

Find:

a) the domain of $f(x)$.

b) the range of $f(x)$.

c) the domain of $g(x)$.

d) the range of $g(x)$.

e) $g \circ f\,(x)$.

Some of the real life applications of rational functions are

(i) *fields and forces in physics*

(ii) *spectroscopy in analytical chemistry*

(iii) *kinetics in biochemistry*

(iv) *electronic circuitry*

(v) *aerodynamics*

(vi) *medicine >concentrations in vivo*

(vii) *wave functions for atoms and molecules*

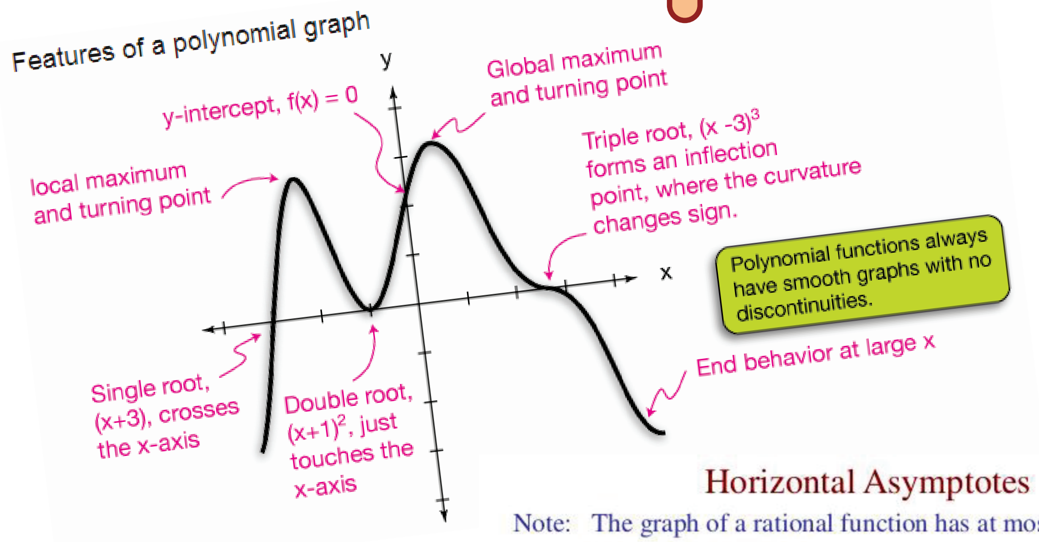
(viii) *optics and >photography to improve image resolution*

(ix) *acoustics and sound.* ~ Wikipedia

(https://math.stackexchange.com/questions/900576/real-world-applications-of-rational-functions)

SUB – STRAND 3.2

Graphs of Functions



Horizontal Asymptotes

Note: The graph of a rational function has at most one horizontal asymptote.

If the degree of the numerator is less than the denominator, the horizontal asymptote is y = 0.

Example: $y = \frac{1}{x}$

If the degree of the numerator is more than the denominator, there is NO horizontal asymptote .

Example: $y = 2x + 1$

If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is y = a / b , where a and b are the leading coefficients of the numerator and the denominator.

Example: $y = \frac{x^2 + x - 2}{2x^2 - x - 1}$

LEARNING OUTCOMES

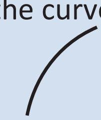
3.2.1 Draw graphs of polynomials.
3.2.2 Draw graphs of rational functions.

3.2.1 Graphs of Polynomials

- What is concave upward / downward?

Concave upward
all the tangents on the interval are below the curve

Concave downward
all tangents are above the curve



- An **Inflection Point** is where a curve changes from **Concave upward** to **Concave downward** (or vice versa).

- Factored Polynomial

The graph of polynomials are smooth, unbroken curve, with no sharp corners.

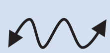
Let a polynomial function be of degree n. To sketch its graph, follow these steps.

Step 1 Find y-intercepts.
Step 2 Find x-intercept.
Step 3 Determine turning point and inflection

> For a factor $(x - a)^n$,
> if n is odd then there is an inflection at $x = a$ and
> if n is even there is a turning point at $x = a$.

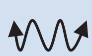
Step 4 Sketch the graph.

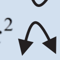
The degree of x in the expansion of the polynomial is odd, the end behavior of $f(x)$ will be similar to the graph of x^3.

$+x^3$  i.e.  tails pointing opposite direction

$-x^3$  i.e. 

The degree of x in the expansion of the polynomial is even, the end behavior of $f(x)$ will be similar to the graph of x^2.

$+x^2$  i.e. 

$-x^2$  i.e. 

<u>Note:</u>
You may also use **test points** within the intervals formed by the x-intercepts to determine the sign of $f(x)$ in the interval. This will determine whether the graph is above or below the x-axis in that interval.
When the values of $f(x)$ is negative, the graph is below the x-axis, and when $f(x)$ has positive value, the graph is above the x-axis.

Example 1 The graph of the polynomial is given by the equation:

$$f(x)=(x+1)^2(x-2)^3.$$

a) Find x and y-intercepts
b) Identify the coordinates of turning point on the x-axis
c) Identify the coordinates of an inflexion point on the x-axis
d) Sketch the graph

✍ **Answers**

a) x-intercepts, let y = 0

$$y=(x+1)^2(x-2)^3$$

$$0=(x+1)^2(x-2)^3$$

$$either\ (x+1)^2=0\ or\ (x-2)^3=0$$

$$x=-1\ and\ \ x=2$$

the coordinates of x-intercepts are (−1,0) and (2,0)

y-intercepts, let x = 0

$$f(x)=(x+1)^2(x-2)^3$$

$$y=(0+1)^2(0-2)^3$$

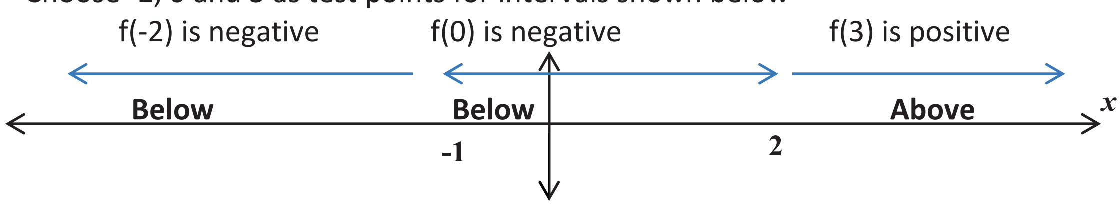
$$=-8$$

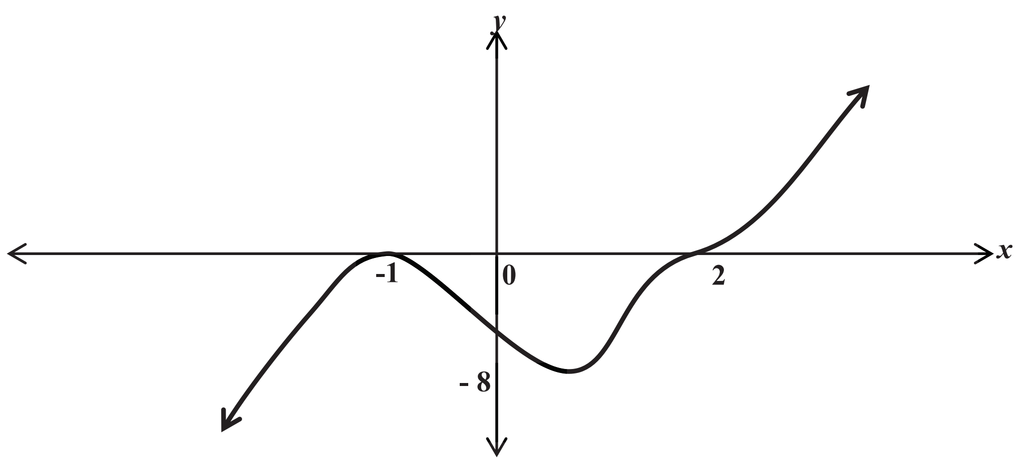
the coordinate of y-intercept is (0, − 8)

b) At $x=-1$, there is a turning point ∴ (−1,0)

c) At $x=2$ there is an inflection ∴ (2,0)

d) Sketch the graph
Sign of $f\ (\ x\ )$ in each interval separated by x-intercepts.
Choose -2, 0 and 3 as test points for intervals shown below





Example 2 Sketch the graph of $y = x^3(x-2)^2(x+1)$ [Clearly show all the intercepts, the turning points and point of inflection.]

Answer

x-intercepts, let y = 0

$y = x^3(x-2)^2(x+1)$

$0 = x^3(x-2)^2(x+1)$

$0 = x^3,\ 0 = (x-2)^2,\ 0 = x+1$

$x = 0,\ \ x = 2,\ \ x = -1$

the coordinates of x-intercepts are
(0,0): inflection point
(2,0): turning point
(−1,0): cuts at x - axis

y-intercepts, let x = 0

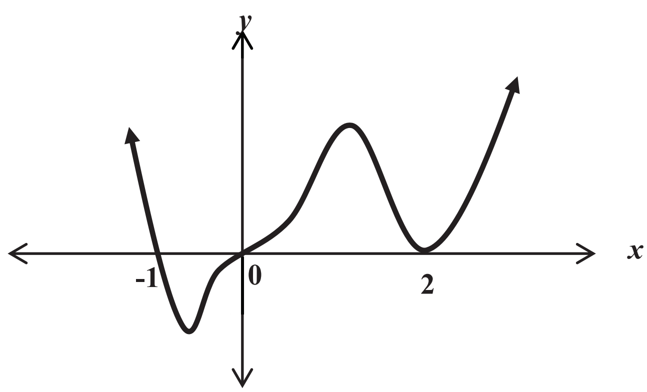
$y = x^3(x-2)^2(x+1)$

$y = 0^3(0-2)^2(0+1)$

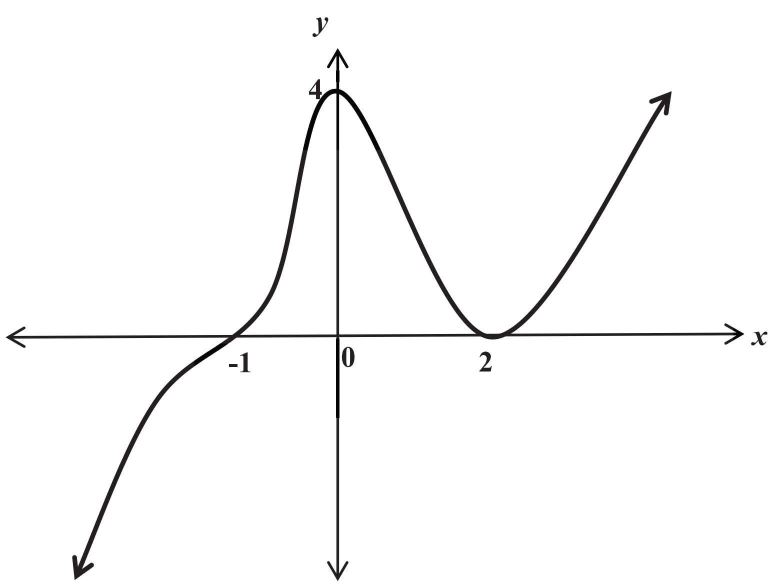
$y = 0$

the coordinate of y-intercept is
(0,0)

Sketch the graph:



Example 3 Write the equation of the polynomial shown.



✍ **Answer**

x-intercepts: $x = -1, \; x = 2$

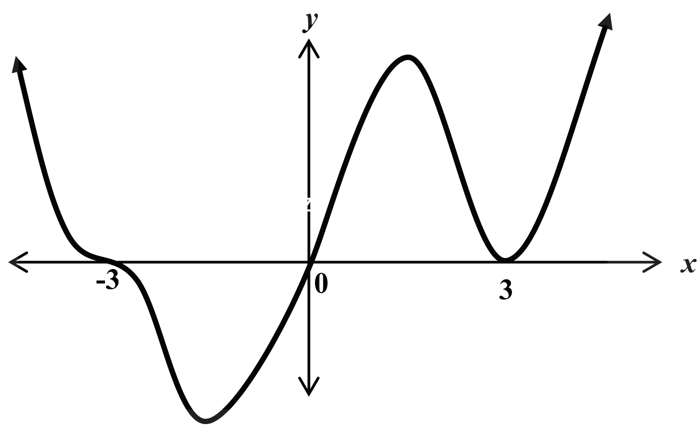
y-intercept = 4

inflection point at x = -1 and turning point at x = 2

$$\therefore y = (x+1)^3(x-2)^2$$

Exercise 3.2.1

1. Write the equation of the polynomial shown.



2. A polynomial function is given by $f(x) = x^2(x+3)^3(2-x)$
   a) Find x and y intercepts of $f(x)$
   b) Sketch the graph of the function $f(x)$ clearly showing the behaviour of the graph around the x-intercepts.

3. A polynomial function is given as $y = f(x)$. Given that the function satisfies $f(0) = f(2) = f(5) = 0$ and $f(3) = 24$,
   a) state the x and y intercepts of the function.
   b) find the equation of the function.
   c) sketch the graph of the function.

4. Sketch the following graphs by clearly showing all the intercepts, the turning points and point of inflection.
   a) $y = x^3(x-2)^2(1-x)$
   b) $y = (x+2)^2(1-x)^3$

3.2.2 Graph of Rational functions

- **Rational functions** are quotients of polynomial functions. This means that rational functions can be expressed as $\boxed{f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0}$

  where $p(x)$ and $q(x)$ are polynomial functions.

- The **domain** of a rational function is the set of all real numbers **except** the x-values that make the denominator zero.

To sketch its graph, follow these steps.

Step 1 Find x-intercepts by letting $p(x) = 0$, i.e. the **numerator equal to 0**

Step 2 Find y-intercept, by letting $x = 0$ that is find $f(0)$.

Step 3 Find the asymptotes using the following procedures:

1. Vertical Asymptotes

Find any vertical asymptotes by setting the **denominator equal to 0** and solving for x. If a is a zero of the denominator, then the **line $x = a$ is a vertical asymptote.**

2. Other Asymptotes

Determine any other asymptotes. Consider three possibilities:

(a) If the denominator has a higher degree than numerator i.e. a **bottom heavy function**, then there is a **horizontal asymptote $y = 0$** (the x-axis).

(b) If the numerator and denominator have the same degree i.e. a **balanced function**,

$$f(x) = \frac{ax^n + \ldots\ldots}{bx^n + \ldots} \qquad \text{where } a, b \neq 0,$$

then the **horizontal asymptote** has equation $y = \dfrac{a}{b}$

(c) If the numerator is of degree exactly one more than the denominator (**top heavy function**) , then there will be an **oblique (slanted) asymptote.**

To find it, divide the numerator by the denominator and disregard the remainder. Set the rest of the quotient equal to *y* to obtain the equation of the asymptote.

Step 4 Sketch the graph

Example 1 A rational function is given as $f(x)=\dfrac{x+3}{(x-3)(x+2)}$

a) Find the x and y intercepts of the graph of $f(x)$.
b) Identify the asymptote(s) and find their equations
c) Sketch the graph of the function $f(x)$ clearly showing the intercepts and asymptotes.

Answers

a) x intercept

let *numerator* = 0 :

$x + 3 = 0$
$x = -3$
x- intercept = (−3,0)

y intercept: let x = 0:

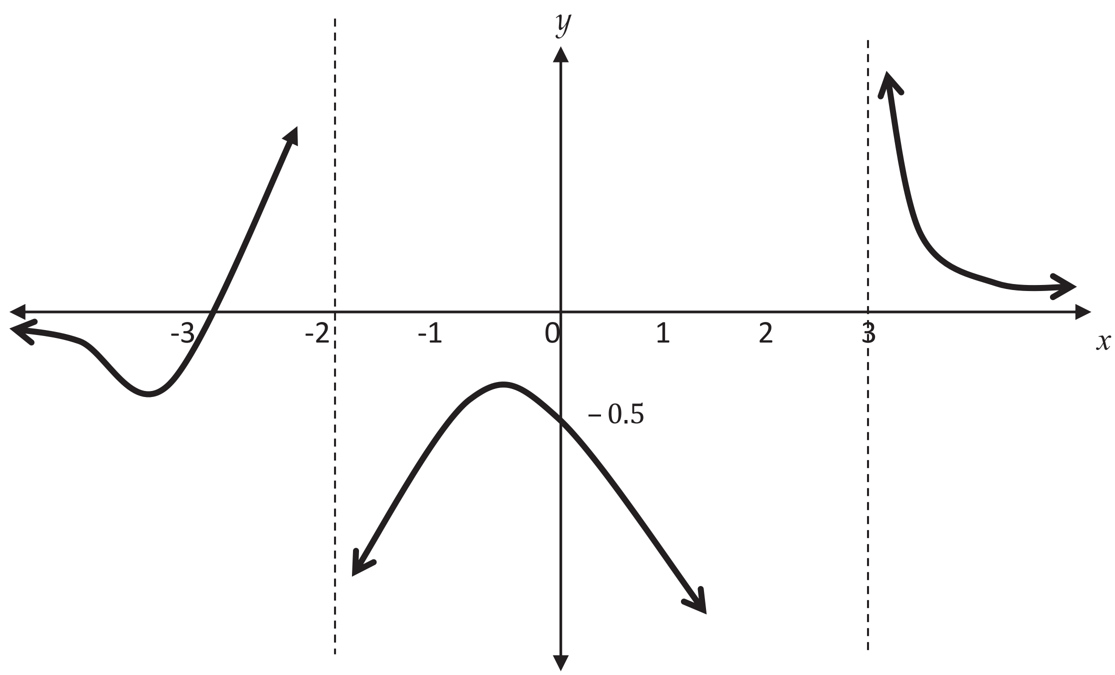
$$f(0)=\frac{(0+3)}{(0-3)(0+2)}=-\frac{1}{2}$$

y-intercept = (0,– 0.5)

b) vertical asymptote
let *denominator* = 0 :
$x - 3 = 0$ and $x + 2 = 0$
$x = 3$ and $x = -2$ are the equations

Since it's a **bottom heavy function**, the horizontal asymptote is $y = 0$

c) Do a sign analysis in each interval separated by vertical asymptotes or x-intercepts. This tells you where each section of the graph appears!
Show asymptotes as dashed lines. The graph must approach the asymptotes.



Example 2 A rational function is given as $f(x) = \dfrac{(x-3)(x+1)}{(x-1)(x+2)}$

a) Sketch the graph of $f(x)$

b) Complete the following, using your graph:

As $x \to +\infty, \quad f(x) \to \boxed{}$

Answers

a)

x-int → let y = 0:

$x - 3 = 0, \; x+1 = 0$

x- intercepts: (3, 0) and (−1, 0)

Vertical Asymptote:

let denominator = 0

$x - 1 = 0, \; x+2 = 0$

$x = 1$ and $x = -2$

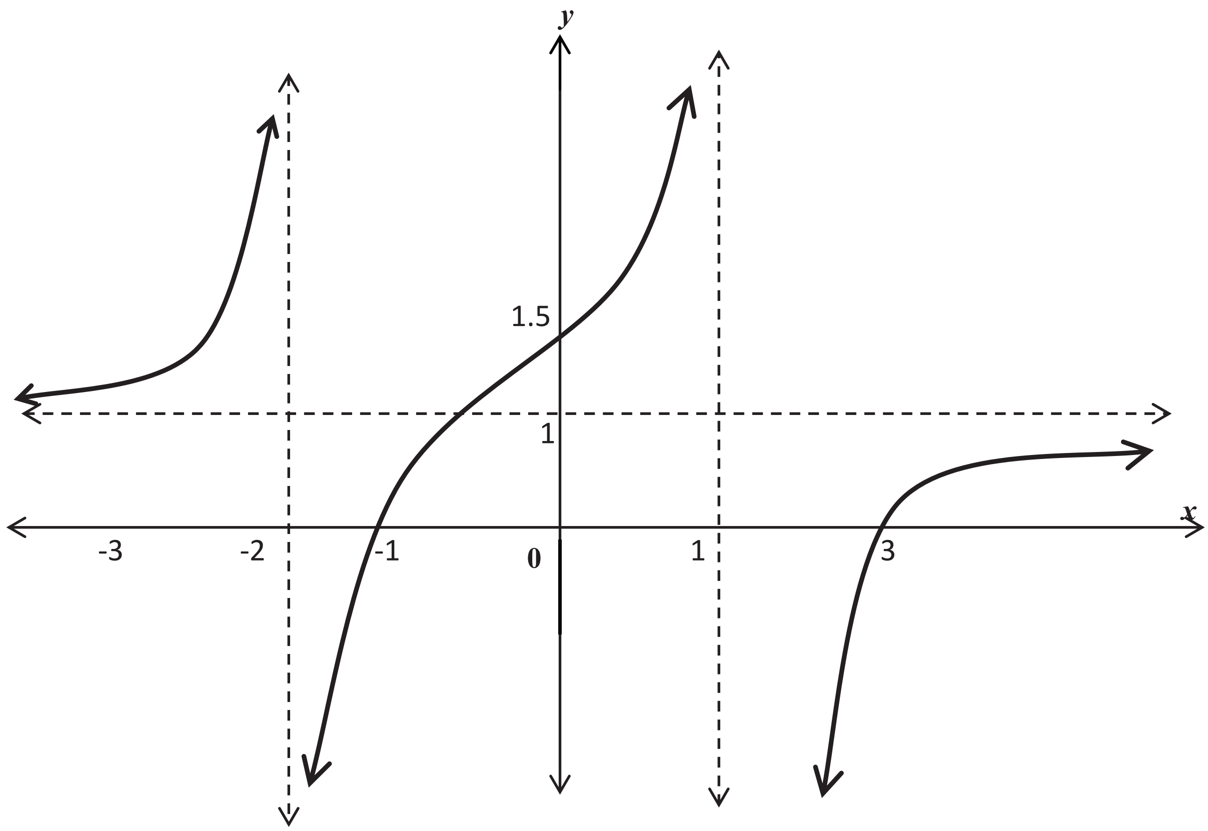
y-int → let x = 0: $f(x) = \dfrac{(0-3)(0+1)}{(0-1)(0+2)}$

y-intercept = (0, 1.5)

Horizontal Asymptote:

It's a balanced function. Consider the ratio of the coefficients of the variable with the highest power:

$y = \dfrac{x^2}{x^2} \Rightarrow y = 1$



b) As $x \to +\infty, \quad f(x) \to \boxed{1}$

Example 3 Sketch the graph of $y = \dfrac{x^2}{x-1}$

Answer

x intercept

let *numerator* = 0 :

$x^2 = 0$

$x = 0$

y intercept: let x = 0:

$f(0) = \dfrac{0^2}{(0-1)} = 0$

Both x- intercept and y-intercept = (0,0)

Vertical Asymptote let denominator = 0 :

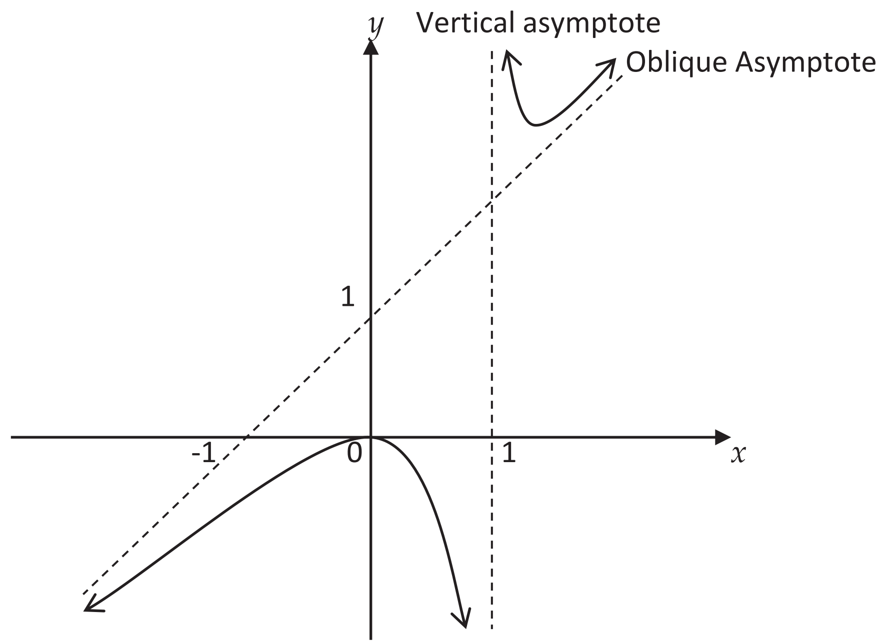
$x - 1 = 0$

$x = 1$

Since it's a top heavy function, carry out the long division

$$\begin{array}{r|l} & \mathbf{x+1} \\ \hline x-1 & x^2 + 0 \\ & \underline{-(x^2 - x)} \\ & \quad x + 0 \\ & \quad \underline{-(x - 1)} \\ & \qquad 1 \end{array}$$

$\therefore$ $y = x + 1$ is the Oblique/slanting Asymptote [recall it's a straight line graph with x-int (-1,0) and y-int (0,1)]



Exercise 3.2.2

For the following functions:

a) Find x and y intercepts.

b) Identify the asymptotes and give their equations.

c) Sketch the graph, clearly showing the intercepts and the asymptote.

1. $f(x) = \dfrac{(x-3)}{(2x+1)(x-2)}$

2. $f(x) = \dfrac{(2-x)}{(x^2-1)}$

3. $g(x) = \dfrac{(1-x)(x+2)}{(x+1)}$

4. $g(x) = \dfrac{(x+4)(2-x)}{(2x+1)(x+1)}$

5. $h(x) = \dfrac{x^2-x-6}{x+1}$

6. $h(x) = \dfrac{x^2+4x+3}{x-1}$

7. $y = \dfrac{(x+1)(x+2)}{(x-1)(x+3)}$

8. $y = \dfrac{(x+1)(3-2x)}{(x-1)(x+2)}$

DID YOU KNOW?

The length of a shadow is a function of its height and the time of day. Shadows can be used to find the height of large objects such as trees or buildings; the same function rule (ratio) by which we compare the length of an upright ruler to its shadow will help us find the unknown input (the height of the large object) when we measure its shadow. ~ Wendy Petti (http://www.educationworld.com/a_curr/mathchat/mathchat010.shtml)

Review Exercise 3

1. Two functions are defined as

   $f : x \to x^2 - 4 \qquad g : x \to x + 3$

   a) State the domain and range of the two functions
   b) Find $(fog)(x)$.
   c) What are the domain and range of $(fog)(x)$?

2. The functions f and g are defined as: $f(x) = \dfrac{1}{3-x}$ and $g(x) = x^2 + 2$

   a) State the domain and range of $f(x)$.
   b) State the range of $g(x)$.
   c) Find $g \circ f(x)$.
   d) Find the range of $g \circ f(x)$.

3. Sketch the polynomial functions shown below:

   a) $y = x(x-2)^2(1-x)^3$
   b) $y = \frac{1}{2}(x+4)^2(2-x)^3$
   c) $y = (x+3)^2(2-x)^3$

4. A rational function is given as $h(x) = \dfrac{2-x}{(x+3)(x-1)}$.

   a) Find the x and y intercepts of the graph of $h(x)$.
   b) Identify the asymptote(s) and give their equations.
   c) Sketch the graph.
   d) Complete the following statement:
      As $x \to \infty, \quad h(x) \to \square$

STRAND FOUR

TRIGONOMETRY

SUB – STRAND 4.1

Trigonometric Identities and Exact Values

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$	$\cos\theta = \frac{1}{\sec\theta}$	$\tan\theta = \frac{1}{\cot\theta}$
$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

LEARNING OUTCOMES

4.1.1 Prove identities.

4.1.2 Find exact values of trigonometric functions.

4.1.1 Identities

- **Basic identities/formulae**

• $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	• $\cos^2\theta + \sin^2\theta = 1$	• $\tan^2\theta + 1 = \sec^2\theta$
• $\cot\theta = \dfrac{1}{\tan\theta}$	• $\sec\theta = \dfrac{1}{\cos\theta}$	• $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

- **Addition Formulae**

• $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
• $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
• $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

- **Product to sum**

• $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
• $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
• $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

- **Sum to product**

• $\sin C + \sin D = 2\sin(\text{half sum}) \cos(\text{half difference})$
• $\sin C - \sin D = 2\cos(\text{half sum}) \sin(\text{half difference})$
• $\cos C + \cos D = 2\cos(\text{half sum}) \cos(\text{half difference})$
• $\cos D - \cos C = 2\sin(\text{half sum}) \sin(\text{half difference})$

- **Double angle**

• $\sin 2A = 2\sin A\cos A$	• $\cos 2A = 2\cos^2 A - 1$ $= \cos^2 A - \sin^2 A$ $= 1 - 2\sin^2 A$	• $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

- **Half angle**

• $\sin(\frac{A}{2}) = \pm\sqrt{\dfrac{1-\cos A}{2}}$	• $\cos(\frac{A}{2}) = \pm\sqrt{\dfrac{1+\cos A}{2}}$	• $\tan(\frac{A}{2}) = \dfrac{1-\cos A}{\sin A}$

*The sign depends on which quadrant the angle is in.

Derivation of Half angle formulae from double angle formula

Consider $\cos 2A = 2\cos^2 A - 1$

Let A = $\frac{\theta}{2}$

$\cos 2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$\cos\theta + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$

$\cos^2\left(\frac{\theta}{2}\right) = \frac{\cos\theta + 1}{2}$

Consider $\cos 2A = 1 - 2\sin^2 A$

Let A = $\frac{\theta}{2}$

$\cos 2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos\theta$

$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$

Proving Identities

Example 1 Show that $\sin 60^o - \sin 40^o = 2\cos 50^o \sin 10^o$

Answer

Sum to product Identity that can be used:

- $\sin C - \sin D = 2\cos(\text{half sum})\ \sin(\text{half difference})$

$$
\begin{aligned}
LHS &= \sin 60^0 - \sin 40^0 \\
&= 2\cos\left(\frac{60^\circ + 40^\circ}{2}\right)\sin\left(\frac{60^\circ - 40^\circ}{2}\right) \\
&= 2\cos 50^\circ \sin 10^\circ \\
&= RHS
\end{aligned}
$$

Example 2 Prove that tan θ cosec θ = sec θ

Answer

Use basic identities

$$
\begin{aligned}
LHS &= \tan\theta \times \text{cosec}\,\theta \\
&= \frac{\cancel{Sin\theta}}{Cos\theta} \times \frac{1}{\cancel{Sin\theta}} \\
&= \frac{1}{Cos\theta} \\
&= \text{Sec}\,\theta \\
&= RHS
\end{aligned}
$$

Example 3 Prove that $\frac{1}{1+\sin^2\theta}+\frac{1}{1+\csc^2\theta}=1$

Answer

Identity that can be used: $\csc^2\theta=\frac{1}{\sin^2\theta}$

$$\frac{1}{1+\sin^2\theta}+\frac{1}{\left[1+\frac{1}{\sin^2\theta}\right]}=1$$

Simplify:

$$\left[1+\frac{1}{\sin^2\theta}\right]$$

$$=\frac{\sin^2\theta+1}{\sin^2\theta}$$

$$LHS=\frac{1}{1+\sin^2\theta}+\left(1\div\left[\frac{\sin^2\theta+1}{\sin^2\theta}\right]\right)$$

$$=\frac{1}{1+\sin^2\theta}+1\times\frac{\sin^2\theta}{\sin^2\theta+1}$$

$$=\frac{1+\sin^2\theta}{\sin^2\theta+1}$$

$$=1$$

$$=RHS$$

Example 4 Prove: $\frac{Sin\,5x-Sin\,3x}{Cos\,5x+Cos\,3x}=\tan x$.

Answer

Identity that can be used: Sum to product

- $\sin C-\sin D=2\cos(\text{half sum})\ \sin(\text{half difference})$
- $\cos C+\cos D=2\cos(\text{half sum})\cos(\text{half difference})$

$$LHS=\frac{Sin\,5x-Sin\,3x}{Cos\,5x+Cos\,3x}$$

$$=\frac{2Cos\left(\frac{5x+3x}{2}\right)Sin\left(\frac{5x-3x}{2}\right)}{2Cos\left(\frac{5x+3x}{2}\right)Cos\left(\frac{5x-3x}{2}\right)}$$

$$=\frac{2Cos4xSinx}{2Cos4xCosx}$$

$$=\tan x$$

$$=RHS$$

Exercise 4.1.1

Prove these identities:

1. $\sin^2 x - \cos^2 x = 2\sin^2 x - 1$

2. csc 2 $\boldsymbol{\theta}$ − cot 2 $\boldsymbol{\theta}$ = tan $\boldsymbol{\theta}$

3. $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \cos\theta$

4. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \sec\theta \cos ec\theta$

5. tan$\boldsymbol{\theta}$ − cot $\boldsymbol{\theta}$ = $\dfrac{\mathbf{1-2cos^2\theta}}{\mathbf{\sin\theta\cos\theta}}$

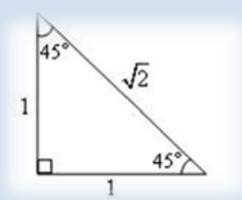
6. $\dfrac{Sin\theta}{Co\sec\theta + Cot\theta}$ = 1 − Cos θ

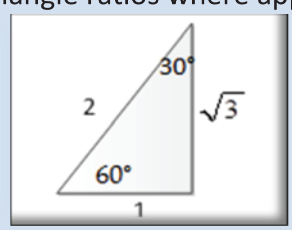
7. $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = 2\sec^2 x$

8. $\dfrac{Sin2\theta + Sin5\theta}{Cos2\theta - Cos5\theta} = \cot\dfrac{3x}{2}$

4.1.2 Exact Values

➢ You may use special triangle ratios where applicable.





Trig Function SOH / CAH / TOA	Angles		
	30^o	45^o	60^o
Sin θ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
Cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
Tan θ	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Example 1 Find the exact value of Sin 75°

Answer

Break down angle preferably as 30°, 45° or 60° as their exact values are known.

Sin 75° = Sin (30° + 45°)

Use Addition Formulae $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

= Sin 30° · Cos 45° + Cos 30° · Sin 45°

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

Example 2 Find the exact value of $\sin 15°$

Answer

$$\sin 15° = \sin \frac{30°}{2}$$

Use half angle formula

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\left(\frac{30}{2}\right) = \sqrt{\frac{1-\cos 30}{2}}$$

Take only positive square root as $15°$ falls in the first quadrant

$$\sin(15°) = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2} \div 2}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2} \div \frac{2}{1}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2} \times \frac{1}{2}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}}$$

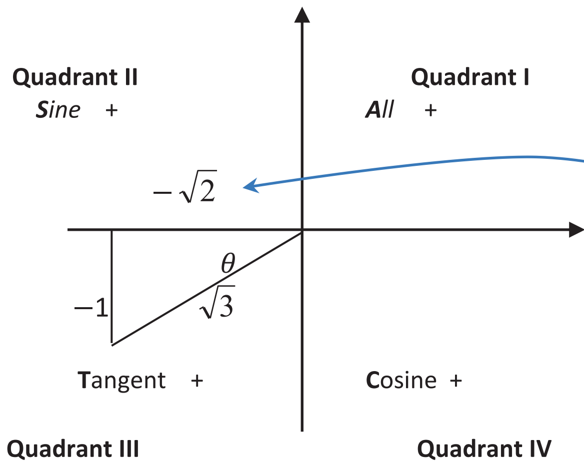
$$\therefore \sin(15°) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

👂 **Example 3** If $\sin\theta = -\frac{1}{\sqrt{3}}$, find sec $\boldsymbol{\theta}$**,** if θ is in quadrant III.

✍ **Answer**

Using SOH/CAH/TOA, the expression for $\sin\theta = \frac{opposite}{hypotenuse} = -\frac{1}{\sqrt{3}}$

Consider a right-angled triangle,



By Pythagoras theorem

$$c^2 = a^2 + b^2$$
$$\sqrt{3}^2 = x^2 + 1^2$$
$$3 = x^2 + 1$$
$$2 = x^2$$
$$x = \sqrt{2}$$

Use identity ***sec θ =*** $\frac{1}{\cos\theta}$

First, get the expression for $\cos\theta$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$= -\sqrt{2}/\sqrt{3}$$

cosine is negative in quadrant III

sec θ = $\frac{1}{\cos\theta}$

$$= \frac{1}{-\sqrt{2}/\sqrt{3}}$$
$$= 1 \div -\frac{\sqrt{2}}{\sqrt{3}}$$
$$= -\frac{\sqrt{3}}{\sqrt{2}}$$

Exercise 4.1.2

1. Find the values of the following in simplest surd form:

   $a)$ $\cos 15°$ $\qquad$ $b)$ $\sin 22\frac{1}{2}°$

   $c)$ $\tan 75°$ $\qquad$ $d)$ $\sin 105°$

   $e)$ $\cos 105°$ $\qquad$ $f)$ $\tan 15°$

2. Find the exact value of R where, $R = 8\cos 75°$.

3. Find the exact value of R where, $R = 4\cos 75° - 4\cos 15°$.

4. Suppose $\tan A = \frac{24}{7}$, $\tan B = \frac{9}{40}$ with A and B as acute angles, find the exact value of $\sin(A - B)$.

5. Suppose $\sin A = \frac{9}{41}$ and $\cos B = \frac{24}{25}$, find the **exact value** of $\cos(A - B)$.

6. If $\cos\theta = -\frac{1}{\sqrt{2}}$ and θ is in the second quadrant, find the following:

   a) $\cos 2\theta$ $\qquad$ b) $\sec\theta$

   c) $\tan\theta$ $\qquad$ d) $\operatorname{cosec}\theta$

7. Suppose $\sin\theta = \frac{\sqrt{5}}{3}$ and θ is in the second quadrant, find the exact value of $\sin 2\theta$.

8. Suppose θ is an angle in the fourth quadrant and $\cos\theta = \frac{20}{29}$. Find the exact value of $\sin 2\theta$ and $\cos 2\theta$

DID YOU KNOW?

Trigonometry is used by astronomers to calculate the distance to the stars. Trigonometry to measure distances between universe objects are at greater distances. ~ anonymous (http://math.tutorvista.com/trigonometry/applications-of-trigonometry.html)

SUB – STRAND 4.2

Trigonometric Graphs

NOTE

Amplitude and Period

For graphs of the form

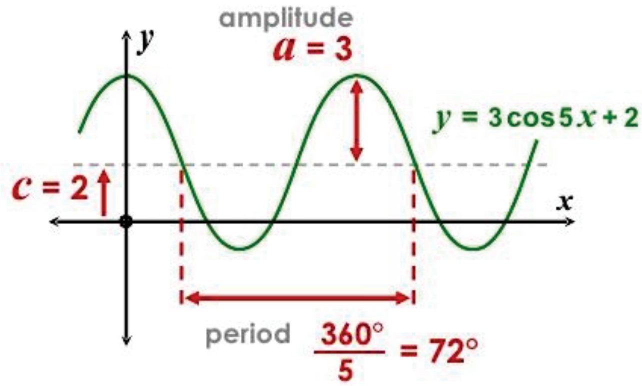
$$y = a \sin bx + c$$

and $$y = a \cos bx + c$$

amplitude $= a$

period $= \dfrac{360°}{b}$

Example



LEARNING OUTCOME

4.2.1 Draw trigonometric graphs.

4.2.1 Trigonometric Graphs

The general form is defined as

$$y = A\ Sin\ (Bx \pm C) \pm k \qquad or \qquad y = A\ Cos\ (Bx \pm C) \pm k$$

$$y = A \ \begin{matrix} Sin \\ Cos \end{matrix} \ (B\,x \pm C) \pm k$$

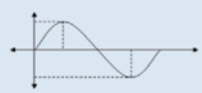
Amplitude: The height / distance

$+A$ means the graph is oriented as usual

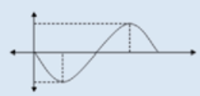
$-A$ means that the graph is inverted

Shape:

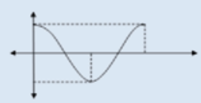
$+$ A Sin



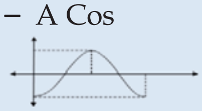
$-$ A Sin



$+$ A Cos



$-$ A Cos



Period: B helps determine the *period* of the graph (the length of the interval needed for the graph of the function to start repeating itself).

$$period = \frac{360}{B} \quad or \quad \frac{2\pi}{B}$$

C shifts the y-axis or the graph by $\frac{C}{B}$ units.

$+\frac{C}{B}$ shifts the y-axis to the right or the graph moves by $\frac{C}{B}$ units to the left.

$-\frac{C}{B}$ shifts the y-axis to the left or the graph moves by $\frac{C}{B}$ units to the right

$\boldsymbol{k}$ shifts the x-axis or the graph by $\boldsymbol{k}$ units.

$+\boldsymbol{k}$ shifts the x-axis down by $\boldsymbol{k}$ units or the graph moves up by $\boldsymbol{k}$ units

$-\boldsymbol{k}$ shifts the x-axis up by $\boldsymbol{k}$ units or the graph moves down by $\boldsymbol{k}$ units

Example 1 A trigonometric function is defined as $f(x) = 3\ sin\ (\ x + \frac{\pi}{4}\ )$

(i) Write the period of the function $f(x)$.

ii) What is the amplitude of $f(x)$?

iii) Sketch $f(x) = 3\ sin\ (\ x + \frac{\pi}{4}\ )\ for\ \ 0 \leq x \leq 2\pi$

iv) Write down the coordinates of the maximum point of $f(x)$ $for\ \ 0 \leq x \leq 2\pi$

✍ **Answers**

Compare with the general form

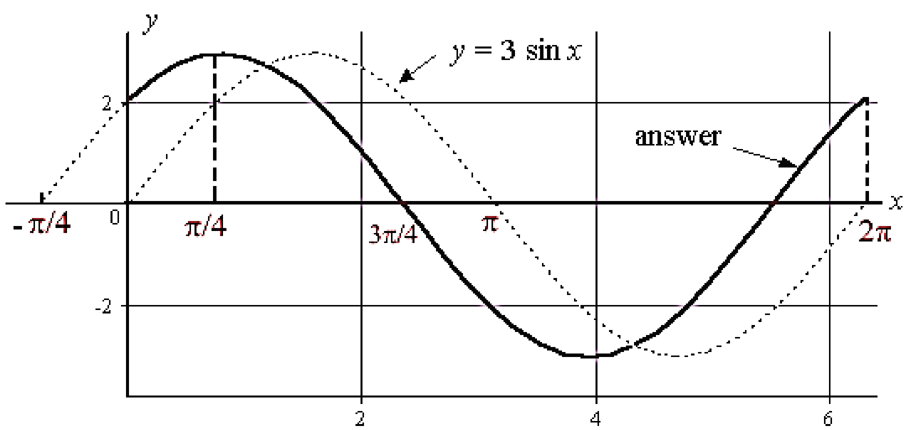
$y = A\ Sin\ (Bx \pm C) \pm k$

$f(x) = 3\ sin\ (\ 1x + \frac{\pi}{4}\ )$

i. period $= \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$

ii. amplitude: A = 3

iii. C: Shift the graph $\frac{\pi}{4}$ units to the left or Shift the y - axis by $\frac{\pi}{4}$ units to the right



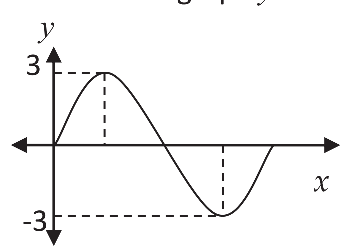
iv. maximum point: reading from the graph, it turns at $x = \frac{\pi}{4}$ and $y = 3$ $\therefore (\frac{\pi}{4}, 3)$

👂 **Example 2** Sketch $y = 3\ sin\ (\ x + \frac{\pi}{2}\ ) + 3\ \ for\ \ 0 \le x \le 2\pi$

✍ **Answer**

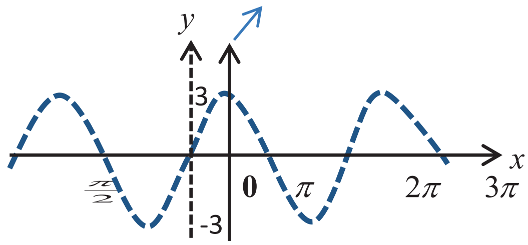
Method 1 – Sifting the axes

First sketch the basic graph $y = 3\ sin\ x$,

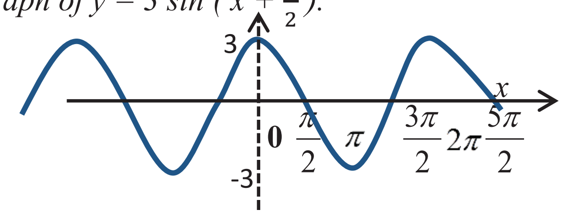


Then shift y axis to right by $\frac{\pi}{2}$ *to get* $y = 3 \sin (x + \frac{\pi}{2})$.

y-axis shifts to the right

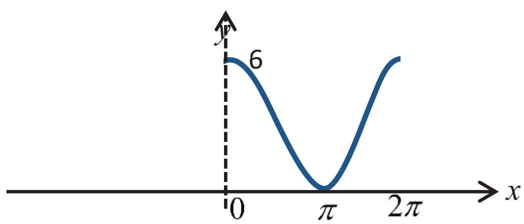


Graph of $y = 3 \sin (x + \frac{\pi}{2})$.



Finally shift x - axis down by 3 unit.

$\therefore$ *graph of* $y = 3 \sin (x + \frac{\pi}{2}) + 3$



Exercise 4.2.1

1. Sketch the following graphs for $0 \le x \le 2\pi$.

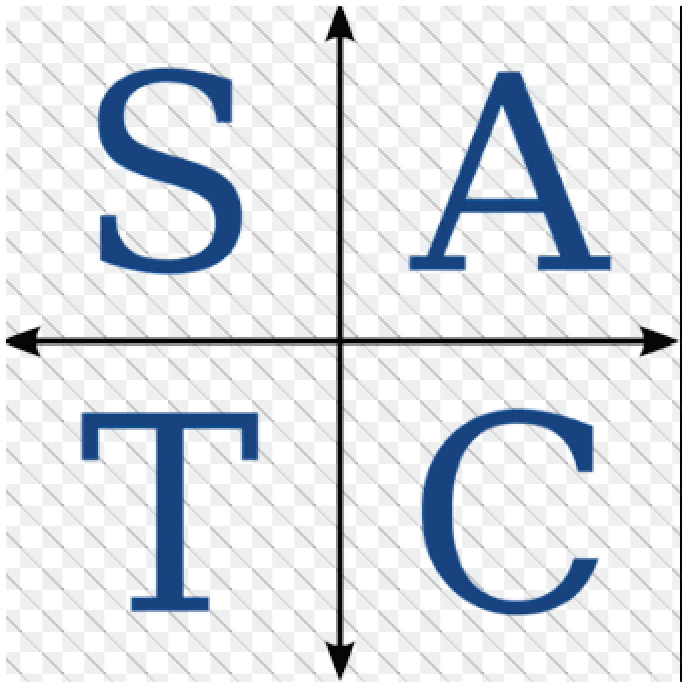
   a) $y = 3 \sin x + 2$
   b) $y = -3 \sin x - 1$
   c) $f(x) = -3 \sin (x + \frac{\pi}{4}) + 1$
   d) $f(x) = 3 \sin (2x + \frac{\pi}{4}) - 1$
   e) $f(x) = 3 \sin (x + \frac{\pi}{4}) + 3$
   f) $f(x) = 3 \sin (x + \frac{\pi}{4}) - 1$

2. Sketch the following graphs for $0^0 \le x \le 360^0$.

   a) $y = 2 \cos x + 1$
   b) $y = ½ \cos x - 1$
   c) $y = 2 \cos (x + 30^0) + 1$
   d) $y = 2 \cos (x + 90^0) - 2$
   e) $y = 2 \cos (x + 45^0) - 2$
   f) $y = 2 \cos (x - 45^0) + 2$

SUB – STRAND 4.3

Trigonometric Equations

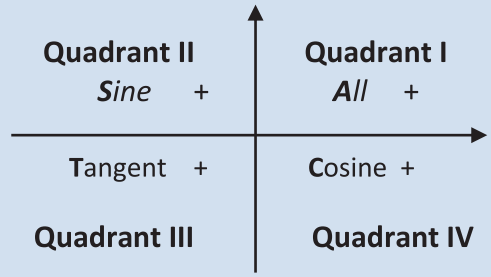


LEARNING OUTCOME

4.3.1 Solve trigonometric equations.

4.3.1 Solving Trigonometric Equations

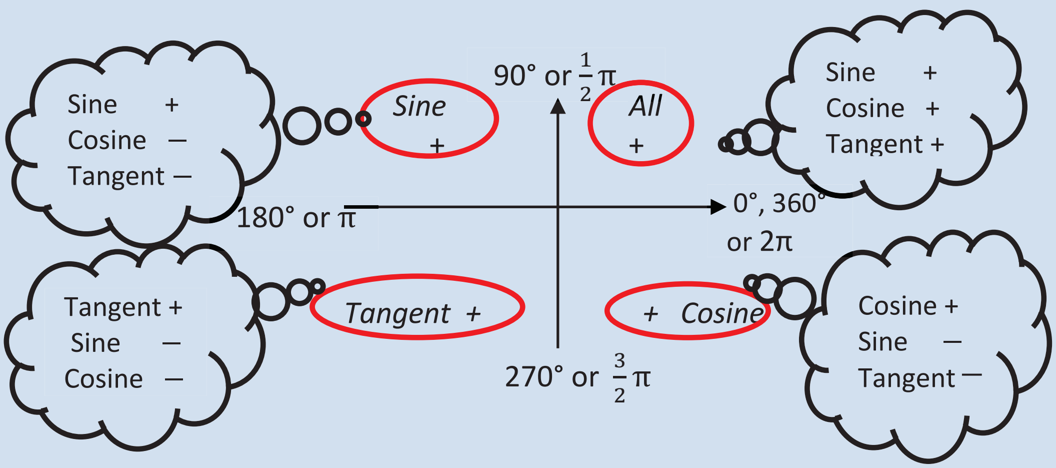
When solving any trigonometric equation, emphasis must be given to the quadrants.



Mnemonic

All **S**cience **T**eachers **C**ry
or
Add **S**ugar **T**o **C**offee

- If you look at the quadrants, the designated trig expressions will be positive, the others will be negative. Further simplifying,



Note: for complex equations, you may use the identities.

Example 1 Find the solution set for $2Cos\theta+\sqrt{3}=0, 0^o\le\theta\le 360^0$

Answer

The angle to be between 0 – 360°

$$2Cos\theta+\sqrt{3}=0, \quad 0^o\le\theta\le 360^0$$

Last

$$2Cos\theta+\sqrt{3}\cancel{-\sqrt{3}}=0-\sqrt{3}$$

$$\frac{\cancel{2}Cos\theta}{\cancel{2}}=\frac{-\sqrt{3}}{2}$$

$$Cos\theta=\frac{-\sqrt{3}}{2}$$

We reached at the trig expression: Consider the two quadrants. But before that, find the acute angle by ignoring the negative sign (–). Note that calculator Mode to be in degrees.

- **Acute angle:**

$$\theta = Cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^o$$

$$\alpha = 30^o$$

When dealing with surds,

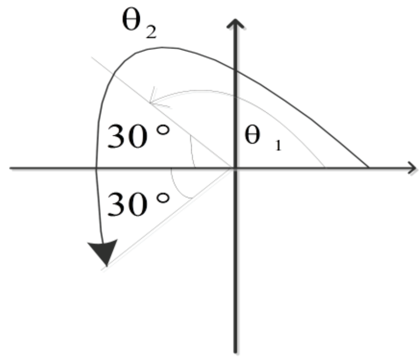
press the division sign (÷), that is Press

Shift	cos	(	$\sqrt{3}$	÷	2	)	=

- Use quadrants to find the angles θ_1 and θ_2. Consider negative sign (–) of Cos, that is in Q II / III

$$\theta_1 = 180 - 30 = 150^\circ$$

$$\theta_2 = 180 + 30 = 210^\circ$$



$$\boxed{\begin{array}{l}\theta = 150^o, 210^o\, or \\ \theta \in \{150^o, 210^o\}\end{array}}$$

Example 2 Solve the equation $3\sin\left(x - 45^\circ\right) = \frac{3\sqrt{3}}{2}$ for $0 \le x \le 360^\circ$.

Answer

$$3\sin\left(x - 45^\circ\right) = \frac{3\sqrt{3}}{2} \quad 0 \le x \le 360^\circ$$

$$\sin\left(x - 45^o\right) = \frac{\cancel{3}\sqrt{3}}{\cancel{3}\times 2}$$

$$(x - 45^o) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

We reached at the trig expression: Consider the two quadrants. But before that, find the acute angle. Note that calculator Mode to be in degrees.

- **Acute angle:**

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= 60^o$$

When dealing with surds,

press the division sign (÷), that is Press

Shift	sin	(	$\sqrt{3}$	÷	2	)	=

Sine is positive, the angle lies in the 1st and 2nd quadrants of the $xy-$plane.

- $\theta_1 = 60^o$ $\qquad\qquad$ $\theta_2 = 180 - 60 = 120^o$

$$(x-45^o) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad (x-45^o) = \theta_2$$

$$(x-45^o) = 60^o \qquad (x-45^o) = 120^o$$

$$x = 60^o + 45^o = 105^o \qquad x = 120^o + 45^o = 165^o$$

$$\boxed{\begin{array}{l} \theta = 105^o, 165^o \; or \\ \theta \in \{105^o, 165^o\} \end{array}}$$

Example 3 Solve $\cos 2x + \cos x = 0$.

Answer

Using identity: • $\cos 2A = 2\cos^2 A - 1$

$$\cos 2x + \cos x = 0$$

$$2\cos^2 x - 1 + \cos x = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

{ factorise trinomial }

<u>A Short-cut Method</u>

Let k be $\cos x$

$2k^2 + k - 1$

factors

$2k$ $\quad -1$ Cross multiply to give

k $\quad 1$ middle term k

$= (2k - 1)(k + 1)$

$= (2\cos x - 1)(\cos x + 1)$

Either

$$(2\cos x - 1) = 0$$

$$\cos x = 1/2$$

$$x = \cos^{-1}(1/2) = \frac{\pi}{3}$$

Or

$$(\cos x + 1) = 0$$

$$\cos x = -1$$

$$x = \cos^{-1}(-1) = \pi$$

From case 1, it can be seen that cosine is positive and hence it lies in the first and the fourth quadrant.

- **Acute angle:**

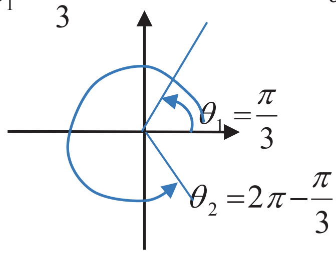
$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Press

Shift | cos | (| 1 | ÷ | 2 |) | =

- Cos is positive, the angle lies in the 1st and 4th quadrants of the $xy-$plane.

$\theta_1 = \frac{\pi}{3}$ $\qquad$ $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{6\pi-\pi}{3} = \frac{5\pi}{3}$



$$\boxed{x = \frac{\pi}{3}, \pi, \frac{5\pi}{3} \ or \quad \theta \in \{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\}}$$

Example 4 Solve $\tan^2\theta = \tan\theta$ for $0° \le \theta \le 180°$.

Answer

Rearranging and factorizing

$\tan^2\theta = \tan\theta$

$\tan^2\theta - \tan\theta = 0$

$\tan\theta \ (\tan\theta - 1) = 0$

Either	Or
$\tan\theta = 0$	$\tan\theta - 1 = 0$
$\theta = \tan^{-1} 0$	$\tan\theta = 1$
$\theta = 0$	$\theta = \tan^{-1}(1) = 45^o$

$$\boxed{\theta = 0^o, 45^o \text{ or } \theta \in \{0^o, 45^o\}}$$

Exercise 4.3.1

Solve the following trigonometric equations:

1. $\cos 2\theta + \sin\theta = 0$ for $0 \le \theta \le 2\pi$.
2. $\cos 2\theta + \cos\theta + 1 = 0$ *for* $0 \le \theta \le 2\pi$.
3. $3\cos 2\theta - \sin\theta = 2$ *for* $0 \le \theta \le 2\pi$.
4. $\cos\theta(2\cos\theta + \sqrt{3}) = 0$ for $0° \le \theta \le 180°$.
5. $3\sin 2x - 2\sin x = 0, 0^0 \le x \le 360^0$.
6. $\sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$.
7. $\tan^2\theta \sec^2\theta = 9 + \tan^2\theta$ for $0 \le \theta \le 2\pi$.
8. $2\cos^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$.

SUB – STRAND 4.4

Application of addition formulae

Original Expression	Combined Expression
$a\sin\theta + b\cos\theta$	$R\sin(\theta+\alpha)$
$a\sin\theta - b\cos\theta$	$R\sin(\theta-\alpha)$
$a\sin\theta + b\cos\theta$	$R\cos(\theta-\alpha)$
$a\sin\theta - b\cos\theta$	$R\cos(\theta+\alpha)$

In each case, a, b and R are positive and α is an a

R is given by:

$$R = \sqrt{a^2 + b^2}$$

LEARNING OUTCOMES

4.4.1 Use addition formulae to write a sum of sine and cosine as either a sine or cosine function.

4.4.2 Find maximum and minimum points on a given interval.

4.4.1 Transformation of trigonometric expressions of the form $A\ Cos\ \theta \pm B\ Sin\theta$

Often trig expressions involve the sum of sine and cosine terms. It is more convenient to write such expressions using one single term by applying the addition formula:

(i) $a\ Cos\ \theta \pm b\ Sin\theta = r\ Cos\ (\theta \pm \alpha)$

(ii) $a\ Cos\ \theta \pm b\ Sin\theta = r\ Sin\ (\theta \pm \alpha)$

where α is an angle to be found and r is the modulus i.e. $r = \sqrt{a^2 + b^2}$ and a and b are ***coefficients*** of **$Cos\ \theta$** and **$Sin\ \theta$** respectively.

This also enables us to solve certain types of trigonometric equations and find maximum and minimum points of complex trigonometric functions.

You will notice that this is very similar to converting rectangular to polar form in Complex Numbers. We can get α and r using calculator, similar to the way we did it in the complex numbers section.

Example 1 Express 2 Cos θ + 3 Sin θ in terms of $r\ Sin\ (\theta + \alpha)$.

Answer

$a = 2, b = 3\ \therefore r = \sqrt{2^2 + 3^2}$

$r = \sqrt{13}$

Apply Addition law:
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$Sin\ (\theta + \alpha) = [\ Sin\ \theta Cos\ \alpha + Cos\ \theta Sin\ \alpha]$

$r\ Sin\ (\theta + \alpha) = r\ Sin\ \theta Cos\ \alpha + r\ Cos\ \theta Sin\ \alpha$

substituting r yields: $\sqrt{13}\ Sin\ \theta Cos\ \alpha + \sqrt{13}\ Cos\ \theta Sin\ \alpha$

Thus in general form, $2\ Cos\ \theta + 3\ Sin\theta$

$= \sqrt{13}\ Sin\ \alpha Cos\ \theta + \sqrt{13}\ Cos\ \alpha\ Sin\ \theta$

Compare Sine & Cosine functions

$2 = \sqrt{13}\, Sin\ \alpha$

$\sin\alpha = \dfrac{2}{\sqrt{13}}$

$\alpha = \sin^{-1}\left(\dfrac{2}{\sqrt{13}}\right)$

$\alpha = 33.69^o$

$3 = \sqrt{13}\, Cos\ \alpha$

$\cos\alpha = \dfrac{3}{\sqrt{13}}$

$\alpha = \cos^{-1}\left(\dfrac{3}{\sqrt{13}}\right)$

$\alpha = 33.69^o$

Since both give the same value for α,

$\therefore \quad 2\ \text{Cos}\ \theta + 3\ \text{Sin}\ \theta = \sqrt{13}\ \text{Sin}\ (\theta + 33.69^o)$

Example 2 Express 2 Cos θ + 3 Sin θ in terms of *r cos* $(\theta + \alpha)$.

Answer

$r = \sqrt{13}$

Apply Addition law: • $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$cos\ (\theta + \alpha) = [cos\ \theta\, Cos\ \alpha + Sin\ \theta\, Sin\ \alpha]$

$r\ cos\ (\theta + \alpha) = r\ cos\ \theta\, Cos\ \alpha + r\ Sin\ \theta\, Sin\ \alpha$

substituting r yields: $\sqrt{13}\ cos\ \theta\, Cos\ \alpha + \sqrt{13}\ Sin\ \theta\, Sin\ \alpha$

Thus in general form, $2\ Cos\ \theta + 3\ Sin\theta$

$= \sqrt{13}\ Cos\ \alpha\ cos\ \theta + \sqrt{13}\ Sin\ \alpha\ Sin\ \theta$

Compare Sine & Cosine functions

$2 = \sqrt{13}\, Cos\ \alpha$

$\cos\alpha = \dfrac{2}{\sqrt{13}}$

$\alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{13}}\right)$

$\alpha = 56.31^o$

$3 = \sqrt{13}\, Sin\ \alpha$

$\sin\alpha = \dfrac{3}{\sqrt{13}}$

$\alpha = \sin^{-1}\left(\dfrac{3}{\sqrt{13}}\right)$

$\alpha = 56.31^o$

Since both give the same value for α,

$\therefore \quad 2\ \text{Cos}\ \theta + 3\ \text{Sin}\ \theta = \sqrt{13}\ \text{cos}\ (\theta + 56.31^o)$

Example 3 Express $y=2\cos\theta+\sqrt{2}\sin\theta$ in the form $R\cos(\theta+\alpha)$.

Answer

$a = 2, b = \sqrt{2} \therefore r = \sqrt{\sqrt{2}^2 + 2^2}$

$r = \sqrt{6}$

Apply Addition law: • $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

And substituting r yields: $\sqrt{6}\cos\theta \, Cos\,\alpha - \sqrt{6}\, Sin\,\theta \, Sin\,\alpha$

Thus in general form, $2\ Cos\ \theta + \sqrt{2}\ Sin\theta$

$= \sqrt{6}\ Cos\ \alpha\ cos\ \theta + \ - \sqrt{6}\ Sin\ \alpha\ \ Sin\ \theta$

Compare Sine & Cosine functions

$2 = \sqrt{6}\ Cos\ \alpha$

$\cos\alpha = \dfrac{2}{\sqrt{6}}$

$\alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{6}}\right)$

$\alpha = 35.26^o$

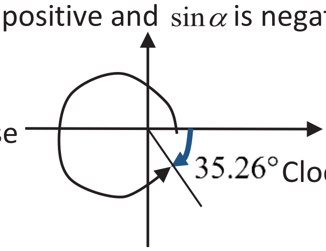
$\sqrt{2} = -\sqrt{6}\ Sin\ \alpha$

$\sin\alpha = \dfrac{\sqrt{2}}{-\sqrt{6}}$

$\alpha = \sin^{-1}\left(-\dfrac{\sqrt{2}}{\sqrt{6}}\right)$

$\alpha = -35.26^o$

As $\cos\alpha$ is positive and $\sin\alpha$ is negative, the angle lies in the 4th quadrant



Anticlockwise angle is positive
Clockwise angle is negative

$\alpha = 360 - 35.26$

$= 324.74° \, or - 35.26°$

$\therefore \quad 2\ Cos\ \theta + \sqrt{2}\ Sin\theta = \sqrt{6}\ \cos(\theta - 35.26^o)$

Example 4 Using the previous example,

a) Give the coordinates of the maximum and minimum point on $y=2\cos\theta+\sqrt{2}\sin\theta$ for $0 \le \theta \le 360°$

b) Find the x and y – intercepts.

c) Sketch the graph of $y=2\cos\theta+\sqrt{2}\sin\theta$ for $0 \le \theta \le 360°$

✍ **Answers**

Since it was reduced to a single trig function i.e.

$2\ Cos\ \theta + \sqrt{2}\ Sin\theta = \sqrt{6}\ \cos(\theta - 35.26^{o})$, so it is easier to analyze

$y = \sqrt{6}\ \cos(\theta - 35.26^{o})$

a) Coordinates of the maximum and minimum point for $0 \le \theta \le 360°$

Cosine is maximum at $0°$

$2\cos\theta + \sqrt{2}\sin\theta = \sqrt{6}\cos(\theta - 35.26°)$

$\theta - 35.26° = 0$

$\theta = 35.26°$

The maximum occurs at

$(35.26°, \sqrt{6})$

Cosine is minimum at $180°$

$2\cos\theta + \sqrt{2}\sin\theta = \sqrt{6}\cos(\theta - 35.26°)$

$\theta - 35.26° = 180°$

$\theta = 215.26°$

The minimum occurs at

$(215.26°, -\sqrt{6})$

b) Intercepts:

Let's find the y-intercept,

By letting $\theta = 0$

$y = \sqrt{6}\cos(\theta - 35.26°)$

$= \sqrt{6}\cos(0 - 35.26°)$

$= \sqrt{6}\cos(-35.26°)$

$\therefore y = 2$

Let's find the x-intercept

By letting $y = 0$

$y = \sqrt{6}\cos(\theta - 35.26°)$

$0 = \sqrt{6}\cos(\theta - 35.26°)$

Cosine is 0 at $90°$ and $270°$

$\theta - 35.26° = 90°$

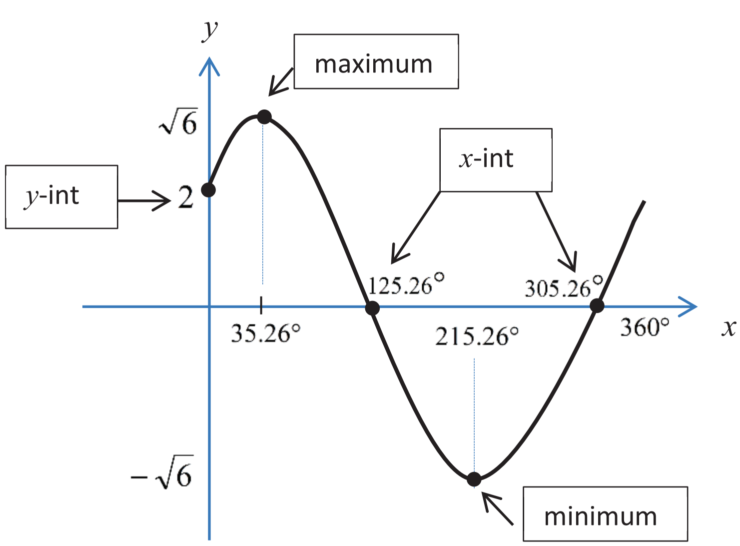
$\theta = 125.26°$

$\theta - 35.26° = 270°$

$\theta = 305.26°$

$\therefore x = 125.26°, 305.26°$

c) Graph of $y = 2\cos\theta + \sqrt{2}\sin\theta$ for $0 \le \theta \le 2\pi$ is shown below



Exercise 4.4.1

1. Express $y=2\sin\theta-\cos\theta$ in the form $R\sin(\theta-\alpha)$.

2. Express $y=\sqrt{3}\cos\theta-\sin\theta$ in the form $R\sin(\theta-\alpha)$.

3. Express $y=\sqrt{3}\cos\theta+\sin\theta$ in the form $R\cos(\theta+\alpha)$.

4. Express $y=2\sin\theta+\cos\theta$ in the form $R\cos(\theta+\alpha)$ and find the coordinates of the minimum and maximum points on the function for $0 \le \theta \le 360°$

5. Express $y=\sin\theta-\sqrt{3}\cos\theta$ in the form $R\sin(\theta-\alpha)$ and find the coordinates of the minimum and maximum points on the function for $0 \le \theta \le 360°$.

6. A function is given by $f(x)=7\sin x+8\cos x$
   a) Express the function $f(x)$ in the form $R\cos(x+\alpha)$ where α is an acute angle.
   b) Hence, sketch the graph of $f(x)=7\sin x+8\cos x$ for $0 \le \theta \le 360°$
   c) Solve the equation $7\sin x+8\cos x=6$ for $0 \le \theta \le 360°$

7. A function is given by $f(x) = 7\cos x - 6\sin x$.
   a) Express the function $f(x)$ in the form $f(x) = R\cos(x + \theta)$, where θ is an acute angle.
   b) Hence, sketch the graph of $f(x) = 7\cos x - 6\sin x$ for $0 \le \theta \le 360°$
   c) Solve the equation $7\cos x - 6\sin x = 5$ for $0 \le \theta \le 360°$

8. A function is given by $f(x) = 4\cos x + 7\sin x$.
   a) Express the function $f(x)$ in the form $f(x) = R\sin(x + \theta)$, where θ is an acute angle.
   b) Hence, sketch the graph of $f(x) = 4\cos x + 7\sin x$ for $0 \le \theta \le 360°$
   c) Solve the equation $4\cos x + 7\sin x = -5$ for $0 \le \theta \le 360°$

DID YOU KNOW?

Trigonometry is the relationships between the sides and angles of triangles. Such relationships are involved in a wide range of engineering problems. Engineers of various types use the fundamentals of trigonometry to design bridges, build structures and solve scientific problems. Trigonometry is very important with engineers who deal with waves, magnetic and electric fields. ~ anonymous
(http://math.tutorvista.com/trigonometry/applications-of-trigonometry.html)

SUB – STRAND 4.5

Inverse Trigonometric Functions

Inverse Trig Function T-Charts

$y = \sin^{-1}(x)$

x	y
-1	$-\frac{\pi}{2}$
0	0
1	$\frac{\pi}{2}$

$y = \cos^{-1}(x)$

x	y
-1	π
0	$\frac{\pi}{2}$
1	0

$y = \tan^{-1}(x)$

x	y
undef	$-\frac{\pi}{2}$
-1	$-\frac{\pi}{4}$
0	0
1	$\frac{\pi}{4}$
undef	$\frac{\pi}{2}$

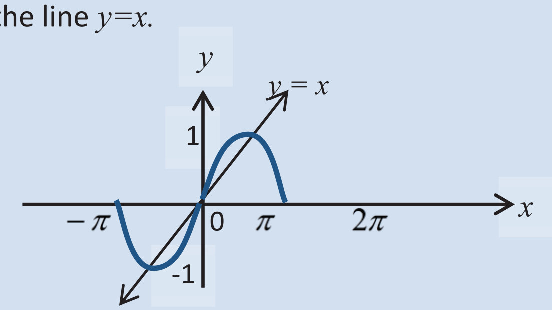
Domain	Range
$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
$-1 \le x \le 1$	$0 \le y \le \pi$
$-\infty < x < \infty$	$-\pi/2 < y < \pi/2$
$x \le -1$ or $x \ge 1$	$-\pi/2 \le y \le \pi/2,\ y \ne 0$
$x \le -1$ or $x \ge 1$	$0 \le y \le \pi,\ y \ne \pi/2$
$-\infty < x < \infty$	$0 < y < \pi$

CONTENT LEARNING OUTCOMES

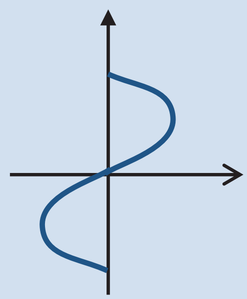
4.5.1 Sketch graphs of inverse trigonometric functions.
4.5.2 Prove inverse trigonometric identities.
4.5.3 Evaluate inverses for the reciprocal trigonometric functions.

4.5.1 Graphs of Inverse Trigonometric Functions

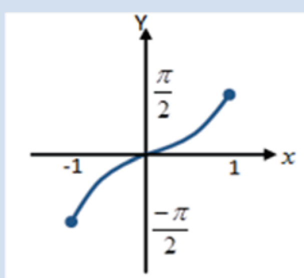
- The graph of inverse trig function is obtained by reflecting the graph in the line $y=x$.
- For example, the graph of $y=\sin^{-1}x$ *is* obtained by reflecting the graph of $y=\sin x$ in the line $y=x$.



$y=\sin^{-1}x$



It can be seen that the graph of $y=\sin^{-1}x$ is not a function. To make it a function we can restrict the range in the interval $\frac{-\pi}{2}\leq y\leq\frac{\pi}{2}$.



- Similarly the graph of $y=\cos^{-1}x$ is not a function. To make it a function we can restrict the range in the interval $0\leq y\leq\pi$
- Likewise the graph of $y=\tan^{-1}x$ is not a function. To make it a function we can restrict the range in the interval $\frac{-\pi}{2}<y<\frac{\pi}{2}$

Summary

The inverse Trig graphs are shown below:

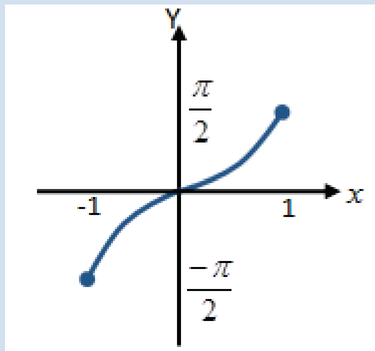
<table>
<tr><th>Equation</th><th>Shape</th><th>Domain</th><th>Range</th></tr>
<tr><td>

$y = \sin^{-1} x$

$y = \sin^{-1}(x)$ or
$y = \arcsin(x)$

<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>−1</td><td>$-\frac{\pi}{2}$</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>$\frac{\pi}{2}$</td></tr>
</table>

</td><td>



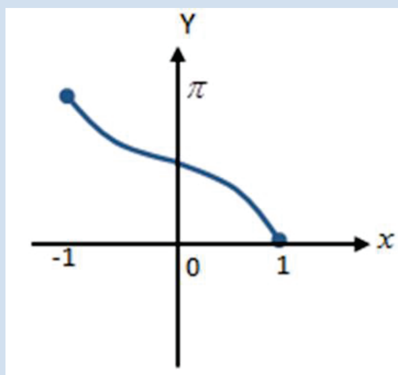
</td><td>$-1 \le x \le 1$</td><td>$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$</td></tr>
<tr><td>

$y = \cos^{-1} x$

$y = \cos^{-1}(x)$ or
$y = \arccos(x)$

<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>−1</td><td>π</td></tr>
<tr><td>0</td><td>$\frac{\pi}{2}$</td></tr>
<tr><td>1</td><td>0</td></tr>
</table>

</td><td>



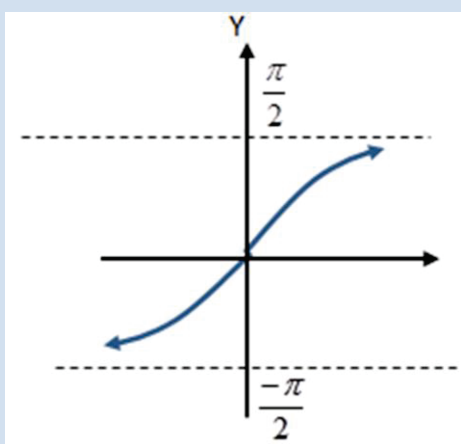
</td><td>$-1 \le x \le 1$</td><td>$0 \le y \le \pi$</td></tr>
<tr><td>

$y = \tan^{-1} x$

$y = \tan^{-1}(x)$ or
$y = \arctan(x)$

<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>und</td><td>$-\frac{\pi}{2}$</td></tr>
<tr><td>−1</td><td>$-\frac{\pi}{4}$</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>$\frac{\pi}{4}$</td></tr>
<tr><td>und</td><td>$\frac{\pi}{2}$</td></tr>
</table>

</td><td>



</td><td>$x \in R$</td><td>$-\frac{\pi}{2} < y < \frac{\pi}{2}$</td></tr>
</table>

4.5.2 Inverse Trigonometric Identities

1. $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$
2. $\sin(\cos^{-1}x) = \sqrt{1-x^2}$
3. $\cos(\sin^{-1}x) = \sqrt{1-x^2}$
4. $\tan(\cos^{-1}x) = \dfrac{\sqrt{1-x^2}}{x}$
5. $\tan(\sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$

Example 1 Prove that $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

Answer

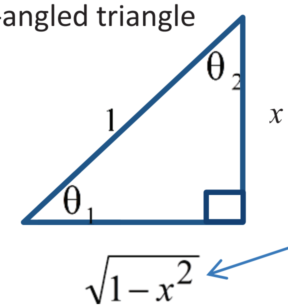
Let $\theta_1 = \sin^{-1}x \Rightarrow \sin\theta_1 = x$

$$= \frac{x}{1} = \frac{opp}{hyp}$$

Let $\theta_2 = \cos^{-1}x \Rightarrow \cos\theta_2 = x$

$$= \frac{x}{1} = \frac{adj}{hyp}$$

Consider a right-angled triangle



By Pythagoras theorem this side is $\sqrt{1-x^2}$

From the triangle $\theta_1 + \theta_2 = \dfrac{\pi}{2}$ $\quad \therefore \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

Example 2 Prove that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Answer

Let $\theta = \sin^{-1}x$

then $\sin\theta = x$

$$\sin\theta = \frac{x}{1} = \frac{opposite}{hypotenuse}$$

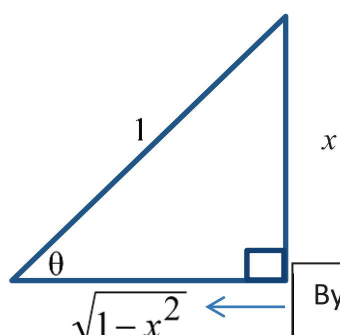
Consider a right-angled triangle, with one side of length 1, and another side of length x

$$\cos(\sin^{-1}x) = \cos\theta$$

$$= \frac{adjacent}{hypotenuse}$$

$$= \frac{\sqrt{1-x^2}}{1}$$

$$\cos(\sin^{-1}x) = \sqrt{1-x^2}$$



By Pythagoras theorem

4.5.3 Inverse for the reciprocal trig Functions

The inverse reciprocal functions are $y=\csc^{-1}x$, $y=\sec^{-1}x$ and $y=\cot^{-1}x$

Let's consider

$$y=\csc^{-1}x$$
$$\csc y=x$$
$$x=\csc y$$
$$x=\frac{1}{\sin y}$$
$$x\sin y=1$$
$$\sin y=\frac{1}{x}$$
$$y=\sin^{-1}\left(\frac{1}{x}\right) \qquad \therefore \csc^{-1}x=\sin^{-1}\left(\frac{1}{x}\right)$$

Similarly, $\sec^{-1}x=\cos^{-1}\left(\frac{1}{x}\right)$ $\qquad$ $\cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)$

Example 2 Evaluate $\csc^{-1}(-2)$

Answer

$$\csc^{-1}(-2)=\sin^{-1}\left(-\frac{1}{2}\right)$$
$$=-30°$$

Exercise 4.4

1. Evaluate

   $a)$ $\sec^{-1}\left(\sqrt{2}\right)$ $\qquad$ $b)$ $\cot^{-1}(-3.903)$

2. Prove that

   $a)$ $\sin(\cos^{-1}3x)=\sqrt{1-9x^2}$

   $b)$ $\tan(\sin^{-1}2x)=\dfrac{2x}{\sqrt{1-4x^2}}$

   $c)$ $\cos(\sin^{-1}5x)=\sqrt{1-25x^2}$

Review Exercise 4

1. Prove that $\dfrac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

2. Solve $\cos^2 x - \cos x - 2 = 0$ for $0° \le x \le 360°$

3. Sketch the graph of $y = 2\sin(x + 30^o) - 1$ for $0° \le x \le 360°$

4. Find the exact value of $\cos 22.5^o$

5. A function is given by $f(x) = 6\sin x + 8\cos x$

   a) Express the function $f(x)$ in the form $R\sin(x+\alpha)$ where α is an acute.

   b) Give the coordinates of the maximum and minimum values of the function $f(x) = 6\sin x + 8\cos x$ for $0° \le x \le 360°$

   c) Hence, sketch the graph of $f(x) = 6\sin x + 8\cos x$ for $0° \le x \le 360°$, showing all the intercepts.

   d) Solve the equation $6\sin x + 8\cos x = 4$ for $0° \le x \le 360°$

6. Sketch the graph of the function $y = \sin^{-1} x$

7. Prove that $\sin(2\sin^{-1} x) = 2x\sqrt{1-x^2}$

STRAND FIVE

LIMITS, CONTINUITY AND DIFFERENTIABILITY

SUB – STRAND 5.1

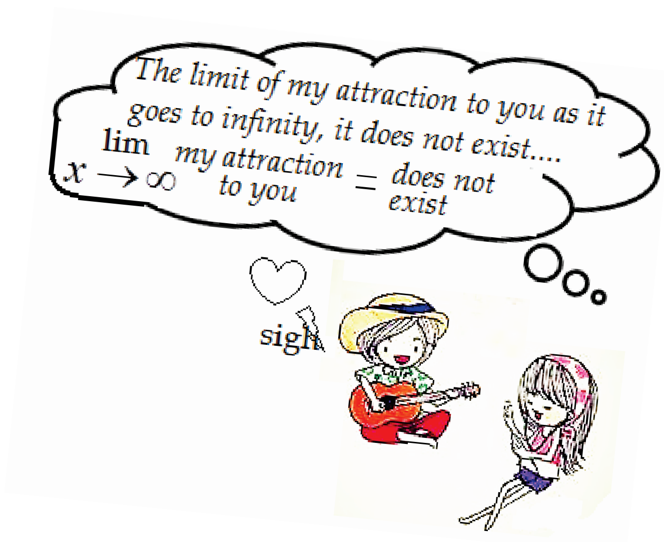
Limits of a Function

The Indeterminate Forms $0/0$ and ∞/∞

L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a,b) containing c, Assume that $g'(x) \neq 0$ for all x in (a,b), except possibly at c itself. If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form 0/0, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$



CONTENT LEARNING OUTCOMES

5.1.1 Calculate limits.

5.1.1 Computing Limits

- To find limits of rational algebraic expressions, directly substitute in the expression.

If upon substitution you get:

- a/b , then this is the limit.
- $0/b$, then the limit is 0.
- $a/0$, then the limit does not exist.
- $0/0$, then the limit may exist.

To find this limit,

- Use Tables: take values closest to a and substitute it in place of the variables in the algebraic expression.
- Algebraic Simplification such as factorizing by common factors, factorize trinomials ($ax^2 + bx + c$) or difference of squares ($a^2 - b^2 = (a-b)(a+b)$).
- L' Hôpital's rule:
  A useful technique of computing limits is using L' Hôpital's rule which involves the following steps:
  - Step 1: Check that limit of $f(x)/g(x)$ is an indeterminate form of the type $0/0$.
  - Step 2: Differentiate f and g separately.
  - Step 3: Find the limit of $f'(x)/g'(x)$.
    If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

Example 1 Find $\lim_{x \to 0} \sin x$

Answer

Directly substitute 0,

$$\lim_{x \to 0} \sin x$$
$$= \sin 0$$
$$= 0$$

Example 2 Find $\lim_{x \to -3} \dfrac{x+3}{x-2}$

Answer

Directly substitute −3,

$$\lim_{x \to -3} \frac{x+3}{x-2}$$

$$= \frac{(-3)+3}{(-3)-2} = \frac{0}{-5}$$

$$= 0$$

Example 3 Evaluate $\lim_{x \to 2} \dfrac{x^2+4x+3}{x-2}$

Answer

Directly substitute 2,

$$\lim_{x \to 2} \frac{x^2+4x+3}{x-2}$$

$$= \frac{2^2+4(2)+3}{2-2}$$

$$= \frac{4+8+3}{0}$$

$$= \frac{15}{0}$$

In this case, limit does not exist.

Example 4 Evaluate $\lim_{x \to 1} \dfrac{1-x^2}{1-x}$

Answer

Directly substitute 1, $\lim_{x \to 1} \dfrac{1-x^2}{1-x} = \dfrac{0}{0}$

In this case, the limit may exist.

<u>Method 1</u> L' Hôpital's rule

$$\lim_{x \to 1} \frac{1-x^2}{1-x} = \lim_{x \to 1} \frac{-2x}{-1}$$

$$= \frac{-2 \times 1}{-1}$$

$$= 2$$

Method 2: Using Tables

Let’s see what happens when we take values close to 1 and substitute it in place of x in the expression $\dfrac{1-x^2}{1-x}$

x	$\dfrac{1-x^2}{1-x}$
0.97	1.97
0.98	1.98
0.99	1.99
1	undefined
1.01	2.01
1.02	2.02
1.03	2.03

The values get closer and closer to 2 as x approaches 1.

It turns out that as x approaches 1, $f(x)$ approaches 2.

Thus, $\lim\limits_{x \to 1} \dfrac{1-x^2}{1-x} = 2$

Method 3: Algebraic Manipulation

Factorize , simplify then substitute $x = 1$,

$\lim\limits_{x \to 1} \dfrac{1-x^2}{1-x}$ using difference of 2 squares to factorise the numerator

$$\lim_{x \to 1} \frac{\cancel{(1-x)}(1+x)}{\cancel{1-x}}$$

$$\lim_{x \to 1} 1+x$$

$$=1+1$$
$$=2$$

Example 5 Evaluate $\lim\limits_{x\to4} \dfrac{2-\sqrt{x}}{4-x}$

Answer

Directly substitute the value 4, $\lim\limits_{x \to 4} \dfrac{2-\sqrt{x}}{4-x} = \dfrac{0}{0}$

Using L' Hôpital's rule: Differentiate

$$f(x)=2-x^{\frac{1}{2}}, \quad f'(x)=-\frac{1}{2}x^{-\frac{1}{2}}$$

$$g(x)=4-x \quad , \quad g'(x)=-1$$

Find limit of $f'(x)/g'(x)$:

$$\lim_{x\to 4}\frac{-\frac{1}{2}x^{-\frac{1}{2}}}{-1}$$

$$=\lim_{x\to 4}\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\lim_{x\to 4}\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{4}}$$

$$=\frac{1}{4}$$

Exercise 5.1.1

1. Evaluate the following limits:

a. $\lim\limits_{x\to 3}\dfrac{2x^2-5x-3}{x-3}$

b. $\lim\limits_{x\to 10} x^2-10$

c. $\lim\limits_{x\to 0} \cos x$

d. $\lim\limits_{x\to 2} \sqrt{x}+1$

e. $\lim\limits_{x\to 0}\dfrac{e^x}{x}$

f. $\lim\limits_{x\to 4}\dfrac{x^2-2x-8}{x-4}$

g. $\lim\limits_{x\to 4}\dfrac{4-\sqrt{x}}{x}$

h. $\lim\limits_{x\to 3}\dfrac{3x-5}{x-3}$

i. $\lim\limits_{x\to \pi} \tan x$

j. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{e^x-1}$

k. $\lim\limits_{x\to 0}\dfrac{(1+x)^2-1}{x}$

l. $\lim\limits_{x\to -1}\dfrac{3x^2-3}{x+1}$

2. Find b if $\lim\limits_{x\to 1}\dfrac{2x-bx^2}{x+3x^3}=2$

DID YOU KNOW

Real-life limits are used any time you have some type of real-world application approach a steady-state solution. As an example, we could have a chemical reaction in a beaker start with two chemicals that form a new compound over time. The amount of the new compound is the limit of a function as time approaches infinity.~ Ifryerda

5.1.2 Limits of Trigonometric Functions

- For the **Indeterminate Form** in trig functions, you probably have to **use** some Trig Identities to compute limits:

  - $\cos^2 x + \sin^2 x = 1$
  - $\sin 2x = 2 \sin x . \cos x$
  - $\tan x = \dfrac{\sin x}{\cos x}$

- L' Hôpital's rule where applicable.
  Some derivatives are given below:

$y = \sin x$	$y = \cos x$	$y = \cos bx$	$y = \sin bx$
$y' = \cos x$	$y' = -\sin x$	$y' = -b \sin bx$	$y' = b \cos bx$

Example 1 Evaluate $\lim\limits_{x \to \pi} \dfrac{\sin^2 x + \cos^2 x}{x}$

Answer

Use identity $\cos^2 x + \sin^2 x = 1$

$$\lim_{x \to \pi} \frac{\sin^2 x + \cos^2 x}{x}$$

$$\lim_{x \to \pi} \frac{1}{x}$$

Directly substitute π,

$$\lim_{x \to \pi} \frac{1}{x} = \frac{1}{\pi}$$

Example 2 Evaluate $\lim\limits_{\theta \to \pi} \dfrac{\sin \theta}{\sin 2\theta}$

Answer

Directly substitute π,

$$\lim_{\theta \to \pi} \frac{\sin \theta}{\sin 2\theta} = \frac{0}{0}$$

The limit may exist.

Method 1: Using Trig Identities

Use identity , cancel then substitute:

$$\lim_{\theta\to\pi}\frac{\sin\theta}{\sin 2\theta}$$

Use identity:
sin $2x$ = 2 sin x. cos x

$$\lim_{\theta\to\pi}\frac{\cancel{\sin\theta}}{2\cancel{\sin\theta}\cos\theta}$$

$$\lim_{\theta\to\pi}\frac{1}{2\cos\theta}$$

$$\therefore\lim_{\theta\to\pi}\frac{1}{2\cos\pi}$$

$$=-\frac{1}{2}$$

Method 2: L' H$\hat{o}$pital's rule

Differentiate:

$$f(x)=\sin\theta \quad\Rightarrow\quad f'(x)=\cos\theta$$

$$g(x)=\sin 2\theta \quad\Rightarrow\quad g'(x)=2\cos 2\theta$$

Find limit $\displaystyle\lim_{\theta\to\pi}\frac{f'(x)}{g'(x)}$

$$\lim_{\theta\to\pi}\frac{\cos\theta}{2\cos 2\theta}$$

$$=\frac{\cos\pi}{2\cos 2\pi}$$

$$=-\frac{1}{2}$$

Example 3 Show that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$

Answers Directly substitute x = 0 we get $\frac{0}{0}$ so we can use L' H$\hat{o}$pital's rule

$$f(x)=\sin x \quad\Rightarrow\quad f'(x)=\cos x$$

$$g(x)=x \quad\Rightarrow\quad g'(x)=1$$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to 0}\frac{\cos x}{1}$$

$$=\frac{\cos 0}{1}$$

$$=1$$

Exercise 5.1.2

1. Find the following limits using identities:

a) $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\cos\theta}{\sin 2\theta}$ b) $\lim\limits_{x \to 0} \dfrac{1-cos^2 x}{x^2}$ c) $\lim\limits_{x \to \pi} \dfrac{\sin 2x}{\tan x}$

2. Evaluate the following limits using L' Hôpital's rule:

a) $\lim\limits_{x \to 0} \dfrac{3\sin 2x}{5x}$ b) $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{2x}$

5.1.3 Limits at Infinity

Limits as $x \to \infty$:

- If *r* is a positive rational number and *c* is any real number, then

$\boxed{\lim\limits_{x \to \infty} \dfrac{c}{x^r} = 0}$ Division by a very large number gives a very small result.

- To use the above property, divide the numerator and denominator by the **highest *power* of x in the denominator**.

Short Cut: cover up rule

- If $f(x) = ax^n + bx^{n-1} + \cdots + c$ is a polynomial of degree n, where coefficient $a \neq 0$, and $g(x) = dx^n + ex^{n-1} + \cdots + f$ is also a polynomial of degree n, where coefficient $d \neq 0$, then

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$$

$$\lim_{x \to \pm\infty} \frac{ax^n + bx^{n-1} + \cdots}{dx^n + ex^{n-1} + \cdots} = \frac{a}{b}$$

In other words, **use the cover up rule**

Example 1 Find $\lim\limits_{x \to \infty} \dfrac{100}{x^2}$

Answer

$$\lim_{x \to \infty} \frac{100}{x^2} = 0$$

Using:

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

Example 2 Find $\lim\limits_{x\to\infty}\dfrac{8x+3}{x^2-9}$

Answer

$$\lim_{x\to\infty}\frac{8x+3}{x^2-9}=\lim_{x\to\infty}\frac{\frac{8x}{x^2}+\frac{3}{x^2}}{\frac{x^2}{x^2}-\frac{9}{x^2}}$$ $\to$ Dividing by the highest power of x in denominator

$$=\lim_{x\to\infty}\frac{\frac{8}{x}+\frac{3}{x^2}}{1-\frac{9}{x^2}}$$

$$=\frac{\frac{8}{\infty}+\frac{3}{\infty}}{1-\frac{9}{\infty}}$$

$$=\frac{0+0}{1-0}$$

$$=0$$

Division by a very large number gives a very small result ≈ 0.

$$\lim_{x\to\infty}\frac{1}{x}=0$$

Example 3 Find $\lim\limits_{x\to+\infty}\left(4-\dfrac{3}{x+1}\right)$

Answer

$$\lim_{x\to+\infty}4-\lim_{x\to+\infty}\frac{3}{x+1}$$

Using: $\lim\limits_{x\to\pm\infty} c=c$

Dividing by the highest power of x

$$=4-\lim_{x\to+\infty}\frac{3/x}{x/x+1/x}$$

$$=4-\lim_{x\to+\infty}\frac{3/x}{1+1/x}$$

$$=4-\left(\frac{0}{1+0}\right)$$

$$=4$$

Exercise 5.1.3

Evaluate the following limits:

a) $\lim\limits_{x\to\infty}\dfrac{5x^3}{x(1-x^2)}$

b) $\lim\limits_{x\to\infty}\dfrac{x^5-2x^4}{x^7}$

c) $\lim\limits_{x\to\infty}\dfrac{x^2+1}{x^3+4}$

d) $\lim\limits_{x\to\infty}\dfrac{x(4x-3)(7-5x)}{(6x^2-1)(8x-3)}$

e) $\lim\limits_{x\to\infty}\dfrac{(x^3-2x)(x+3)}{(4x^2-1)(7x-2x^2)}$

f) $\lim\limits_{x\to\infty}\dfrac{5x^2+3x+1}{4x^2}$

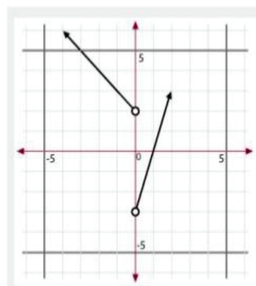
SUB – STRAND 5.2

Piece-wise Functions

What is a Piecewise Function?

- A function that combines *pieces* of different equations.
- Each piece is for a different **domain** (set of x values).
- Example:

$$f(x)=\begin{cases}-x+2, & if\ x<0\\ 3x-3, & if\ x>0\end{cases}$$



LEARNING OUTCOME

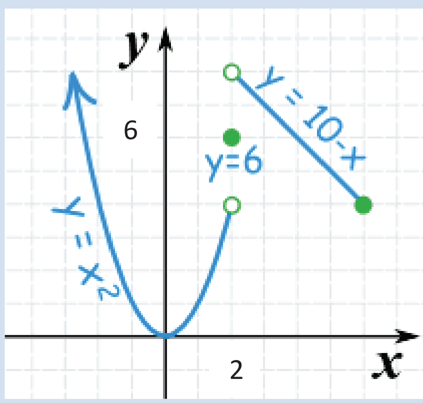
5.2.1 Sketch piecewise-defined functions.

5.2.1 Piecewise Functions

Piecewise Function is a **function** which is defined by multiple **sub-functions**, each **sub-function** applying to a certain interval.
Consider the function $f(x)$ defined below.

$$f(x)=\begin{cases} x^2 & , \quad x<2 \\ 6 & , \quad x=2 \\ 10-x, & 2<x\le 6 \end{cases}$$

The graph representing the function is



Unshaded circle means that the point is not included
Shaded circle means the point is included.

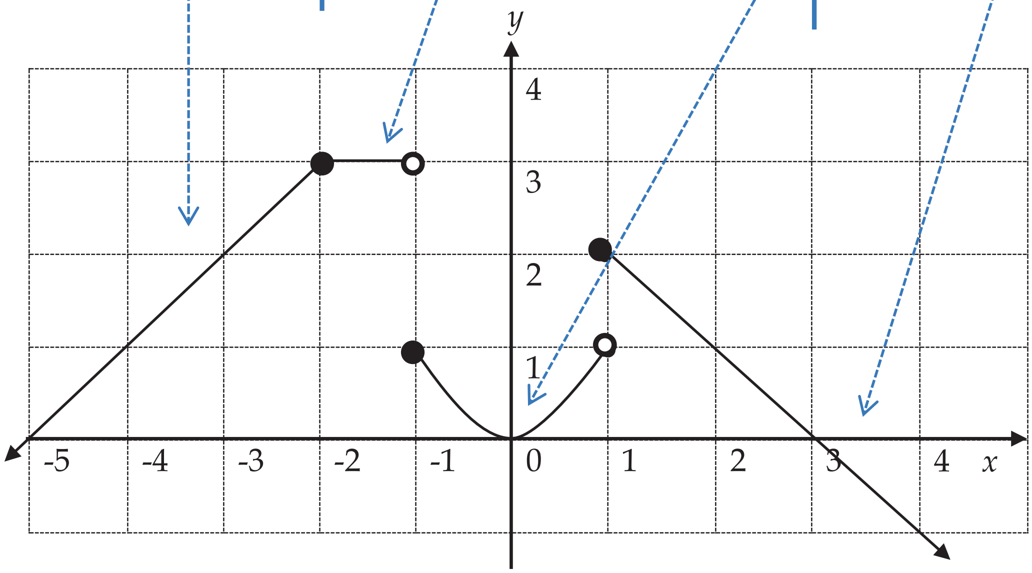
Example 1 Sketch the graph of $g(x)$ defined below

$$g(x)=\begin{cases} x+5, & x\le -2 \\ 1 \quad , & -2<x<-1 \\ x^2 \quad , & -1\le x<1 \\ 3-x, & x\ge 1 \end{cases}$$

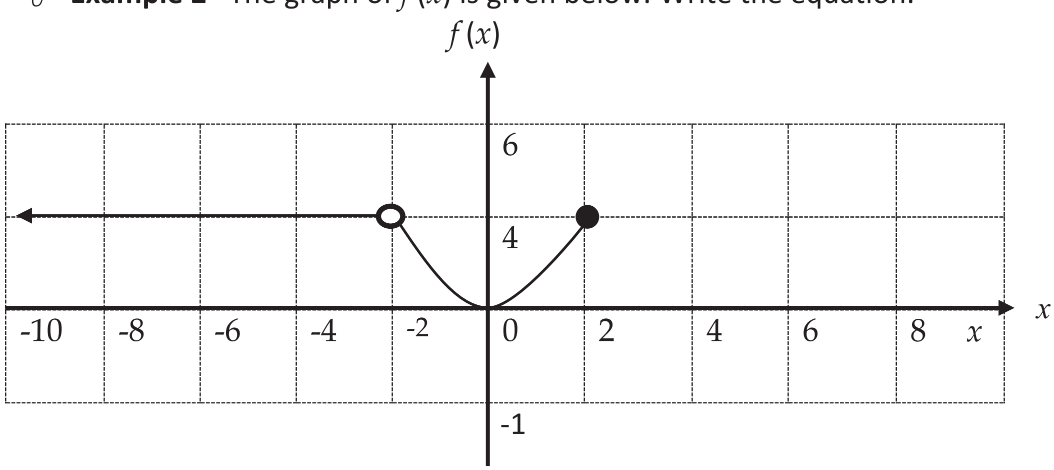
Answer

Let's look at different x intervals:

$x \le -2$	$-2 < x < -1$	$-1 \le x < 1$	$x \ge 1$
$g(x) = x + 5$	$g(x) = 1$	$g(x) = x^2$	$g(x) = 3 - x$
Linear graph	Linear (horizontal line)	quadratic	Linear graph



Example 2 The graph of $f(x)$ is given below. Write the equation.



Answer

$$f(x) = \begin{cases} 4, & x < -2 \\ x^2, & -2 < x \le 2 \end{cases}$$

Exercise 5.2.1

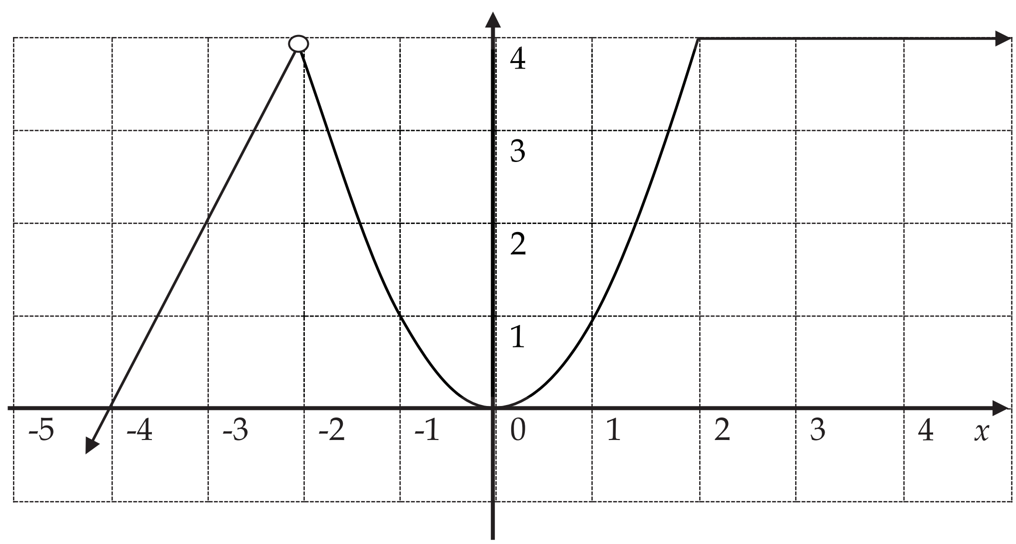
a) Sketch the following Functions:

1.
$$g(x) = \begin{cases} x+1, & x > 2 \\ x-1, & x \leq 2 \end{cases}$$

2.
$$f(x) = \begin{cases} 2 & , \quad x \leq -2 \\ x^2 & , \quad -2 < x \leq 2 \\ 1 & , \quad x \geq 2 \end{cases}$$

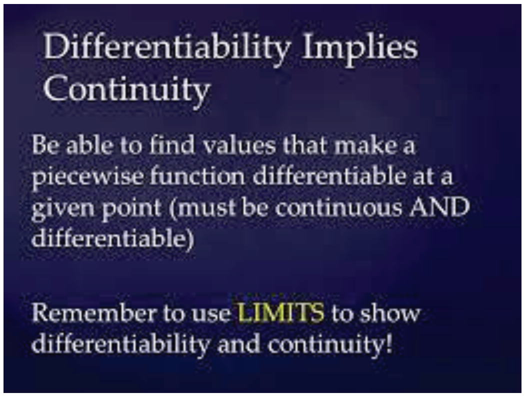
3.
$$f(x) = \begin{cases} -2(x+4) & , \quad x \leq -3 \\ -x^2 & , \quad -3 < x \leq 2 \\ -4 & , \quad x \geq 2 \end{cases}$$

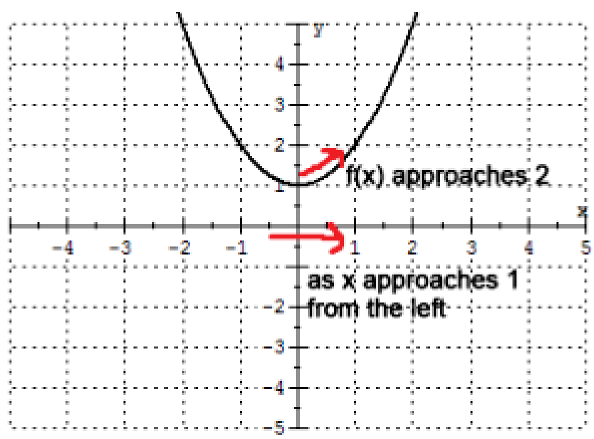
b) Give the equation of the function shown below



SUB – STRAND 5.3

Limits, Continuity and Differentiability from Graphs





LEARNING OUTCOME

5.3.1 Find limits, points of discontinuity and points of non-differentiability from graphs.

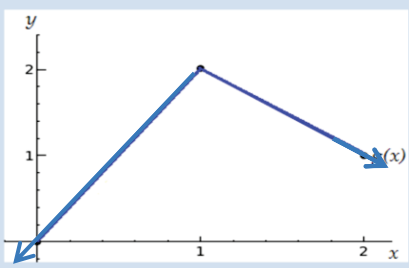
5.3.1 Limits, continuity and differentiability of Piecewise-defined Functions

If we are interested in what is happening to the function $f(x)$ as x gets close to some value c from the right, we write $\lim\limits_{x \to c^+} f(x)$. This is called the right handed limit.

Similarly, if we are interested in what is happening to the function $f(x)$ as x gets close to some value c from the left, we write $\lim\limits_{x \to c^-} f(x)$. This is called the left handed limit.

- **Discontinuity**- A function is discontinuous if it has a **jump** or has a **hole** in the graph or has **asymptotes**. Otherwise, the function is **continuous**.

- A function is **Non-differentiable** if it is discontinuous or has a sharp corner or has end points.



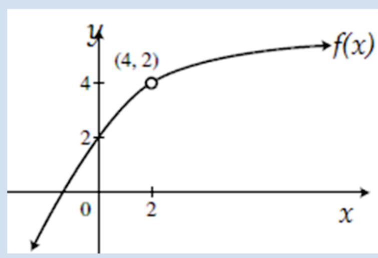
Continuity:
Continuous graph since there is no jump or hole in the graph.

Not differentiable at
$x = 1$(sharp corner)

Limits:
Evaluate $\lim\limits_{x \to 1} f(x)$:
Look for the value that y gets close to as x approaches 1 from left and right side

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 2$

$\therefore \lim\limits_{x \to 1} f(x) = 2$



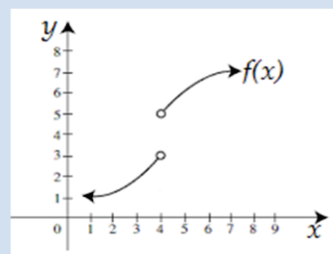
Continuity:
Discontinuous at $x = 2$, since there is a hole in the graph.

Not differentiable at
$x = 2$*(discontinuous)*

Limits:
Evaluate $\lim\limits_{x \to 2} f(x)$:
Look for the value that y gets close to as x approaches 2 from left and right side

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = 4$

$\therefore \lim\limits_{x \to 2} f(x) = 4$



Continuity:
Discontinuous at $x = 4$, since there is a jump.

Not differentiable at
$x = 4$*(discontinuous)*

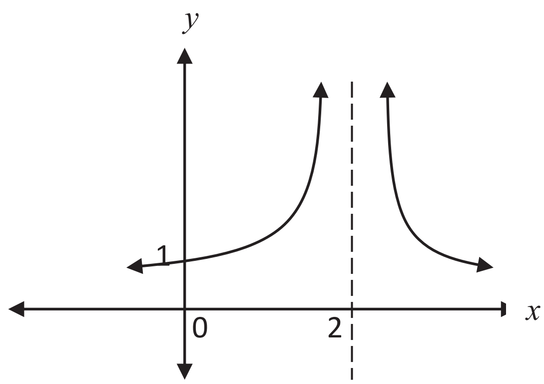
Limits:
Evaluate $\lim\limits_{x \to 4} f(x)$:
Look for the value that y gets close to as x approaches 4 from left and right side

$\lim\limits_{x \to 4^-} f(x) = 3 \neq$

$\lim\limits_{x \to 4^+} f(x) = 5$

$\therefore \lim\limits_{x \to 4} f(x)$ *does not exist.*

Example 1 Given below is the graph of $y = f(x)$



Find:

a) $\lim\limits_{x \to 0} f(x)$

b) $\lim\limits_{x \to 2} f(x)$

Answers

Look for the value that y gets close to as x approaches 0 from left and right

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = 1$

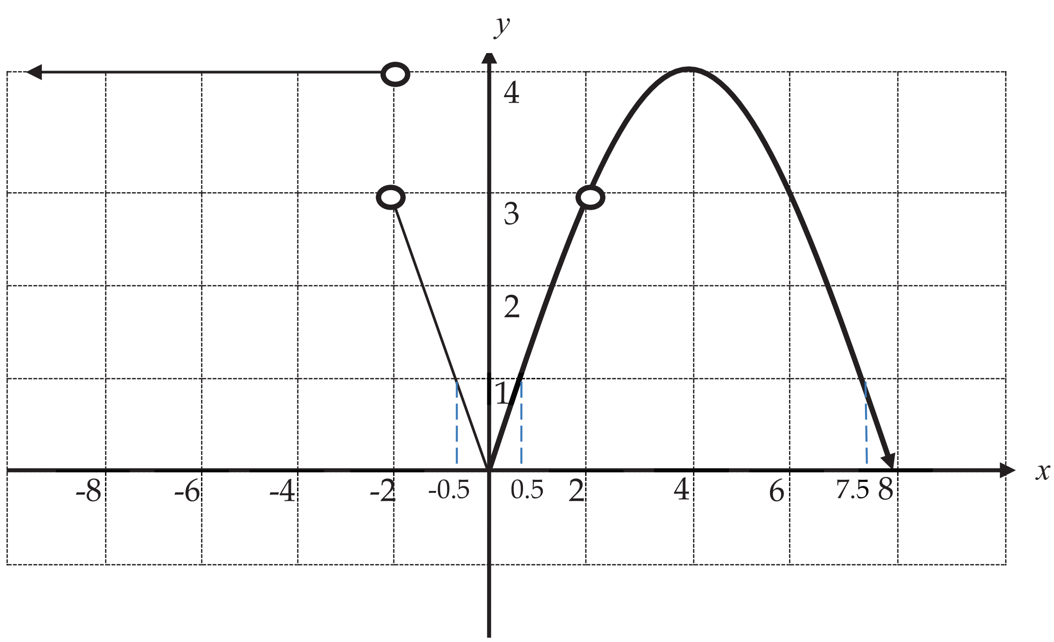
$\therefore \lim\limits_{x \to 0} f(x) = 1$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = +\infty$

y becomes very large as x approaches 2

$\therefore \lim\limits_{x \to 0} f(x) = +\infty$

Example 2 The graph of $g(x)$ is given below. Use the graph to answer the questions that follow.



a) Find the following limits:

i. $\lim\limits_{x \to -2^-} g(x)$ ii. $\lim\limits_{x \to -2^+} g(x)$ iii. $\lim\limits_{x \to -2} g(x)$

b) For what values of x is

i. $g(x)$ discontinuous?

ii. $g(x)$ not differentiable?

iii. $g(x) = 1$?

iv. $g(x)=3$?

Answers

a) i. $\lim\limits_{x \to -2^-} g(x) = 4$ ii. $\lim\limits_{x \to -2^+} g(x) = 3$

iii. Limit *does not exist*

b)

i. $g(x)$ discontinuous: $x \in \{-2,2\}$

ii. $g(x)$ not differentiable: $x \in \{-2, 0, 2\}$

iii. $g(x)= 1$ i.e $y = 1$

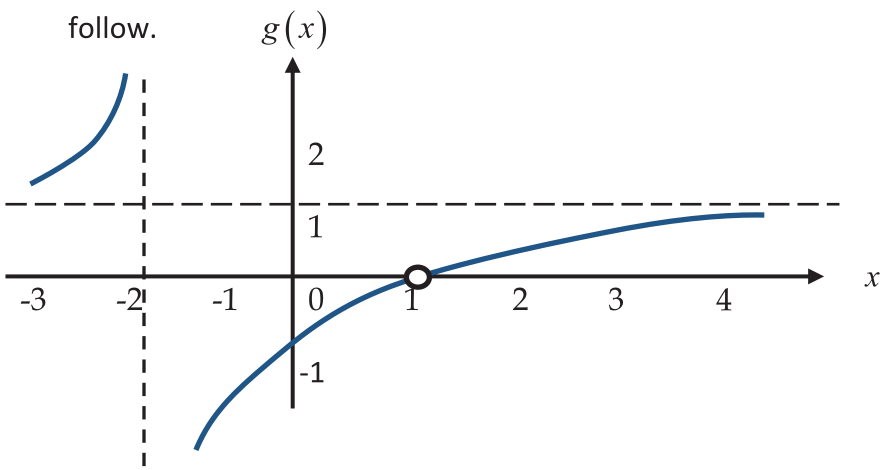
Make a straight horizontal line and look at the places it cuts

$x \in \{-0.5, 0.5, 7.5\}$

iv. $x = 6$

Exercise 5.3.1

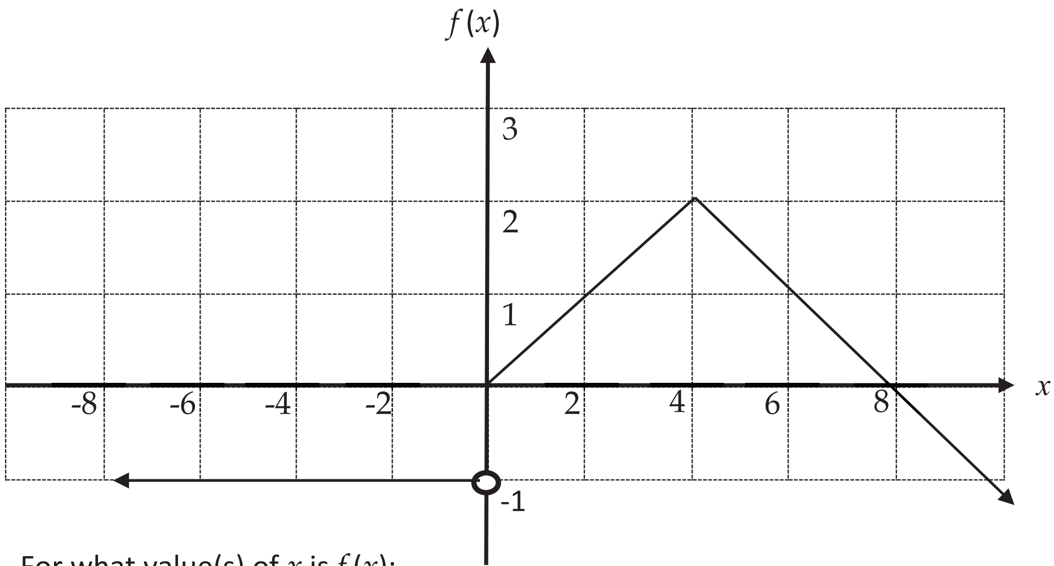
1. The graph of $g(x)$ is given below. Use the graph to answer the questions that follow.



a) Find $\lim\limits_{x \to \infty} g(x)$

b) For what values of x is the function discontinuous?

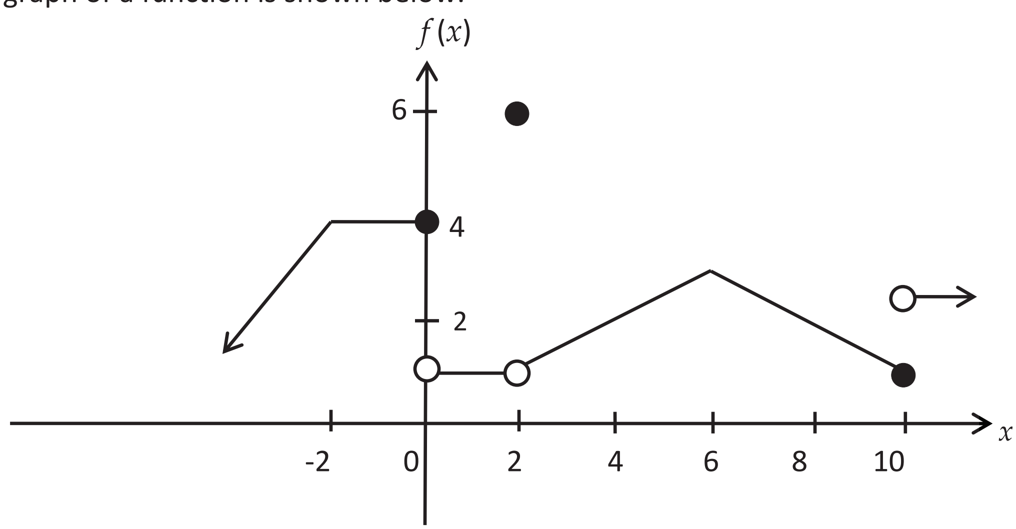
2. The graph of $f(x)$ is given below.



For what value(s) of x is $f(x)$:

a) discontinuous
b) not differentiable
c) equal to zero

3. The graph of a function is shown below.



Use the graph to give the value(s) of x for which $f(x)$

a) is continuous but not differentiable.
b) does not have a limit.
c) is equal to 6.

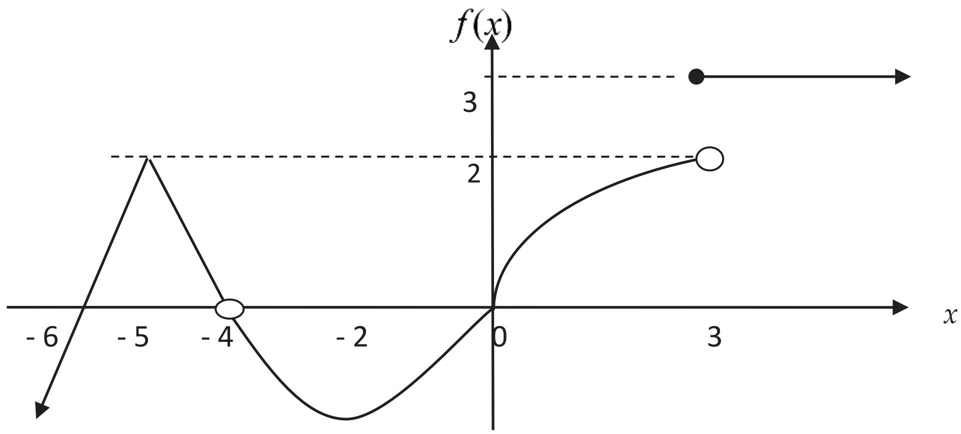
DID YOU KNOW? *If you drop an ice cube in a glass of warm water and measure the temperature with time, the temperature eventually approaches the room temperature where the glass is stored. Measuring the temperature is a limit again as time approaches infinity.* ~ Ifryerda

Review Exercise 5

1. Evaluate the following limits:

a) $\lim\limits_{x\to 2} \dfrac{x^2-3}{x+2}$ b) $\lim\limits_{x\to\infty} \dfrac{5x(x-2)}{3x^2-2x+4}$ c) $\lim\limits_{a\to 9} \dfrac{a-9}{\sqrt{a}-3}$ d) $\lim\limits_{x\to\infty} \dfrac{4x^3-x}{2x^3-x}$

2. The graph of the function $f(x)$ is given below.

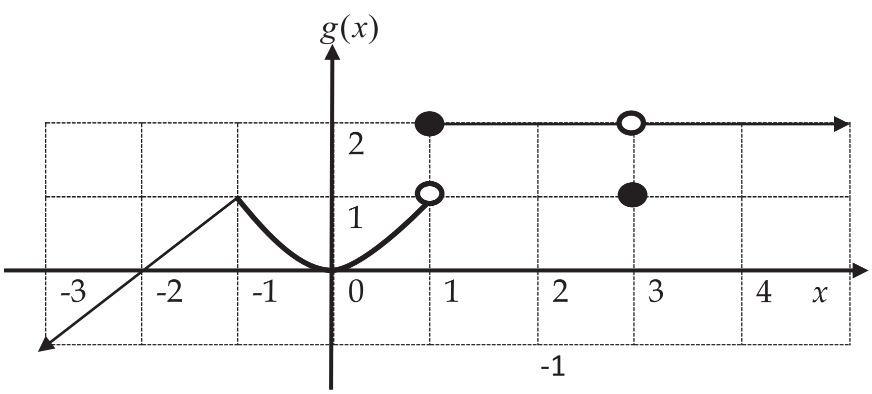


a) Use the graph to find the value (s) of x for which $f(x)$ is:

i. Discontinuous;
ii. Not differentiable;
iii. Equal to zero.

b) Evaluate $\lim\limits_{x\to -4} f(x)$ c) Evaluate $\lim\limits_{x\to 3^-} f(x)$

4. The graph of a piece-wise function, $g(x)$ is given below. Use the graph to answer the questions that follow.



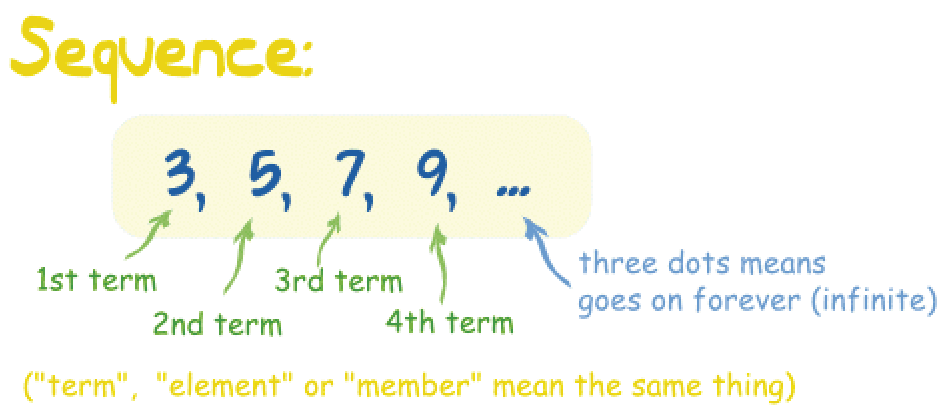
a) For what values of x is g(x) discontinuous?
b) For what values of x is g(x) not differentiable?
c) For what values of x is $g(x)=1$?
d) Find $g(3)$

STRAND SIX

ALGEBRA

SUB – TOPIC 6.1

Sequences



LEARNING OUTCOMES

6.1.1 List terms of sequences.
6.1.2 List sequence of partial sum.
6.1.3 Find limits of a sequence.
6.1.4 Determine convergence and divergence.

6.1.1 Sequences

(i) Sequence : is an ordered list of numbers.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots\ldots$$

(ii) Series : obtained by adding terms in a sequence.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots$$

Partial Sum, S_n

The *partial sum* of a sequence is the sum of a finite number of consecutive terms beginning with the first term. The first four *partial sums* are computed below, where S_n represents the sum of the first n terms of the sequence.

$S_1 = T_1$

$S_2 = T_1 + T_2$

$S_3 = T_1 + T_2 + T_3$

$\vdots$

Convergence and Divergence

Sequences which approach a **definite value** are said to **converge**. If a sequence converges we call the value it approaches the **limit**.

If a sequence has a limit, we say the sequence is convergent and that the sequence converges to the limit. Otherwise, the sequence is divergent.

Example 1 A sequence is defined by $T_n = 2n$.

a) Find the first four terms of this sequence.
b) Write as partial sum.

Answers

a) $T_n = 2n$

$T_1 = 2(1) = 2$

$T_2 = 2(2) = 4$

$T_3 = 2(3) = 6$

$T_4 = 2(4) = 8$

The terms of the sequence are < 2, 4, 6, 8, ……. >.

b) $S_1 = 2$

$S_2 = 2 + 4 = 6$

$S_3 = 2 + 4 + 6 = 12$

$S_4 = 2 + 4 + 6 + 8 = 20$

The sequence of partial sum are < 2, 6, 12, 20, ……. >.

Example 2 A sequence is defined as $a_n = \frac{1}{n}$.

a) List the first six terms.

b) What is the $\lim\limits_{n \to \infty} \frac{1}{n}$?

Answers

a) The terms of the sequence $\frac{1}{n} = \{1, \ 0.5, \ 0.33, \ 0.25, \ 0.2, \ 0.17....\}$

b) $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

Example 3 Determine whether the *sequence* $a_n = \frac{7n+3}{n-9}$ converge or diverge, and if it converges, give the value to which it converges to.

Answers

Let's find the limit:

Method 1:
Divide each term by variable with the highest power in the denominator

$$\lim_{n\to\infty} \frac{7n+3}{n-9} = \lim_{n\to\infty} \frac{\frac{7n}{n}+\frac{3}{n}}{\frac{n}{n}-\frac{9}{n}}$$

$$= \lim_{n\to\infty} \frac{7+\frac{3}{n}}{1-\frac{9}{n}}$$

$$= \frac{7+\frac{3}{\infty}}{1-\frac{9}{\infty}}$$

$$= \frac{7+0}{1-0}$$

$$=7$$

Method 2:
L' Hôpital's Rule or Cover Up Rule to find the horizontal asymptote of the hyperbola

$$\lim_{n\to\infty} \frac{7n+3}{n-9}$$

$$= \lim_{n\to\infty} \frac{7}{1}$$

$$= 7$$

This is clearly a **converging sequence**. It **converges to 7**.

Example 4 A sequence < a_n > is defined by $a_n = 2n+1$

a) List the first six terms.
b) What is the limit of this sequence?
c) Determine whether the *sequence* converge or diverge.

Answers

a) The terms of the sequence $2n+1 = \{3, \ 5, \ 7, \ 9, 11, 13....\}$

b)

$$\lim_{n\to\infty} 2n+1 = 2\times\infty+1$$
$$=\infty$$

c) The terms of the sequence increase without bounds (goes to infinity) so the sequence **diverges.**

Note: If the limit is infinity (positive or negative) the sequence diverges

Example 5 A sequence < a_n > is defined by $a_n = \dfrac{n^2}{2n+1}$

a) List the first three terms of the sequence.
b) List the first two terms of the sequence of partial sums.
c) Show that $a_n = \dfrac{n^2}{2n+1}$ is divergent.

Answers

a)

$$a_1 = \frac{1^2}{2(1)+1} \qquad a_2 = \frac{2^2}{2(2)+1} \qquad a_2 = \frac{3^2}{2(3)+1}$$
$$= \frac{1}{3} \qquad\qquad = \frac{4}{5} \qquad\qquad = \frac{9}{7}$$

Hence the sequence is < a_n > = < $\frac{1}{3}, \ \frac{4}{5}, \ \frac{9}{7},$ .... >

b)

$$S_1 = T_1 \qquad S_2 = T_1 + T_2$$
$$= \frac{1}{3} \qquad = \frac{1}{3}+\frac{4}{5}$$
$$= \frac{17}{15}$$

Hence the sequence of partial sum is < S_n > = < $\frac{1}{3}, \ \frac{17}{15},$ .... >

c) Find the limit: Apply L' Hôpital's Rule

$$\lim_{n \to \infty} \frac{n^2}{2n+1} = \lim_{n \to \infty} \frac{2n}{2}$$

$$= \lim_{n \to \infty} n \quad = \infty$$

This is clearly a **diverging sequence** since the limit is infinity.

Exercise 6.1.1

1. A sequence < a_n > is defined by $a_n = \dfrac{7n+3}{n-9}$

   a) Find the first four terms of the sequence.
   b) Find the first three terms of the sequence of partial sums.
   c) Find $\displaystyle\lim_{n \to \infty} \frac{7n+3}{n-9}$
   d) Explain why the sequence converges.

2. A sequence < a_n > is defined by $a_n = \dfrac{n+2}{4n+1}$

   a) Find the first four terms of the sequence.
   b) Determine whether a *sequence* converges or diverges, and if it converges, give the value to which it converges to.

3. A sequence < a_n > is defined by $a_n = \dfrac{n+2}{n^2}$

   a) Find the first two terms of the sequence of partial sums.
   b) Determine whether a *sequence* converges or diverges, and if it converges, give the value to which it converges to.

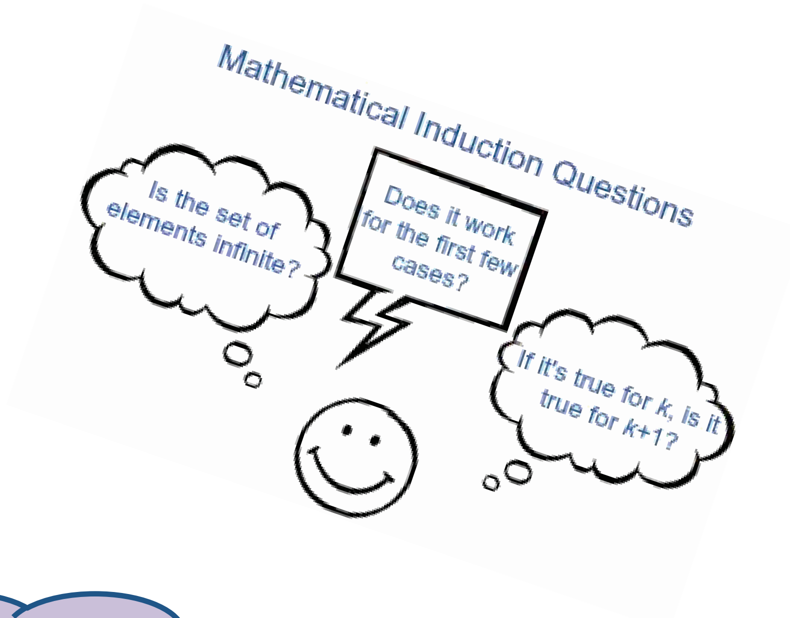
4. A sequence < a_n > is defined by $a_n = \dfrac{12n+1}{3n+2}$

   a) How many terms of the sequence are less than 3.99?
   b) State the limit of the sequence.
   c) Does it converge or diverge. Explain

5. A sequence < a_n > is defined by $a_n = 3n+4$
   Does it converge or diverge. Explain

SUB – STRAND 6.2

Mathematical Induction



LEARNING OUTCOME

6.2.1 Prove by mathematical induction.

6.2.1 Mathematical Induction

Mathematical Induction is a method of proving statements that involves a variable which takes on the values of the Natural numbers, N.

Also note that $\sum_{i=1}^{n} t_i = t_1 + t_2 + t_3 + ... + t_n$

Basically there are 4 steps involved in proving a statement:

Step 1: Prove that it is true for $n = 1$

Step 2: Assume that it is true for $n = k$

Step 3: Prove that it is true for $n = k + 1$

Step 4: Conclusion

Example 1 Prove that $1+2+3+...+n = \frac{1}{2} n(n+1)$ for $n \geq 1$.

Answer

Step 1: Prove that the statement is true for $n = 1$.

$LHS = n$
$= 1$

$RHS = \frac{1}{2} n (n + 1)$
$= \frac{1}{2} \times 1 \times (1 + 1)$
$= \frac{1}{2} \times 2$
$= 1$

$\therefore LHS = RHS$

Step 2: Assume that the statement is true for $n = k$.

$$1+2+3+...+k = \frac{1}{2} k(k+1)$$

Step 3 : Prove that it is true for $n = k + 1$

$$\underbrace{1+2+3+\ldots+k+(k+1)}_{\text{LHS}} = \underbrace{\frac{1}{2}(k+1)(k+1+1)}_{\text{RHS}}$$

$1 + 2 + 3 + \ldots + k + (k + 1) = ½ (k + 1)(k + 2)$

LHS = $\underbrace{1 + 2 + 3 + \ldots + k}_{\text{From step 2}} + (k + 1)$

$= \frac{1}{2}k(k+1) + (k + 1)$

= (k + 1) [½ k + 1] factorise (k + 1)

= ½ (k + 1) (k + 2)

= RHS

Step 4: **Conclusion**

Thus by mathematical induction the formula is valid for all $n \in N$.

Example 2 Prove that $\sum_{r=1}^{n} 5r-3=\frac{1}{2}n(5n-1)$.

Answer

$2+7+\ldots\ldots\ldots\ldots+ 5n - 3 = \frac{1}{2}n(5n-1)$

Step 1: Prove that the statement is true for $n=1$.

LHS = $(5n - 3)$	RHS= $½\, n\, (5n - 1)$
$= 5\times1 - 3$	$= ½ \times 1\, (5\times1 - 1)$
$= 5 - 3$	$= ½\, (5 - 1)$
$= 2$	$= 2$

$\therefore$ LHS = RHS

Step 2: Assume that the statement is true for $n=k$.

$2+7+\ldots\ldots\ldots\ldots+ 5k-3 = \frac{1}{2}k(5k-1)$

Step 3 : Prove that it is true for $n = k + 1$

$$2+7+\dots\dots\dots+5k-3 \quad +5(k+1)-3 \quad = \quad \frac{1}{2}(k+1[5(k+1)-1]$$

LHS

$$\underbrace{2+7+\dots\dots\dots+5k}-3+5(k+1)-3$$

$$=\frac{1}{2}k(5k-1) \qquad + [\,5(k+1)-3\,]$$

$$=\frac{1}{2}k(5k-1) \qquad +5k+5-3$$

$$= \frac{1}{2}k(5k-1) + 5k+2$$

$$= \frac{1}{2}k(5k-1) + \frac{5k+2}{1}\,\frac{\times 2}{\times 2}$$

$$= \frac{1}{2}k(5k-1) + \frac{1}{2}(10k + 4)$$

$= ½\ (5k^2 + 9k + 4)$

$=½\ (k + 1)\ (5k + 4)$

= RHS

RHS = ½ (k + 1) [5(k + 1) – 1]

= ½ (k + 1) (5k + 4)

A Short-cut Method

$5k^2 + 9k + 4$

factors

$5k \quad 4$

$k \quad 1$

Cross multiply to give middle term

$=½\ (k + 1)\ (5k + 4)$

Step 4: **Conclusion**

Thus by mathematical induction the formula is valid for all $n \in N$.

Exercise 6.2

Prove by mathematical induction that:

1. $3 + 7 + 11 \ldots.+ (4n - 1) = n(2n+1)$ is true for $n \in N$.

2. $1^2 + 2^2 + 3^2 + \ldots.+ n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true for $n \in N$.

3. $1{\cdot}2 + 2{\cdot}3 + \ldots.+ n(n+1) = \dfrac{n(n+1)(n+2)}{3}$ is true for $n \in N$.

4. $1 + 3 + 5 \ldots.+ (2n - 1) = n^2$ is true for $n \in N$.

5. $\displaystyle\sum_{r=1}^{n} 2^{r+1} = 2^2\left(2^n - 1\right)$ is true for $n \in N$.

6. $\displaystyle\frac{1}{2}\sum_{r=1}^{n} 2^r = 2^n - 1$ is true for $n \in N$.

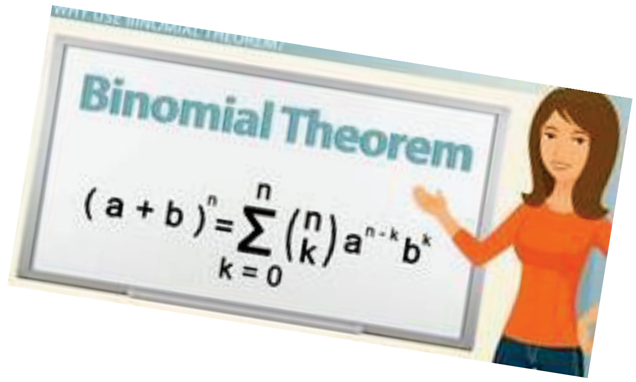
7. $\displaystyle\sum_{r=1}^{n} 3r^2 + r = n(n+1)^2$ is true for $n \in N$.

DID YOU KNOW?

A truly real-life example of mathematical induction is the sinking of the Titanic: The crew of the Titanic realized that the ship was doomed when they realized that the bulkhead that was being flooded would be completely flooded, and that when a given bulkhead was completely flooded, the next bulkhead would undergo the same fate, thus sinking the whole ship. ~ S. Shenoy (https://www.quora.com/What-is-the-use-of-Mathematical-Induction-in-real-life)

SUB – STRAND 6.3

Binomial Theorem



LEARNING OUTCOMES

6.3.1 Expand using Binomial Theorem.

6.3.2 Find particular terms, coefficients and the independent term.

6.3.1 Binomial Theorem

- Factorial !

  The factorial of a number (symbol $n!$) is defined as:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \ldots 3 \times 2 \times 1,\ n \in N$$

Example $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

This can be directly found using the calculator which has the key !

Press 5 ! = ________

- Combinations ^nc_r

  A combination is a selection of a certain number of elements from a set where the order of elements is not taken into account. The number of possible combinations of "r" elements from "n" things is denoted by:

$$^nc_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Example $^5C_2 = \dfrac{5!}{(5-2)!\,2!} = 10$

This can also be found using the ^nc_r key on the calculator.

Press $5\ ^nc_r\ 2$ = ________

- The Binomial Theorem

  The **binomial theorem** provides a useful method for raising any binomial to a nonnegative integral power:

$$(x+a)^n = \binom{n}{0}x^{n-0}a^0 + \binom{n}{1}x^{n-1}a^1 + \binom{n}{2}x^{n-2}a^2 + \ldots + \binom{n}{n}x^{n-n}a^n$$

Note:

- The power of x decreases from n to 0 while the power of a increases from 0 to n.
- The number of terms in the expansion is one greater than the power
- The sum of the powers of x and a (first and second term) always equals the power of the binomial n.
- The $(r+1)^{th}$ term which is the **general term** is given by $T_{r+1} = \binom{n}{r}x^{n-r}a^r$

  and $\binom{n}{r}$ is the binomial coefficient.

Example 1 Evaluate the following using calculator or otherwise:

a) $\frac{7!}{4!}$ b) ${}^{10}C_6$

Answers

a) $\frac{7!}{4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}} = 210$

b) ${}^{10}C_6 = \frac{10\,!}{(10-6)!\,6\,!} = 210$

Example 2 Expand $(x+a)^3$ using the binomial theorem.

Answer

$$(x+a)^3 = \binom{3}{0}x^{3-0}a^0 + \binom{3}{1}x^{3-1}a^1 + \binom{3}{2}x^{3-2}a^2 + \binom{3}{3}x^{3-3}a^3$$

$$= 1\times x^3 \times 1 \quad + 3\times x^2 \times a \quad + 3\times x^1 \times a^2 \quad + 1\times 1\times a^3$$

$$= x^3 + 3x^2a + 3xa^2 + a^3$$

Example 3 Expand $(2x - \frac{1}{x^2})^3$ using binomial theorem

Answer

$$(2x-\frac{1}{x^2})^3 = \binom{3}{0}(2x)^3(-\frac{1}{x^2})^0 + \binom{3}{1}(2x)^2(-\frac{1}{x^2})^1 + \binom{3}{2}(2x)^1(-\frac{1}{x^2})^2 + \binom{3}{3}(2x)^0(-\frac{1}{x^2})^3$$

$$= 1.8x^3.1 \quad + 3.(4x^2)(-\frac{1}{x^2}) \quad + 3.(2x).(\frac{1}{x^4}) \quad + (-\frac{1}{x^6})$$

$$= 8x^3 - 12 + \frac{6}{x^3} - \frac{1}{x^6}$$

Example 4 A term in an expansion of $(a+b)^n$ is $\binom{p}{k}(x^2)^3\left(-\frac{2}{x}\right)^5$.

Find a, b, n, p and k

Answer

The general term in the expansion of $(a+b)^n$ is $\binom{n}{k}a^{n-k}b^k$.

Now, comparing with the term $\binom{p}{k}(x^2)^3\left(-\frac{2}{x}\right)^5$

we have $a = x^2$, $b = -\frac{2}{x}$, $k = 5$ and

$$n - k = 3$$
$$\Rightarrow n = p = 8$$

Exercise 6.3.1

1. Use the binomial theorem to expand and simplify $(x + y)^4$

2. Expand and simplify $\left(2x - \frac{1}{2}\right)^4$

3. Use the binomial theorem to find the first three terms of $\left(\frac{3x^3}{4} - \frac{2}{x^2}\right)^5$

4. Use the binomial theorem to expand and simplify $\left(2x - \frac{1}{x^2}\right)^3$

5. A term in an expansion of $(a + b)^c$ is $\binom{n}{k}(3x^2)^4(-2y)^3$.
   a) Give the values of a, b, c, n and k.
   b) Write out the given term in simplified form.

DID YOU? KNOW

Binomial theorem is used in forecast services. The disaster forecast also depends upon the use of binomial theorems. Moreover, it allows engineers, to calculate the magnitudes of the projects and thus delivering accurate estimates of not only the costs but also time required to construct them. For contractors, it is a very important tool to help ensuring the costing projects is competent enough to deliver profits.~ Y. Bhalgat (https://www.quora.com/What-are-some-real-world-examples-of-the-use-of-the-binomial-theorem)

6.3.2 Finding particular terms, coefficients and the independent term

- To find a particular term in $(x+a)^n$, use the general formula and substitute the values of power (n), r, a and b.

$$T_{r+1}=\binom{n}{r}x^{n-r}a^r$$

- To Find the coefficient of variables with desired power or independent term
  - Use the general formula mentioned above and simplify
  - Equate power of the variable, with the desired power. In case of constant / independent term, equate power of variable to zero.
  - Solve for r
  - Go back and substitute in the original formula the value of r to find the answer

Finding a particular term

Example 1 What is the third term of $(a-3)^{10}$?

Answer

The general term in the expansion of $(x+a)^n$ is given by $\mathrm{T}_{r+1}=\binom{n}{r}x^{n-r}a^r$

For the third term r = 2,

$$\mathrm{T}_{2+1}=\binom{10}{2}a^{10-2}(-3)^2$$

$$\mathrm{T}_3=45\times a^8\times 9$$

$$=405a^8$$

Finding the coefficient

Example 2 Use the **binomial theorem** to find coefficient of x^4 in the expansion of $(x+1)^{10}$.

Answer

Use the general formula mentioned above and simplify

$$T_{r+1} = \binom{10}{r} x^{10-r} (1)^r$$

Equate power of the variable x, with the desired power.

$$x^{10-r} = x^4$$

$$10 - r = 4$$

Solve for r, the term

$$10 - \cancel{r +} r = 4 + r$$

$$r = 10 - 4$$

$$r = 6$$

Go back and substitute into the original formula the value of r

The required term is $T_{6+1} = \binom{10}{6} x^{10-6} (1)^6$

$$= \binom{10}{6} (x)^4 (1)^6$$

$$= (210)\, x^4 (1)$$

$\therefore$ Coefficient is 210

Finding Constant Term (term independent of x)

Example 3 Finding the term independent of x in the expansion of

$$\left(3x^2 - \frac{1}{x}\right)^{12}$$

Answer

Use the general formula mentioned above and simplify

$$T_{r+1} = \binom{12}{r} (3x^2)^{12-r} \times \left(-\frac{1}{x}\right)^r$$

$$= \binom{12}{r} \bullet 3^{12-r} \bullet x^{24-2r} \bullet \frac{(-1)^r}{x^r}$$

$$T_{r+1} = \binom{12}{r} \bullet 3^{12-r} \bullet x^{24-2r} \bullet x^{-r} \bullet (-1)^r$$

$$= \binom{12}{r} \bullet 3^{12-r} \bullet x^{24-2r-r} \bullet (-1)^r$$

$$= \binom{12}{r} \bullet 3^{12-r} \bullet x^{24-3r} \bullet (-1)^r$$

Equate power of the variable x to zero.

$$x^{24-3r} = x^0$$

$$24 - 3r = 0$$

Solve for r, the term

$$24 - 3r = 0$$

$$24 = 3r$$

$$\frac{3r}{3} = \frac{24}{3}$$

$$r = 8$$

Go back and substitute in the original formula the value of r

$$T_{8+1} = \binom{12}{8} \bullet 3^{12-8} \bullet x^{24-3\times 8} \bullet (-1)^8$$

$$= 495 \bullet 81$$

$$= 40\,095$$

Exercise 6.3.2

1. Find the constant term in the expansion of $\left(x^2 + \frac{1}{x}\right)^6$

2. Use the **binomial theorem** to find the coefficient of x^{-2} in the expansion of $\left(x^2 + \frac{1}{x}\right)^8$

3. Find the 15th term of $\left(3x^2 - \frac{1}{x}\right)^{15}$

4. Find the coefficient of x^{10} in the expansion of $(2x - 3x^2)^7$.

SUB – STRAND 6.4

Partial Fractions

Partial fraction decomposition table

Type	*Factor example*	*Decomposition*
Linear factor	$(x-4)$	$\frac{A}{x-4}$
Repeated linear factor	$(x-4)^2$	$\frac{A}{(x-4)}+\frac{B}{(x-4)^2}$
Quadratic irreducible factor	(x^2+4)	$\frac{Ax+B}{(x^2+4)}$

LEARNING OUTCOMES

6.4.1 Write fractions with distinct linear factors in the denominator as partial fractions.

6.4.2 Write fractions with repeated linear factors in the denominator as partial fractions.

6.4.3 Write fractions with irreducible quadratic factors in the denominator as partial fractions.

6.4.1 Denominator with distinct linear factors

If we add $\frac{1}{(x+4)}+\frac{2}{(x-3)}$, we get $\frac{3x+5}{(x+4)(x-3)}$

The reverse process i.e. $\frac{3x+5}{(x+4)(x-3)}$ splitting as $\frac{1}{(x+4)}+\frac{2}{(x-3)}$ is known as decomposition into partial fractions.

If the denominator of the terms in the partial fraction is **linear**, we make the numerator a constant.

Follow the following steps while decomposing into partial fraction with distinct linear factors:

1. Factorize the denominator so that you get the distinct linear factors
2. Separate the factors in denominator. Let their constant to be A, B, C, etc
3. Make denominators same.
4. Equate the numerators.
5. Solve for the variables.

Example 1 Express $\frac{3x+5}{x^2+x-12}$ as a sum of partial fractions.

Answer

$$\frac{3x+5}{x^2+x-12}=\frac{3x+5}{(x+4)(x-3)}$$ Factorize the denominator

$$\frac{3x+5}{(x+4)(x-3)}=\frac{A}{(x+4)}+\frac{B}{(x-3)}$$ Separate the factors in denominator. Let their constant to be A and B

$$\frac{3x+5}{(x+4)(x-3)}=\frac{A(x-3)+B(x+4}{(x-3)(x+4)}$$ Make denominators same

$$3x+5=A(x-3)+B(x+4)$$ Since the denominator on both sides is same, we can equate the numerators

Let x = 3 as it will eliminate A ;

$$3x+5=A(x-3)+B(x+4)$$
$$3(3)+5=A(3-3)+B(3+4)$$
$$14=7B$$
$$B=2$$

Let x = - 4 as this will eliminate B;

$$3x+5=A(x-3)+B(x+4)$$
$$3(-4)+5=A(-4-3)+B(-4+4)$$
$$-7=-7A$$
$$A=1$$

$$\therefore \frac{3x+5}{x^2+x-12}=\frac{1}{(x+4)}+\frac{2}{(x-3)}$$

Method 2: Equating Coefficients

$3x+5=A(x-3)+B(x+4)$

$3x+5=Ax-3A+Bx+4B$

Consider coefficients of x : $3= A+B$.
Consider constants : $5= -3A+4B$

Solve simultaneously:

$3=A+B \rightarrow A=3-B$ [for first equation, Make A the subject and substitute in second equation]

$5= -3A+4B$

$5= -3(3-B)+4B$

$5= -9+7B$

$B=2$

$A=3-B$ [go back and substitute in any equation]

$A= 3-2$

$=1$

$$\therefore \frac{3x+5}{x^2+x-12}=\frac{1}{(x+4)}+\frac{2}{(x-3)}$$

Example 2 Express $\frac{-2\left(x^2+4x+7\right)}{(x^2-1)(x+3)}$ as a sum of partial fractions.

Answer

$$\frac{-2\left(x^2+4x+7\right)}{(x^2-1)(x+3)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+3}$$

$$=\frac{A(x-1)(x+3)+B(x+1)(x+3)+C(x+1)(x-1)}{(x^2-1)(x+3)}.$$

Since the denominator on both sides is the same, we can equate the numerators

$$-2\left(x^2+4x+7\right)=A(x-1)(x+3)+B(x+1)(x+3)+C(x+1)(x-1)$$

Let $x=1$,

$$-2\left(1^2+4(1)+7\right)=A(1-1)(1+3)+B(1+1)(1+3)+C(1+1)(1-1)$$

$$-24=8B \qquad \Rightarrow B=-3$$

Let $x=-3$,

$$-2\left((-3)^2+4(-3)+7\right)=A(-3-1)(-3+3)+B(-3+1)(-3+3)+C(-3+1)(-3-1)$$

$$-8=8C \qquad \Rightarrow C=-1$$

Now substitute any other value for x to solve for A.

let $x = 0$,

$$-2\left(0^2+4(0)+7\right) = A(0-1)(0+3) - B(0+1)(0+3) - C(0+1)(0-1)$$

$$\Rightarrow -14 = -3A + 3B - C$$

$$\Rightarrow -14 = -3A + 3(-3) - (-1) \quad (\text{recall that } B = -3 \text{ and } C = -1)$$

$$-14 = -3A - 9 + 1$$

$$\Rightarrow A = 2$$

Therefore, $\dfrac{-2\left(x^2+4x+7\right)}{(x^2-1)(x+3)} = \dfrac{2}{x+1} - \dfrac{3}{x-1} - \dfrac{1}{x+3}$

6.4.2 Repeated linear factors

Repeated linear factors refer to a factor in the denominator that occurs more than once.
The process for repeated factors is slightly different from the process for distinct linear. For each non-repeated factor in the denominator, follow the process for distinct linear factors.

Given

$$f(x) = \frac{P(x)}{(x-r)^n}$$

- Where polynomial $P(x)$ has degree $< n$

then $f(x)$ can be decomposed into

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \ldots + \frac{A_n}{(x-r)^n}$$

Example 3 Express $\dfrac{3x+2}{(x+1)(x-1)^2}$ as a sum of partial fractions.

Answer

$$\frac{3x+2}{(x+1)(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\frac{3x+2}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

Since the denominator on both sides is same, we can equate the numerators:

$$3x+2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

Carefully select x values to obtain the values of the coefficients that makes the expression zero.

$$3x+2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

Let x = 1,

$$3(1)+2 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$$

$$5 = 2C$$

$$C = \frac{5}{2}$$

Let x = -1,

$$3(-1)+2 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$$

$$-1 = 4A$$

$$A = -\frac{1}{4}$$

Let x = 0,

$$3x+2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$$3(0)+2 = A(0-1)^2 + B(0+1)(0-1) + C(0+1)$$

$$2 = A - B + C$$

$$2 = -\frac{1}{4} + \frac{5}{2} - B$$

$$B = \frac{1}{4}$$

Therefore $\dfrac{3x+2}{(x+1)(x-1)^2} = -\dfrac{1}{4(x+1)} + \dfrac{1}{4(x-1)} + \dfrac{5}{2(x-1)^2}$

Example 4 Express $\dfrac{6x^2-x-2}{x^2(x+1)}$ as a sum of partial fractions.

Answer

$$\frac{6x^2-x-2}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

Make denominators same

$$\frac{6x^2-x-2}{x^2(x+1)}=\frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}.$$

Since the denominator on both sides is same, we can equate the numerators:

$$6x^2-x-2=Ax(x+1)+B(x+1)+Cx^2$$

Let x = 0, then

$$6(0^2)-0-2=B(0+1)$$

$$\Rightarrow B=-2$$

Let $x=-1$, then

$$6(-1)^2-(-1)-2=C(-1)^2$$

$$\Rightarrow C=5$$

Let $x=1$, then

$$6(1^2)-1-2=A(1)(1+1)+B(1+1)+C(1^2) \quad \Rightarrow 3=2A+2B+C$$

$$\Rightarrow 3=2A+2(-2)+5 \ (\text{recall that } B=-2 \text{ and } C=5) \quad \Rightarrow A=2/2=1$$

Therefore, $\dfrac{6x^2-x-2}{x^2(x+1)}=\dfrac{1}{x}-\dfrac{2}{x^2}+\dfrac{5}{x+1}$

6.4.3 Quadratics which cannot be factorised

Quadratics which cannot be factorized refers to a quadratic that is in the denominator that cannot be factored.
When setting up the partial fraction decomposition, the numerator should have a linear term Ax + B.
This is because the degree of the denominator is 2, so the degree of the numerator should be 1 (linear term).

Example 5 Express $\dfrac{x^2-1}{(x^2+1)(x-2)}$ as a sum of partial fractions.

Answer

$$\frac{x^2-1}{(x^2+1)(x-2)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-2)}$$

$$\frac{x^2-1}{(x-2)(x^2+1)}=\frac{(x-2)(Ax+B)+C(x^2+1)}{(x-2)(x^2+1)}$$

Since the denominator on both sides is same, we can equate the numerators.

$$x^2-1=(x-2)(Ax+B)+C(x^2+1)$$

Let x = 2,

$$2^2-1=(2-2)(A\times 2+B)+C(2^2+1)$$

$$3=5C$$

$$C=\frac{3}{5}$$

Let x = 0,

$$0^2-1=(0-2)(A(0)+B)+C(0^2+1)$$

$$-1=-2B+C$$

$$-1=-2B+\frac{3}{5}$$

$$B=\frac{4}{5}$$

Let x = 1,

$$(1)^2-1=(1-2)(A+B)+C(1^2+1)$$

$$0=-1(A+\frac{4}{5})+2\left(\frac{3}{5}\right)$$

$$=-A-\frac{4}{5}+\frac{6}{5}$$

$$0=-A+\frac{2}{5}$$

$$A=\frac{2}{5}$$

Hence $\dfrac{x^2-1}{(x^2+1)(x-2)}=\dfrac{2x+4}{5(x^2+1)}+\dfrac{3}{5(x-2)}$

Example 6 Express $\dfrac{2x+1}{(x-2)(x^2+1)}$ as a sum of partial fractions.

Answer

Make denominators same

$$\frac{2x+1}{(x-2)(x^2+1)}=\frac{A}{x-2}+\frac{Bx+C}{\left(x^2+1\right)}=\frac{A\left(x^2+1\right)+\left(Bx+C\right)\left(x-2\right)}{(x-2)(x^2+1)}$$

Since the denominator on both sides is same, we can equate the numerators

$$2x+1=A\left(x^2+1\right)+\left(Bx+C\right)\left(x-2\right)$$

Let $x=2$,

$$2\left(2\right)+1=A\left(\left(2\right)^2+1\right)+\left(B\left(2\right)+C\right)\left(2-2\right)$$

$$5=5A \qquad \Rightarrow A=1$$

Let $x=0$,

$$2\left(0\right)+1=1\left(\left(0\right)^2+1\right)+\left(B\left(0\right)+C\right)\left(0-2\right)$$

$$1=1-2C \qquad \Rightarrow C=0$$

Let $x=1$,

$$2\left(1\right)+1=A\left(\left(1\right)^2+1\right)+\left(B\left(1\right)+C\right)\left(1-2\right)$$

$$\Rightarrow 3=2A-B-C \qquad \Rightarrow 3=2\left(1\right)-B+0 \qquad \text{(recall that } A=1 \text{ and } C=0)$$

$$3=2-B \qquad \Rightarrow B=-1$$

Therefore

$$\frac{2x+1}{(x-2)(x^2+1)}=\frac{1}{x-2}-\frac{x}{\left(x^2+1\right)}$$

Exercise 6.4

Express as a sum of partial fractions.

1. $\dfrac{-4x^2+11x-6}{x^2(x-3)}$
2. $\dfrac{15-4x-x^2}{(x+1)(x-2)^2}$
3. $\dfrac{3x+2}{(x+1)(x-1)^2}$
4. $\dfrac{8x+11}{x^2-x-2}$
5. $\dfrac{3}{(x-1)\left(x^2+2x+1\right)}$
6. $\dfrac{-x-2}{x\left(x^2+1\right)}$
7. $\dfrac{10x+24}{(x-3)\left(x^2+9\right)}$
8. $\dfrac{-x-2}{x\left(x^2-4\right)}$

Review Exercise 6

1. Use **mathematical induction** to prove that the following formula is valid for all positive integers, n.

$$8+72+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+8\times 9^{n-1}=9^{n}-1$$

2. Prove by induction that $5+10+15+20+\ldots\ldots+5n=\dfrac{5n(n+1)}{2}$

3. Express the following as a sum of partial fractions:

   a) $\dfrac{2x}{(x-3)(x-5)}$

   b) $\dfrac{2x}{(x-3)^{2}(x-5)}$

   c) $\dfrac{2x}{(x^{2}-9)(x-5)}$

4. Find the values of the constants **A** and **B** in the expression

$$\frac{5x}{(x+2)(x^{2}+1)}=\frac{\mathbf{A}}{x+2}+\frac{2x+\mathbf{B}}{x^{2}+1}$$

5. An expression is given as $\left(3x^{2}-\dfrac{1}{x}\right)^{12}$

   a) Find the term independent of x in the expansion of $\left(3x^{2}-\dfrac{1}{x}\right)^{12}$

   b) Use the **binomial theorem** to find the coefficient of x^{-2} in the expansion of $\left(3x^{2}-\dfrac{1}{x}\right)^{12}$

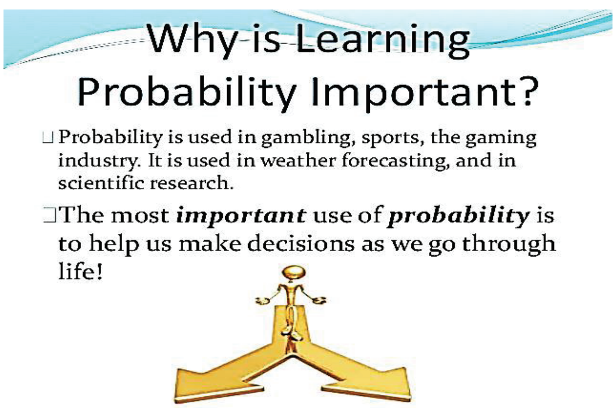
   c) Find the 5$^{\text{th}}$ term of $\left(3x^{2}-\dfrac{1}{x}\right)^{12}$

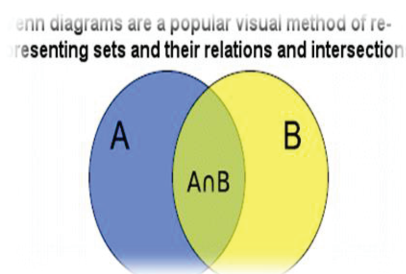
STRAND SEVEN

PROBABILITY AND INFERRENTIAL STATISTICS

SUB – STRAND 7.1

Probability





LEARNING OUTCOMES

7.1.1 Identify types of events.
Calculate probabilities associated with these events.

7.1.2 Calculate probabilities using the addition rule.

7.1.1 Types of Events

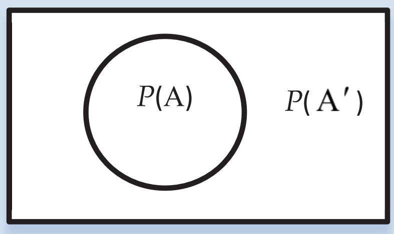
7.1.1.1 Complementary events

- The complement of event A (symbol A') means every outcome which is not in event A.

  Example — If event A is getting a pass in a test, then A' is getting a fail in a test.

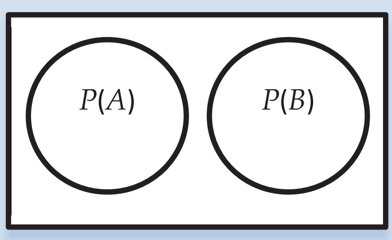
- Two events are complementary if their probabilities add up to one.

$$P(A) + P(A') = 1$$



7.1.1.2 Mutually exclusive events

Mutually Exclusive Events have no outcomes in common.



- Intersection

  Since no outcome is common, then there will not be any intersection, i.e.

$A \cap B = \emptyset \; or \; \{\}$ null set	$P(A \cap B) = 0$

- Union

$$\boldsymbol{P(A \cup B) = P(A) + P(B)}$$

Example 1 Given $P(A) = 0.15$ and $P(B) = 0.43$. Find $P(A \cup B)$ given events A and B are mutually exclusive?

Answer

$$\boldsymbol{P(A \cup B) = P(A) + P(B)}$$

$$= 0.15 + 0.43$$
$$= 0.58$$

7.1.1.3 Independent events

Events A and B are independent if their occurrence do not affect each other.
$\Rightarrow$*Can happen together*

- Probability of A and B occurring together

$$\boxed{P(A \cap B) = P(A).\ P(B)}$$

Example 2 Given $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.12$. Are events A and B independent?

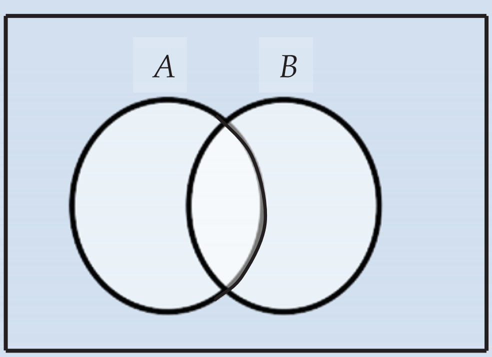
Answer

$P(A \cap B) = P(A).\ P(B)$
$= 0.3 \times 0.4$
$= 0.12$

Since $P(A \cap B) = P(A).\ P(B)$, the events A and B are Independent.

7.1.2 Addition rule

If A and B are not disjoint then



$$\boxed{\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}}$$

Example 3 $P(A) = 0.2$, $P(B) = 0.3$ and $P(A \cap B) = 0.06$.

Find:

a) $P(A')$
b) $P(A \cup B)$

Answers

a) $P(A')$
$= 1 - P(A)$
$= 1 - 0.2$
$= 0.8$

b) $P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$
$= 0.2 + 0.3 - 0.06$
$= 0.44$

Exercise 7.1.1

1. The probability that event A occurs is 0.60 while the probability that event B occurs is 0.25. The probability that both A and B occur is 0.12.
   a) Explain why events A and B are not independent.
   b) Find the probability of neither A nor B occurring.

2. The National Bank has 2 computers. The probability that Computer A will break down once in a month is 0.05. The probability that Computer B will break down once in a month is 0.1. In a given month and assuming that the events are independent,
   a) what is the probability that either Computer A or Computer B will break down?
   b) what is the probability that neither Computer A nor Computer B will break down?

3. An experiment consists of rolling 2 fair dice. Event A is "the number on the first die is 5" and event B is "the sum of the numbers is 10".
   a) Explain why events A and B are not mutually exclusive.
   b) What is the probability that event A or B occurs?

4. Box A and Box B contain identical items. Box A has 10 items while Box B has 8. Three items from each box are defective.
   If an item is drawn from each box, find the probability that:
   a) both items are good.
   b) only one of the items is defective.

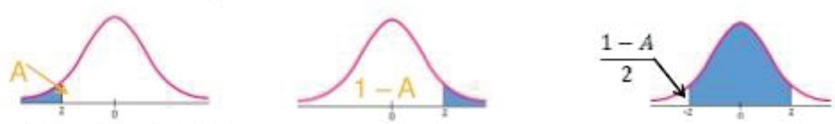
SUB – STRAND 7.2

Inverse Normal Problems

Inverse Normal Distribution

- The **inverse normal distribution** is used to find z-values given an area/probability.

1. Draw a sketch and find the area to the left based on which case you have



2. Use the table

- Table: Find the area in the body of the table and list the corresponding z-score (you may need to convert to an x value)

LEARNING OUTCOME

7.2.1 Solve applied inverse normal distribution problems.

7.2.1 Inverse Normal Distribution

➢ Table to be used:

INVERSE NORMAL DISTRIBUTION

For a listed value of P, the table gives the value of z such that the standardised normal variate Z has a probability P of lying between 0 and z. e.g. P = 0.30, z = .8416 means P(0 < Z < 0.8416) = .30

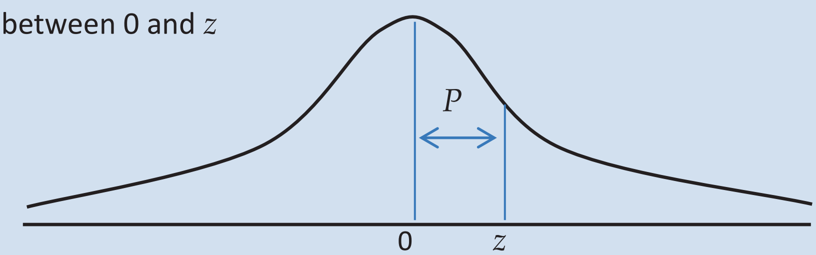
P	z	P	z	P	z	P	z	P	z	P	z
.00	.0000	.10	.2533	.20	.5244	.30	.8416	.40	1.2816	.470	1.8808
.01	.0251	.11	.2793	.21	.5534	.31	.8779	.41	1.3408	.471	1.8957
.02	.0502	.12	.3055	.22	.5828	.32	.9154	.42	1.4051	.472	1.9110
.03	.0753	.13	.3319	.23	.6128	.33	.9542	.43	1.4758	.473	1.9268
.04	.1004	.14	.3585	.24	.6433	.34	.9945	.44	1.5548	.474	1.9431
.05	.1257	.15	.3853	.25	.6745	.35	1.0364	.45	1.6449	.475	1.9600
.06	.1510	.16	.4125	.26	.7063	.36	1.0803	.46	1.7507	.476	1.9774
.07	.1764	.17	.4399	.27	.7388	.37	1.1264			.477	1.9954
.08	.2019	.18	.4677	.28	.7722	.38	1.1750			.478	2.0141
.09	.2275	.19	.4959	.29	.8064	.39	1.2265			.479	2.0335

P	z	P	z	P	z	P	z	P	z
.480	2.0537	.485	2.1701	.490	2.3263	.495	2.5758	.4995	3.2905
.481	2.0749	.486	2.1973	.491	2.3656	.496	2.6521	.4999	3.7190
.482	2.0969	.487	2.2262	.492	2.4089	.497	2.7478	.49995	3.8906
.483	2.1201	.488	2.2571	.493	2.4573	.498	2.8782	.49999	4.2649
.484	2.1444	.489	2.2904	.494	2.5121	.499	3.0902		

➢ Conversion formula

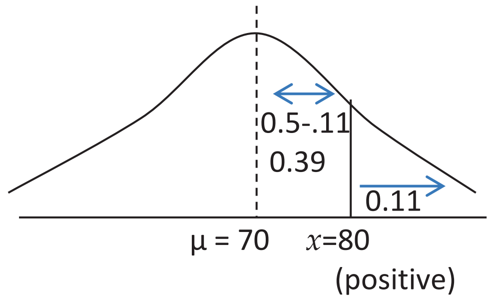
$$Z = \frac{x-\mu}{\sigma}$$

➢ P is between 0 and z



Example 1 X has a **normal distribution** with mean = 70.
Given $P(X > 80) = 0.11$, find the standard deviation.

Answer



P	z
.30	.8416
.31	.8779
.32	.9154
.33	.9542
.34	.9945
.35	1.0364
.36	1.0803
.37	1.1264
.38	1.1750
.39	1.2265

$P(X > 80) = 0.11$

P (z > $\frac{x-\mu}{\sigma}$) = 1.2265

z = 1.2265, substituting values

$1.2265 = \frac{80-70}{\sigma}$ Solve σ

$1.2265 \times \sigma = 80 - 70$

$\sigma = \frac{10}{1.2265}$

σ = 8.15

Example 2 Intelligence quotient (IQ) scores of people are normally distributed with standard deviation of 10. The probability that a person, selected at random, has an IQ greater than 95 is 0.69. Calculate the mean IQ score.

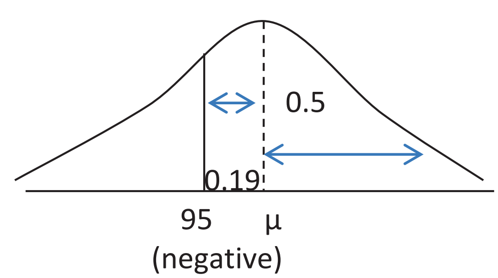
Answer

P (X > 95) = 0.69

P (z > $\frac{x-\mu}{\sigma}$) = 0.69

P (z > $\frac{95-\mu}{10}$) = 0.69

Normal Curve



Look for P= 0.19 in the inverse table

INVERSE NORMAL DISTRIBUTION			
P	z	P	z
.00	.0000	.10	.2533
.01	.0251	.11	.2793
.02	.0502	.12	.3055
.03	.0753	.13	.3319
.04	.1004	.14	.3585
.05	.1257	.15	.3853
.06	.1510	.16	.4125
.07	.1764	.17	.4399
.08	.2019	.18	.4677
.09	.2275	.19	.4959

z = — 0.4959, substituting values

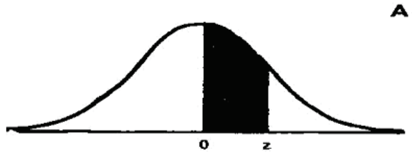
$- 0.4959 = \frac{95-\mu}{10}$ Solve μ

$- 0.4959 \times 10 = 95 - \mu$

μ = 95 + 4.959

μ= 99.96

Not all P values are given in the inverse normal table, so we have to use the normal distribution table as shown below.



AREAS UNDER NORMAL PROBABILITY CURVE

The tabulated value is the probability that the standardized normal variate Z (with μ=0, σ=1) lies between 0 and z.
e.g. P(0 < Z < 1.43) = 42.36%

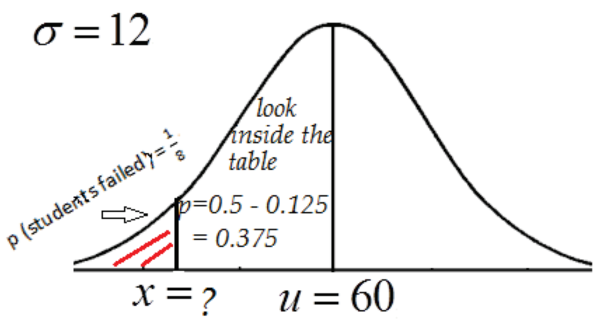
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359	4	8	12	16	20	24	28	32	36
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754	4	8	12	16	20	24	28	32	36
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141	4	8	12	15	19	22	27	31	35
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549	3	6	10	13	16	19	23	26	29
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852	3	6	9	12	15	18	21	24	27
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133	3	6	8	11	14	17	19	22	25
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389	3	5	8	10	13	15	18	20	23
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621	2	5	7	9	12	14	16	18	21
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830	2	4	6	8	10	12	14	16	19
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015	2	4	5	7	9	11	13	15	16
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177	2	3	5	6	8	10	11	13	14
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319	1	3	4	6	7	8	10	11	13
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441	1	2	4	5	6	7	8	10	11
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545	1	2	3	4	5	6	7	8	9
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633	1	2	3	3	4	5	6	7	8
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706	1	1	2	3	4	4	5	6	6
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767	1	1	2	2	3	4	4	5	5
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817	0	1	1	2	2	3	3	4	4
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857	0	1	1	2	2	2	3	3	4
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890	0	1	1	1	2	2	2	3	3
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916	0	0	1	1	1	2	2	2	2
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936	0	0	1	1	1	1	1	2	2
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952	0	0	0	1	1	1	1	1	1
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964	0	0	0	0	1	1	1	1	1
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974	0	0	0	0	0	1	1	1	1
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981	0	0	0	0	0	0	0	0	1
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986	0	0	0	0	0	0	0	0	1
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990	0	0	0	0	0	0	0	0	0
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993	0	0	0	0	0	0	0	0	0
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995	0	0	0	0	0	0	0	0	0
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997	0	0	0	0	0	0	0	0	0
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998	0	0	0	0	0	0	0	0	0
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	0	0	0	0	0	0	0	0	0
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	0	0	0	0	0	0	0	0	0

EXAMPLE 3: Fifteen thousand students sat an exam and their marks were normally distributed with a mean of 64 % and a standard deviation of 12.

If one in eight students failed, what was the minimum mark required for a pass?

Answer

p (students failed) = $\frac{1}{8}$ or 0.125 , x= ?



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359	4	8	12	16	20	24	28	32	36
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754	4	8	12	16	20	24	28	32	36
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141	4	8	12	15	19	22	27	31	35
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549	3	6	10	13	16	19	23	26	29
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852	3	6	9	12	15	18	21	24	27
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133	3	6	8	11	14	17	19	22	25
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389	3	5	8	10	13	15	18	20	23
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621	2	5	7	9	12	14	16	18	21
1.1	.3643	.3665	.3686	.3708	.3729	$\boxed{.3749}$	.3770	.3790	.3810	.3830	2	4	6	8	10	12	14	16	19
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015	2	4	5	7	9	11	13	15	16
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177	2	3	5	6	8	10	11	13	14
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319	1	3	4	6	7	8	10	11	13

The closest entry **below** 0.375 is 0.3749 which corresponds to z value of 1.15. This differs from 0.375 by 0.0001. The closest value to 0.0001 in the difference column on the right is '2' in the column headed '1'.

z = 1.151

Thus z = -1.151, use formula to find the minimum mark, x

$$z = \frac{x - \mu}{\sigma}$$

$$-1.151 = \frac{x - 64}{12}$$

$$x = 50.19$$

Therefore, the minimum mark required for a pass is 50.19%

Exercise 7.2.1

1. The final exam marks of a class of 500 students are normally distributed with a mean of 62 marks & standard deviation of 15 marks.

   a) What is the cut – off mark for A grade if the top 16.6% of the students in the class are awarded an A grade?

   b) The E grade is the lowest grade in the examination. What is the cut – off mark for E grade if 2.55% of the students in the class are awarded an E grade?

2. Find k if $P(0<z<k)=0.12$

3. A normal random variable X has **mean** of 70. Given that $\mathrm{P}(X>80)=0.05$, find the **standard deviation**.

4. A normal random variable X has **mean** of 50. Given that $P(X<80)=0.75$, find the **standard deviation**.

5. A normal random variable X has **standard deviation** of 50. Given that $P(X<50)=0.01$, find the **mean**.

6. A normal distribution, X, has a **standard deviation** of 4. Given that P (X < 70) = 0.7, calculate the **mean**.

7. The marks for a college examination are normally distributed with a mean of 56. If **6%** of all the students who sat for the examination had marks **greater than 80**, find the standard deviation for the distribution.

Probabilities are based on long-term percentages (over thousands of trials), so when you apply them to a group, the group has to be large enough (the larger the better, but at least 1,500 or so items or individuals) for the probabilities to really apply. Here's an example where long-term interpretation makes sense in place of short-term interpretation.~ anonymous (http://catalogimages.wiley.com/images/db/pdf/0471751413.excerpt.pdf)

SUB – STRAND 7.3

Binomial Distribution

Formula of Binomial Probability Distribution

$$P(X = x) = C_x^n p^x (1-p)^{n-x} = C_x^n p^x q^{n-x}, \textit{where } x = 0,\ 1,\ 2,\ \cdots,\ n$$

Where

p:	probability of success
q:	probability of failure
note:	q=1-p
n:	number of trials
x:	number of successes

This starts the count of number of ways event can occur.

This is the probability of success for x trials.

$$P(x) = \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x}$$

This ends the count of number of ways event can occur.

This deletes duplications.

This is the probability of failure for the x trials.

LEARNING OUTCOMES

7.3.1 Define binomial experiments.
Calculate binomial probabilities using formulae and tables.

7.3.1 Binomial probabilities

The binomial distribution has **two possible outcomes** (the prefix “bi” means two).

Example: A coin has only two possible outcomes: heads or tails and taking a test has two possible outcomes: pass or fail.
The two outcomes are called **success or failure.**

- Properties of Binomial Experiment:
  1. The experiment consists of ‘n’ repeated trials
  2. Only two possible outcomes (Success or Failure)
  3. Probability of success is the same for each trial.
  4. Each trial is independent of each other

- The binomial distribution formula is:

$$P(X=x) = \binom{n}{x} p^x q^{n-x} \quad , x = 0, 1, 2, 3, \ldots\ldots\ldots n$$

where:

P = binomial probability
x = total number of “successes”
p = probability of success
q = probability of failure ($q = 1 - p$)
n = number of trials

Example 1 If 20% of the bulbs produced by a factory are faulty, determine the probability that out of sample of 12 randomly chosen bulbs, one will be faulty.

Answer

Given : $n = 12$, $p = 20\% = 0.20$, $q = 1 - 0.20 = 0.80$,
$x = 1$ (one will be faulty)

$$\text{Probability} = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{12}{1} (0.2)^1 (0.8)^{12-1}$$

$$= 0.2062$$

7.3.2 Table of Binomial probabilities for Individual terms

➢ You may use the Binomial distribution table for individual terms:

BINOMIAL DISTRIBUTION; INDIVIDUAL TERMS

Tabulated values are $P(X = x) = \binom{n}{x} \pi^x (1 - \pi)^{n-x}$ for certain values of n, π. If $\pi > .50$ use $P(Y = y) = \binom{n}{y} \pi_1^y (1 - \pi_1)^{n-y}$ where $\pi_1 = 1 - \pi$, $y = n - x$.

						π						
n	x	.01	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
2	0	.9801	.9025	.8100	.7225	.6400	.5625	.4900	.4225	.3600	.3025	.2500
	1	.0198	.0950	.1800	.2550	.3200	.3750	.4200	.4550	.4800	.4950	.5000
	2	.0001	.0025	.0100	.0225	.0400	.0625	.0900	.1225	.1600	.2025	.2500
3	0	.9703	.8574	.7290	.6141	.5120	.4219	.3430	.2746	.2160	.1664	.1250
	1	.0294	.1354	.2430	.3251	.3840	.4219	.4410	.4436	.4320	.4084	.3750
	2	.0003	.0071	.0270	.0574	.0960	.1406	.1890	.2389	.2880	.3341	.3750
	3		.0001	.0010	.0034	.0080	.0156	.0270	.0429	.0640	.0911	.1250
4	0	.9606	.8145	.6561	.5220	.4096	.3164	.2401	.1785	.1296	.0915	.0625
	1	.0388	.1715	.2916	.3685	.4096	.4219	.4116	.3845	.3456	.2995	.2500
	2	.0006	.0135	.0486	.0975	.1536	.2109	.2646	.3105	.3456	.3675	.3750
	3		.0005	.0036	.0115	.0256	.0469	.0756	.1115	.1536	.2005	.2500
	4			.0001	.0005	.0016	.0039	.0081	.0150	.0256	.0410	.0625
5	0	.9510	.7738	.5905	.4437	.3277	.2373	.1681	.1160	.0778	.0503	.0313
	1	.0480	.2036	.3281	.3915	.4096	.3955	.3602	.3124	.2592	.2059	.1563
	2	.0010	.0214	.0729	.1382	.2048	.2637	.3087	.3364	.3456	.3369	.3125
	3		.0011	.0081	.0244	.0512	.0879	.1323	.1811	.2304	.2757	.3125
	4			.0004	.0022	.0064	.0146	.0283	.0488	.0768	.1128	.1563
	5				.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0313
6	0	.9415	.7351	.5314	.3771	.2621	.1780	.1176	.0754	.0467	.0277	.0156
	1	.0571	.2321	.3543	.3993	.3932	.3560	.3025	.2437	.1866	.1359	.0938
	2	.0014	.0305	.0984	.1762	.2458	.2966	.3241	.3280	.3110	.2780	.2344
	3		.0021	.0146	.0415	.0819	.1318	.1852	.2355	.2765	.3032	.3125
	4		.0001	.0012	.0055	.0154	.0330	.0595	.0951	.1382	.1861	.2344
	5			.0001	.0004	.0015	.0044	.0102	.0205	.0369	.0609	.0938
	6					.0001	.0002	.0007	.0018	.0041	.0083	.0156
7	0	.9321	.6983	.4783	.3206	.2097	.1335	.0824	.0490	.0280	.0152	.0078
	1	.0659	.2573	.3720	.3960	.3670	.3115	.2471	.1848	.1306	.0872	.0547
	2	.0020	.0406	.1240	.2097	.2753	.3115	.3177	.2985	.2613	.2140	.1641
	3		.0036	.0230	.0617	.1147	.1730	.2269	.2679	.2903	.2918	.2734
	4		.0002	.0026	.0109	.0287	.0577	.0972	.1442	.1935	.2388	.2734
	5			.0002	.0012	.0043	.0115	.0250	.0466	.0774	.1172	.1641
	6				.0001	.0004	.0013	.0036	.0084	.0172	.0320	.0547
	7						.0001	.0002	.0006	.0016	.0037	.0078
8	0	.9227	.6634	.4305	.2725	.1678	.1001	.0576	.0319	.0168	.0084	.0039
	1	.0746	.2793	.3826	.3847	.3355	.2670	.1977	.1373	.0896	.0548	.0313
	2	.0026	.0515	.1488	.2376	.2936	.3115	.2965	.2587	.2090	.1569	.1094
	3	.0001	.0054	.0331	.0839	.1468	.2076	.2541	.2786	.2787	.2568	.2188
	4		.0004	.0046	.0185	.0459	.0865	.1361	.1875	.2322	.2627	.2734
	5			.0004	.0026	.0092	.0231	.0467	.0808	.1239	.1719	.2188
	6				.0002	.0011	.0038	.0100	.0217	.0413	.0703	.1094
	7					.0001	.0004	.0012	.0033	.0079	.0164	.0313
	8							.0001	.0002	.0007	.0017	.0039
9	0	.9135	.6302	.3874	.2316	.1342	.0751	.0404	.0207	.0101	.0046	.0020
	1	.0830	.2985	.3874	.3679	.3020	.2253	.1556	.1004	.0605	.0339	.0176
	2	.0034	.0629	.1722	.2597	.3020	.3003	.2668	.2162	.1612	.1110	.0703
	3	.0001	.0077	.0446	.1069	.1762	.2336	.2668	.2716	.2508	.2119	.1641
	4		.0006	.0074	.0283	.0661	.1168	.1715	.2194	.2508	.2600	.2461
	5			.0008	.0050	.0165	.0389	.0735	.1181	.1672	.2128	.2461
	6			.0001	.0006	.0028	.0087	.0210	.0424	.0743	.1160	.1641
	7					.0003	.0012	.0039	.0098	.0212	.0407	.0703
	8						.0001	.0004	.0013	.0035	.0083	.0176
	9								.0001	.0003	.0008	.0020
10	0	.9044	.5987	.3487	.1969	.1074	.0563	.0282	.0135	.0060	.0025	.0010
	1	.0914	.3151	.3874	.3474	.2684	.1877	.1211	.0725	.0403	.0207	.0098
	2	.0042	.0746	.1937	.2759	.3020	.2816	.2335	.1757	.1209	.0763	.0439
	3	.0001	.0105	.0574	.1298	.2013	.2503	.2668	.2522	.2150	.1665	.1172
	4		.0010	.0112	.0401	.0881	.1460	.2001	.2377	.2508	.2384	.2051
	5		.0001	.0015	.0085	.0264	.0584	.1029	.1536	.2007	.2340	.2461
	6			.0001	.0012	.0055	.0162	.0368	.0689	.1115	.1596	.2051
	7				.0001	.0008	.0031	.0090	.0212	.0425	.0746	.1172

BINOMIAL DISTRIBUTION, INDIVIDUAL TERMS

						π						
n	x	.01	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
10	8					.0001	.0004	.0014	.0043	.0106	.0229	.0439
	9							.0001	.0005	.0016	.0042	.0098
	10									.0001	.0003	.0010
11	0	.8953	.5688	.3138	.1673	.0859	.0422	.0198	.0088	.0036	.0014	.0005
	1	.0995	.3293	.3835	.3248	.2362	.1549	.0932	.0518	.0266	.0125	.0054
	2	.0050	.0867	.2131	.2866	.2953	.2581	.1998	.1395	.0887	.0513	.0269
	3	.0002	.0137	.0710	.1517	.2215	.2581	.2568	.2254	.1774	.1259	.0806
	4		.0014	.0158	.0536	.1107	.1721	.2201	.2428	.2365	.2060	.1611
	5		.0001	.0025	.0132	.0388	.0803	.1321	.1830	.2207	.2360	.2256
	6			.0003	.0023	.0097	.0268	.0566	.0985	.1471	.1931	.2256
	7				.0003	.0017	.0064	.0173	.0379	.0701	.1128	.1611
	8					.0002	.0011	.0037	.0102	.0234	.0462	.0806
	9						.0001	.0005	.0018	.0052	.0126	.0269
	10								.0002	.0007	.0021	.0054
	11										.0002	.0005
12	0	.8864	.5404	.2824	.1422	.0687	.0317	.0138	.0057	.0022	.0008	.0002
	1	.1074	.3413	.3766	.3012	.2062	.1267	.0712	.0368	.0174	.0075	.0029
	2	.0060	.0988	.2301	.2924	.2835	.2323	.1678	.1068	.0639	.0339	.0161
	3	.0002	.0173	.0852	.1720	.2362	.2581	.2397	.1954	.1419	.0923	.0537
	4		.0021	.0213	.0683	.1329	.1936	.2311	.2367	.2128	.1700	.1208
	5		.0002	.0038	.0193	.0532	.1032	.1585	.2039	.2270	.2225	.1934
	6			.0005	.0040	.0155	.0401	.0792	.1281	.1766	.2124	.2256
	7				.0006	.0033	.0115	.0291	.0591	.1009	.1489	.1934
	8				.0001	.0005	.0024	.0078	.0199	.0420	.0762	.1208
	9					.0001	.0004	.0015	.0048	.0125	.0277	.0537
	10							.0002	.0008	.0025	.0068	.0161
	11								.0001	.0003	.0010	.0029
	12										.0001	.0002
15	0	.8601	.4633	.2059	.0874	.0352	.0134	.0047	.0016	.0005	.0001	
	1	.1303	.3658	.3432	.2312	.1319	.0668	.0305	.0126	.0047	.0016	.0005
	2	.0092	.1348	.2669	.2856	.2309	.1559	.0916	.0476	.0219	.0090	.0032
	3	.0004	.0307	.1285	.2184	.2501	.2252	.1700	.1110	.0634	.0318	.0139
	4		.0049	.0428	.1156	.1876	.2252	.2186	.1792	.1268	.0780	.0417
	5		.0006	.0105	.0449	.1032	.1651	.2061	.2123	.1859	.1404	.0916
	6			.0019	.0132	.0430	.0917	.1472	.1906	.2066	.1914	.1527
	7			.0003	.0030	.0138	.0393	.0811	.1319	.1771	.2013	.1964
	8				.0005	.0035	.0131	.0348	.0710	.1181	.1647	.1964
	9				.0001	.0007	.0034	.0116	.0298	.0612	.1048	.1527
	10					.0001	.0007	.0030	.0096	.0245	.0515	.0916
	11						.0001	.0006	.0024	.0074	.0191	.0417
	12							.0001	.0004	.0016	.0052	.0139
	13								.0001	.0003	.0010	.0032
	14										.0001	.0005
	15											
20	0	.8179	.3585	.1216	.0388	.0115	.0032	.0008	.0002			
	1	.1652	.3774	.2702	.1368	.0576	.0211	.0068	.0020	.0003	.0001	
	2	.0159	.1887	.2852	.2293	.1369	.0669	.0278	.0100	.0031	.0008	.0002
	3	.0010	.0596	.1901	.2428	.2054	.1339	.0716	.0323	.0123	.0040	.0011
	4		.0133	.0898	.1821	.2182	.1897	.1304	.0738	.0350	.0139	.0046
	5		.0022	.0319	.1028	.1746	.2023	.1789	.1272	.0746	.0365	.0148
	6		.0003	.0089	.0454	.1091	.1686	.1916	.1712	.1244	.0746	.0370
	7			.0020	.0160	.0545	.1124	.1643	.1844	.1659	.1221	.0739
	8			.0004	.0046	.0222	.0609	.1144	.1614	.1797	.1623	.1201
	9			.0001	.0011	.0074	.0271	.0654	.1158	.1597	.1771	.1602
	10				.0002	.0020	.0099	.0308	.0686	.1171	.1593	.1762
	11					.0005	.0030	.0120	.0336	.0710	.1185	.1602
	12					.0001	.0008	.0039	.0136	.0355	.0727	.1201
	13						.0002	.0010	.0045	.0146	.0366	.0739
	14							.0002	.0012	.0049	.0150	.0370
	15								.0003	.0013	.0049	.0148
	16									.0003	.0013	.0046
	17										.0002	.0011
	18											.0002
	19											
	20											

Example 1 For the previous example, use the table to find the probability.

Answer

Given : n = 12, p or π = 20% = 0.20, q = 1 – 0.20 = 0.80, x = 1

		π 0.20
n	x	
12	0	↓
	1 ⟶	0.2062
	.	
	.	
	.	
	11	
	12	

∴ Probability = 0.2062

Example 2 If 20% of the bulbs produced by a factory are faulty, determine the probability that out of sample of 12 randomly chosen bulbs, at least 8 will be faulty.

Answer

Given : n = 12, p = 20% = 0.20, x = (8, 9, 10, 11, 12)
Reading Probability from the table of individual terms and add up the probability values corresponding to x = 8 to x = 12.

		π 0.20
n	x	
12	.	.
	.	.
	.	.
	8 ⟶	0.0005
	9 ⟶	0.0001
	10	
	11	
	12	

Probability = 0.0005 + 0.0001

= 0.0006

7.3.3 Table of Cumulative Binomial Distribution

- You may also use the Cumulative Binomial distribution table:

CUMULATIVE BINOMIAL DISTRIBUTION

Tabulated values are $P(X \geq x)$ for certain values of n, π. If $\pi > .50$ use $P(Y \leq y) = 1 - P(Y \geq y + 1)$, where $y = n - x$, $\pi_1 = 1 - \pi$.

n	x	.01	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
2	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0199	.0975	.1900	.2775	.3600	.4375	.5100	.5775	.6400	.6975	.7500
	2	.0001	.0025	.0100	.0225	.0400	.0625	.0900	.1225	.1600	.2025	.2500
3	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0297	.1426	.2710	.3859	.4880	.5781	.6570	.7254	.7840	.8336	.8750
	2	.0003	.0072	.0280	.0608	.1040	.1562	.2160	.2818	.3520	.4253	.5000
	3		.0001	.0010	.0034	.0080	.0156	.0270	.0429	.0640	.0911	.1250
4	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0394	.1855	.3439	.4780	.5904	.6836	.7599	.8215	.8704	.9085	.9375
	2	.0006	.0140	.0523	.1095	.1808	.2617	.3483	.4370	.5248	.6090	.6875
	3		.0005	.0037	.0120	.0272	.0508	.0837	.1265	.1792	.2415	.3125
	4			.0001	.0005	.0016	.0039	.0081	.0150	.0256	.0410	.0625
5	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0490	.2262	.4095	.5563	.6723	.7627	.8319	.8840	.9222	.9497	.9687
	2	.0010	.0226	.0815	.1648	.2627	.3672	.4718	.5716	.6630	.7438	.8125
	3		.0012	.0086	.0266	.0579	.1035	.1631	.2352	.3174	.4069	.5000
	4			.0005	.0022	.0067	.0156	.0308	.0540	.0870	.1312	.1875
	5				.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0312
6	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0585	.2649	.4686	.6229	.7379	.8220	.8824	.9246	.9533	.9723	.9844
	2	.0015	.0328	.1143	.2235	.3446	.4661	.5798	.6809	.7667	.8364	.8906
	3		.0022	.0158	.0473	.0989	.1694	.2557	.3529	.4557	.5585	.6562
	4		.0001	.0013	.0059	.0170	.0376	.0705	.1174	.1792	.2553	.3437
	5			.0001	.0004	.0016	.0046	.0109	.0223	.0410	.0692	.1094
	6					.0001	.0002	.0007	.0018	.0041	.0083	.0156
7	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0679	.3017	.5217	.6794	.7903	.8665	.9176	.9510	.9720	.9848	.9922
	2	.0020	.0444	.1497	.2834	.4233	.5551	.6706	.7662	.8414	.8976	.9375
	3		.0038	.0257	.0738	.1480	.2436	.3529	.4677	.5801	.6836	.7734
	4		.0002	.0027	.0121	.0333	.0706	.1260	.1998	.2898	.3917	.5000
	5			.0002	.0012	.0047	.0129	.0288	.0556	.0963	.1529	.2266
	6				.0001	.0004	.0013	.0038	.0090	.0188	.0357	.0625
	7						.0001	.0002	.0006	.0016	.0037	.0078
8	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0773	.3366	.5695	.7275	.8322	.8999	.9424	.9681	.9832	.9916	.9961
	2	.0027	.0572	.1869	.3428	.4967	.6329	.7447	.8309	.8936	.9368	.9648
	3	.0001	.0058	.0381	.1052	.2031	.3215	.4482	.5722	.6846	.7799	.8555
	4		.0004	.0050	.0214	.0563	.1138	.1941	.2936	.4059	.5230	.6367
	5			.0004	.0029	.0104	.0273	.0580	.1061	.1737	.2604	.3633
	6				.0002	.0012	.0042	.0113	.0253	.0498	.0885	.1445
	7					.0001	.0004	.0013	.0036	.0085	.0181	.0352
	8							.0001	.0002	.0007	.0017	.0039
9	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0865	.3698	.6126	.7684	.8658	.9249	.9596	.9793	.9899	.9954	.9980
	2	.0034	.0712	.2252	.4005	.5638	.6997	.8040	.8789	.9295	.9615	.9805
	3	.0001	.0084	.0530	.1409	.2618	.3993	.5372	.6627	.7682	.8505	.9102
	4		.0006	.0083	.0339	.0856	.1657	.2703	.3911	.5174	.6386	.7461
	5			.0009	.0056	.0196	.0489	.0988	.1717	.2666	.3786	.5000
	6			.0001	.0006	.0031	.0100	.0253	.0536	.0994	.1658	.2539
	7					.0003	.0013	.0043	.0112	.0250	.0498	.0898
	8						.0001	.0004	.0014	.0038	.0091	.0195
	9								.0001	.0003	.0008	.0020
10	0	1	1	1	1	1	1	1	1	1	1	1
	1	.0956	.4013	.6513	.8031	.8926	.9437	.9718	.9865	.9940	.9975	.9990
	2	.0043	.0861	.2639	.4557	.6242	.7560	.8507	.9140	.9536	.9767	.9893
	3	.0001	.0115	.0702	.1798	.3222	.4744	.6172	.7384	.8327	.9004	.9453
	4		.0010	.0128	.0500	.1209	.2241	.3504	.4862	.6177	.7340	.8281
	5		.0001	.0016	.0099	.0328	.0781	.1503	.2485	.3669	.4956	.6230
	6			.0001	.0014	.0064	.0197	.0473	.0949	.1662	.2616	.3770
	7				.0001	.0009	.0035	.0106	.0260	.0548	.1020	.1719

CUMULATIVE BINOMIAL DISTRIBUTION

						π						
n	x	.01	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
10	8					.0001	.0004	.0016	.0048	.0123	.0274	.0547
	9							.0001	.0005	.0017	.0045	.0107
	10									.0001	.0003	.0010
11	0	1	1	1	1	1	1	1	1	1	1	1
	1	.1047	.4312	.6862	.8327	.9141	.9578	.9802	.9912	.9964	.9986	.9995
	2	.0052	.1019	.3026	.5078	.6779	.8029	.8870	.9394	.9698	.9861	.9941
	3	.0002	.0152	.0896	.2212	.3826	.5448	.6873	.7999	.8811	.9348	.9673
	4		.0016	.0185	.0694	.1611	.2867	.4304	.5744	.7037	.8089	.8867
	5		.0001	.0028	.0159	.0504	.1146	.2103	.3317	.4672	.6029	.7256
	6			.0003	.0027	.0117	.0343	.0782	.1487	.2465	.3669	.5000
	7				.0003	.0020	.0076	.0216	.0501	.0994	.1738	.2744
	8					.0002	.0012	.0043	.0122	.0293	.0610	.1133
	9						.0001	.0006	.0020	.0059	.0148	.0327
	10								.0002	.0007	.0022	.0059
	11										.0002	.0005
12	0	1	1	1	1	1	1	1	1	1	1	1
	1	.1136	.4596	.7176	.8578	.9313	.9683	.9862	.9943	.9978	.9992	.9998
	2	.0062	.1184	.3410	.5565	.7251	.8416	.9150	.9576	.9804	.9917	.9968
	3	.0002	.0196	.1109	.2642	.4417	.6093	.7472	.8487	.9166	.9579	.9807
	4		.0022	.0256	.0922	.2054	.3512	.5075	.6533	.7747	.8655	.9270
	5		.0002	.0043	.0239	.0726	.1576	.2763	.4167	.5618	.6956	.8062
	6			.0005	.0046	.0194	.0544	.1178	.2127	.3348	.4731	.6128
	7			.0001	.0007	.0039	.0143	.0386	.0846	.1582	.2607	.3872
	8				.0001	.0006	.0028	.0095	.0255	.0573	.1117	.1938
	9					.0001	.0004	.0017	.0056	.0153	.0356	.0730
	10							.0002	.0008	.0028	.0079	.0193
	11								.0001	.0003	.0011	.0032
	12										.0001	.0002
15	0	1	1	1	1	1	1	1	1	1	1	1
	1	.1399	.5367	.7941	.9126	.9648	.9866	.9953	.9984	.9995	.9999	.1000
	2	.0096	.1710	.4510	.6814	.8329	.9198	.9647	.9858	.9948	.9983	.9995
	3	.0004	.0362	.1841	.3958	.6020	.7639	.8732	.9383	.9729	.9893	.9963
	4		.0055	.0556	.1773	.3518	.5387	.7031	.8273	.9095	.9576	.9824
	5		.0006	.0127	.0617	.1642	.3135	.4845	.6481	.7827	.8796	.9408
	6		.0001	.0022	.0168	.0611	.1484	.2784	.4357	.5968	.7392	.8491
	7			.0003	.0036	.0181	.0566	.1311	.2452	.3902	.5478	.6964
	8				.0006	.0042	.0173	.0500	.1132	.2131	.3465	.5000
	9				.0001	.0008	.0042	.0152	.0422	.0950	.1818	.3036
	10					.0001	.0008	.0037	.0124	.0338	.0769	.1509
	11						.0001	.0007	.0028	.0093	.0255	.0592
	12							.0001	.0005	.0019	.0093	.0176
	13								.0001	.0003	.0011	.0037
	14										.0001	.0005
	15											
20	0	1	1	1	1	1	1	1	1	1	1	1
	1	.1821	.6415	.8784	.9612	.9885	.9968	.9992	.9998	.1000	.1000	.1000
	2	.0169	.2642	.6083	.8244	.9308	.9757	.9924	.9979	.9995	.9999	.1000
	3	.0010	.0755	.3231	.5951	.7939	.9087	.9645	.9879	.9964	.9991	.9998
	4		.0159	.1330	.3523	.5886	.7748	.8929	.9556	.9840	.9951	.9987
	5		.0026	.0432	.1702	.3704	.5952	.7625	.8818	.9490	.9811	.9941
	6		.0003	.0113	.0673	.1958	.3828	.5836	.7546	.8744	.9447	.9793
	7			.0024	.0219	.0867	.2142	.3920	.5834	.7500	.8701	.9423
	8			.0004	.0059	.0321	.1018	.2277	.3990	.5841	.7480	.8684
	9			.0001	.0013	.0100	.0409	.1133	.2376	.4044	.5857	.7483
	10				.0002	.0026	.0139	.0480	.1218	.2447	.4086	.5881
	11					.0006	.0039	.0171	.0532	.1275	.2493	.4119
	12					.0001	.0009	.0051	.0196	.0565	.1308	.2517
	13						.0002	.0013	.0060	.0210	.0580	.1316
	14							.0003	.0015	.0065	.0214	.0577
	15								.0003	.0016	.0064	.0207
	16									.0003	.0015	.0059
	17										.0003	.0013
	18											.0002
	19											
	20											

Example 1 For the previous example 2, use the Cumulative Binomial distribution table to find the probability.

Answer

n	x	π 0.20
12	.	.
	.	.
	.	.
	8	0.0006
	9	
	10	
	11	
	12	

Example 2

There are 12 equivalent units in the Lagoon Motel. The manager knows that the probability that a unit will be occupied on any one night is 0.40. Find the probability that **at most** 10 units will be occupied on any one night.

Answer

Let X be the number of units occupied on any one night.

$p = 0.4$, $q = 1 - 0.4 = 0.6$ and $n = 12$.

$$P(X \leq 10) = 1 - P(X > 10)$$
$$= 1 - [P(X = 11) + P(X = 12)].$$
$$= 1 - [0.0003 + 0.0000]$$
$$= 0.9997$$

Using the binomial distribution individual terms

n	x	π 0.40
12	.	.
	.	.
	.	.
	11	0.0003
	12	

Example 3

A shop owner has found out that 85% of the people who come to this shop on any day buy something. If in one day 15 people go to this shop, find the probability that 11 of them will be buying something.

Answer

Method 1

n = 15 p = 0.85, x = 11

0.85 is not in the tables, so using the formula

- $P(X=x) = \binom{n}{x} p^x q^{n-x}$

$$= \binom{15}{11} 0.85^{11}\ 0.15^{15-11}$$

= 0.1156

Method 2:

n = 15 p = 0.85, x = 11

0.85 is not in the tables, so instead we find the probability that 4 do not buy something (**which is exactly the same as 11 buying something**) with probability 0.15

n = 15, p = 0.15, X = 4

P(X = 4) = 0.1156 from the tables

Example 4

The probability that Roy scores a goal in a soccer game is 0.30. What is the probability that Roy scores a goal in **at least 3** of the next 5 soccer games?

✍ **Answer**

Using the binomial distribution individual terms
to derive the probability.
Since p = 0.3 and $n = 5$, we have:

$$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$$
$$= 0.1323 + 0.0283 + 0.0024 = 0.163$$

Note this can be directly obtained from cumulative distribution table.

Exercise 7.3

1. A maths teacher sets up study groups in her maths class. Each study group has 3 students. If 20 % of the maths students in her class are females, what is the probability that at least one member of a group is a female?

2. A hospital with a heart transplant unit finds that the probability that a patient is still alive after 3 years is 30%. Find the probability that if the unit operates on 10 patients in a year, more than seven patients will not live up to 3 years after the operation.

3. A survey in a country shows that 95% of the people love listening to music. What is the probability that from 12 people interviewed on the streets, at least 11 will be found to have love for music?

4. A shop owner has found out that 80% of the people who come to this shop on any day buy something. If in one day 20 people go to this shop, find the probability that at least 75% of them will be buying something.

5. A primary student is going to guess the answers to all the questions in a 10 question multiple choice test where there are 5 choices for each answer. What is the probability of getting at least 8 correct answers?

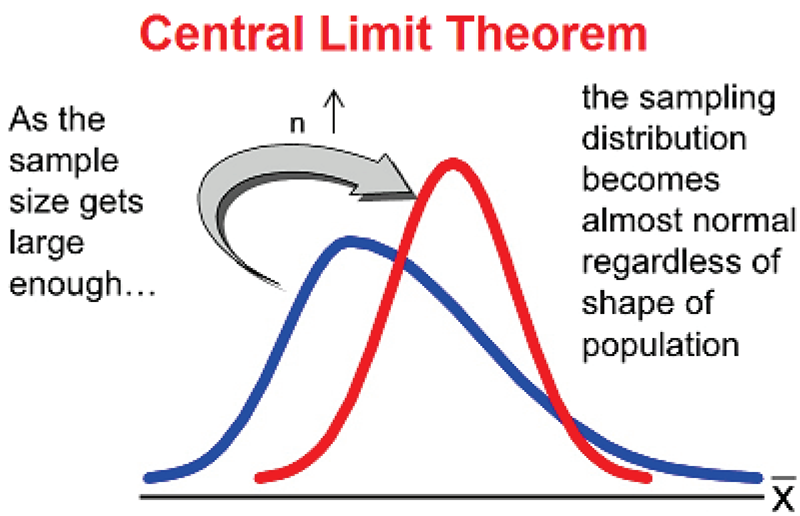
6. A survey on the streets of Suva on a sunny day showed that 80% of the people wore sunglasses. What is the probability that exactly 10 out of 15 people will be wearing sunglasses on a particular sunny day?

7. A student is going to guess the answers to all the questions on a 5 question - multiple-choice test where there are four choices for each answer. What is the probability of getting at least four correct answers?

8. It is found that 20% of light bulbs produced in a factory are faulty. In a random sample of 10 bulbs, what is the probability that less than four bulbs are faulty?

SUB – STRAND 7.4

Estimation



Confidence level	Z value
90%	1.65
95%	1.96
99%	2.58
99,9%	3.291

LEARNING OUTCOMES

7.4.1 Explain central limit theorem.
7.4.2 Find confidence intervals.
7.4.3 Determine sample size for estimating the population mean.

7.4.1 Estimation

- **Central limit theorem**

If a random sample of size n is drawn from a large or infinite population with mean, μ and standard deviation, σ, then the distribution of sample mean, $\bar{x}$, is approximately normally distributed with $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ and $\mu_{\bar{x}} = \mu$.

Hence $\boxed{z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}$

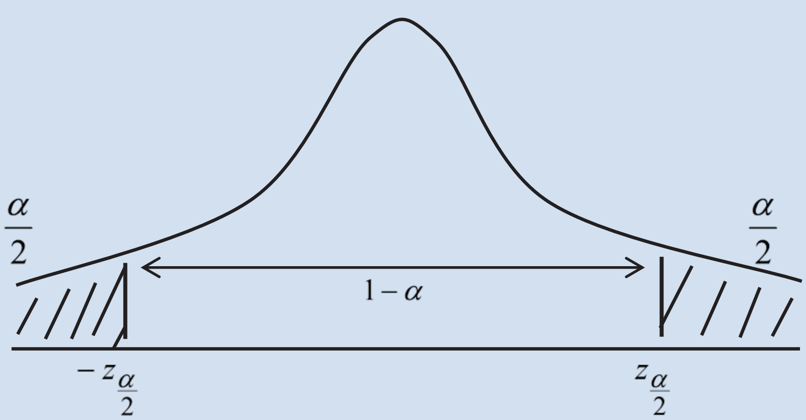
One very important application of central limit theorem is the determination of the reasonable values of the population mean.
Hypothesis testing and estimation will use the central limit theorem.

- **Interval Estimate for the population mean μ**

Since we know something about the distribution of sample mean $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ and $\mu_{\bar{x}} = \mu$, we can make statements about how confident we are that the true mean is within a given interval about our sample mean.

Statisticians use α to represent the probability that μ does not lie between the upper and the lower limits in the interval. The probability that the μ lies between the upper and the lower limits is then $1-\alpha$

Suppose k and m are the lower and upper limits respectively of the interval: $k < \mu < m$ then $P(k < \mu < m) = 1-\alpha \ for \ 0 < \alpha < 1$
So we have a probability of $1-\alpha$ that the population mean lies between k and m. This is the $(1-\alpha)100\%$ confidence interval



$z_{\frac{\alpha}{2}}$ is the z-value leaving an area of $\frac{\alpha}{2}$ to the right

7.4.2 Confidence Interval

- To calculate the Confidence Interval, use the formula

$$\boxed{\bar{x} - z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}}$$

where:

$\bar{x}$: sample mean

σ : standard deviation

n : sample size

$z_{\alpha/2}$: z-value leaving an area of $\frac{\alpha}{2}$ to the right in a standard normal distribution

- If $\alpha = 0.05$ we have 95% confidence interval for μ (population *mean)*

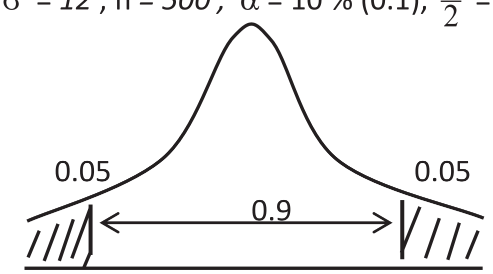
- In summary,

99% Confidence Interval for μ is	$\bar{x} - 2.576\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.576\frac{\sigma}{\sqrt{n}}$
95% Confidence Interval for μ is	$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$
90% Confidence Interval for μ is	$\bar{x} - 1.645\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}$

Example 1 A sample of 500 Year 13 students showed that the average time they spent studying math is 35 minutes with a standard deviation of 12 minutes. Find a 90% confidence interval for estimating the average time a student will spent studying math. Assume the sample is taken from a normal population.

Answer

$\bar{x} = 35$; $\sigma = 12$; $n = 500$, $\alpha = 10\ \% \ (0.1)$, $\frac{\alpha}{2} = 0.05$



Look for P= 0.45 in the inverse normal table to get $z_{\sigma/2}$ = 1.6449

$$\bar{x} - z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

$$35 - 1.6449 \times \frac{12}{\sqrt{500}} < \mu < 35 + 1.6449 \times \frac{12}{\sqrt{500}}$$

$$\underline{34.12 < \mu < 35.88}$$

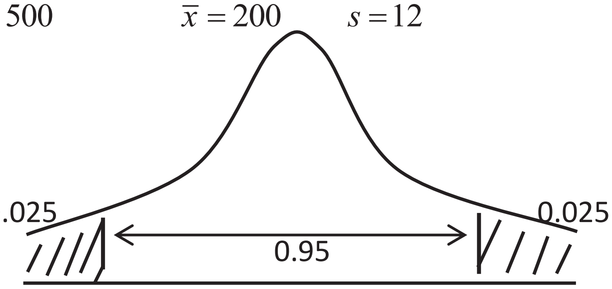
There is a probability of 0.90 that the population mean would fall between 34.12 and 35.88.

Note:
When the population standard deviation σ is not known we can use the sample standard deviation *s* if the sample size is $\underline{\geq 30}$

Example 2 When a sample of 500 coconuts is graded, a mean weight of 200 g is recorded. The standard deviation of the weight of these coconuts is known to be 12 g. Find a 95% confidence interval for the mean of coconuts. [Give your answer to 1 decimal place.]

Answer

$n = 500$ $\quad\bar{x} = 200$ $\quad s = 12$



$\alpha = 0.05 \qquad \Rightarrow \alpha/2 = 0.025$ and $z_{\alpha/2} = 1.96$

$$\bar{x} - z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

$$200 - 1.96 \times \frac{12}{\sqrt{500}} < \mu < 200 + 1.96 \times \frac{12}{\sqrt{500}}$$

$$\Rightarrow 198.9 \text{ g} < \mu < 201.1 \text{ g}$$

7.4.3 Choosing a sample size for estimating the population mean

- How large a sample should we take to obtain a reliable estimate of the population mean?

The sample size can be calculated by the formula:

$$\boxed{n=\left(\frac{z_{\frac{\alpha}{2}}\sigma}{e}\right)^2}$$

where

σ = standard deviation

n = sample size

e = error

$z_{\frac{\alpha}{2}}$ = the z - value leaving an area of $\frac{\alpha}{2}$ to the right

The sample size in the formula is the smallest sample size that will satisfy the accuracy requirement. Any larger sample size will also satisfy the requirements.

When finding the sample size, n, all fractional values are rounded up to the next whole number. This will reduce the error.

Example 3 A population has a standard deviation of 16. What minimum sample size should be taken to estimate the mean within 2 units of true value with 95% confidence?

Answer

σ= 4; e = 2; 95% = 0.95 $\Rightarrow \alpha$ =0.05; $\frac{\alpha}{2}=0.025$

Using the inverse normal table with P = 0.475 (1- $\frac{\alpha}{2}$) $\Rightarrow z_{\sigma/2}$ = 1.960

$$n=\left(\frac{z_{\frac{\alpha}{2}}\sigma}{e}\right)^2$$

$$=\left(\frac{1.96\times 4}{2}\right)^2=15.37$$

$$=16$$

rounded up to the next whole number

Exercise 7.4

1. A sample of 100 students took a mean time of 40 minutes with a standard deviation of 10 minutes to travel to school. Determine the 95 % confidence interval for the mean time taken by the students to travel to school.

2. A sample of size 120 items is taken from a population with an unknown mean mass, μ, and standard deviation 7.7g. The sample mean mass is found to be 562g. Construct a 99% confidence interval for the population mean mass, μ.

3. A sample of 225 students of Lomavata High school took a mean time of 30 minutes, with a standard deviation of 6 minutes, to travel to school. Determine the 98% confidence interval for the mean time taken by the students to travel to school.

4. A car reaches a speed of 100 km/h on a straight narrow road and then suddenly it had to stop. If the standard deviation of the length taken to stop is 15m, find how large a sample is required to be 95% confident that the error in the estimated mean will not exceed 2m.

5. A population of adult height has a normal distribution with a standard deviation of 3.6cm. Determine the sample size that is required to allow the mean of population to be estimated within 0.3cm with 98% confidence.

6. Determine the sample size that is required to estimate the mean weight of boys in Class 2, with a standard deviation of 3 kg, if we want the estimate to be accurate to within 1kg, with 95% confidence.

7. Determine the sample size that is required from a population of light bulbs with a bulb life that has a standard deviation of 20 hours, to estimate the mean bulb life to within 5 hours with 98% confidence.

8. Determine the sample size required to estimate weight of Form Seven Boys to within 1.0 kg with 95% confidence. Assume that the standard deviation of such weights is 3.0 kg.

SUB – STRAND 7.5

Hypothesis Testing

State H_0 as well as H_a

↓

Specify the level of significance (or the value)

↓

Decide the correct sampling distribution

↓

Sample a random sample(s) and work

↓

Calculate the probability that sample

↓

Is this probability equal to or smaller than value in case of one tailed test and

Yes	No
Reject H_0	Accept H_0
Thereby run the risk of committing Type I error	Thereby run some risk of committing Type II error

LEARNING OUTCOMES

7.5.1 Define terms used in hypothesis testing.
7.5.3 Perform hypothesis testing.

7.5.1 Hypothesis Testing

- A **hypothesis test** is a statistical test where a sample data is used to decide whether statements made about population parameters are true or false.

Examples of the type of statements to be tested are

- The average price of a school bag in Fiji is $10.90
- The mean wage of workers in Fiji is $42 per week.

These statements are examples of **hypothesis**.

- An assumption about the **existing situation** or value of a population parameter is called the **null hypothesis** (H_0). This is expressed in the form

$H_0 : \mu = 42$, if the assumption is that the mean wage of workers is $42.

- The **alternative hypothesis** (H_a) is a new belief about the population parameter which one will have to accept if there is a significant difference between the **sample results** and the **expected results**. This is expressed in either of the forms shown below

$H_a : \mu \neq 42$ or $H_a : \mu > 42$ or $H_a : \mu < 42$

- **Significance level of the test**

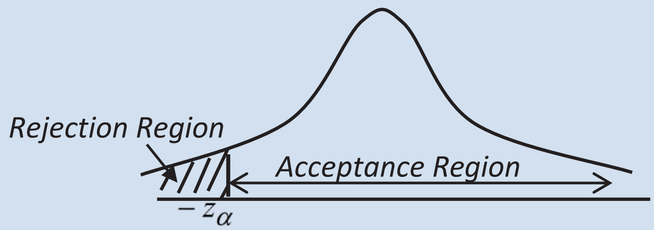
The probability of rejecting the null hypothesis when it is in fact correct is the significance level of the test. The significance level of the test is denoted by α .

- **Type I Error**
  Rejecting a null hypothesis when in fact it is true
  The probability of Type I Error is α
  It is also called level of significance of the test
- **Type II Error**
  Accepting a null hypothesis when in fact it is false

7.5.2 One-tailed and two-tailed test

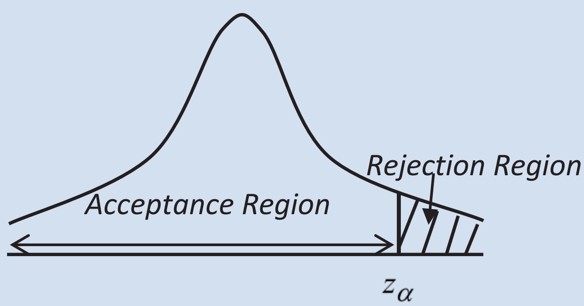
- CASE 1: One – tailed test
  - When a '<' sign appears in the alternative hypothesis, the test is called a **left-tailed test**, i.e.

One – Tail $(H_A : \mu < \mu_o)$



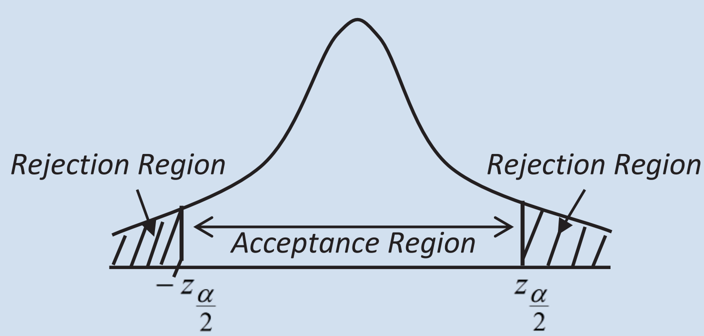
  - When a '>' sign appears in the alternative hypothesis, the test is called a **right-tailed test**, i.e.

One – Tail $(H_A : \mu > \mu_o)$



- CASE 2: Two– tailed test
  An alternate hypothesis with a '≠' sign is called a **two-tailed test.**

Two– tail $(H_A : \mu \neq \mu_o)$



7.5.3 Steps in hypothesis Testing

➢ Steps for Hypothesis testing:

- List down all given variables
- Set up the acceptance and rejection regions. Check the alternative hypothesis (H_A) and decide on the type of graph to sketch.
- Use the inverse normal table to find the z-score. (If $\mu>\mu_o$ or $\mu<\mu_o$ find 'z_α' else if $\mu \neq \mu_o$ find $z_{\frac{\alpha}{2}}$)
- Calculate the value of z: $$\boxed{z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}$$

  where
  $\bar{x}$: sample mean
  μ : population mean
  σ : standard deviation
  n : sample size

- State the conclusion.

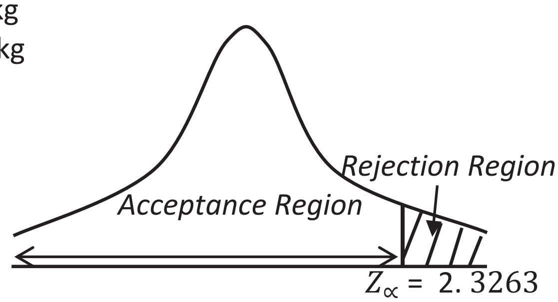
Example 1 A manufacturer of sports equipment has developed a new fishing line that he claims has a **mean** breaking strength of 10kg with a **standard deviation** of 0.6kg.
Test the null hypothesis $H_0: \mu = 10kg$ against the alternative hypothesis $H_a: \mu > 10kg$ if a random sample of 100 lines is tested and found to have a mean breaking strength of 10.2kg. Use **0.01** level of significance and state your **conclusion** clearly.

Answer

$\bar{x}$ = *10.2kg* ; σ *=0.6* ; n = *100* ; α = *0.01;*

H_o: μ = 10kg

H_a: μ > 10kg



Critical region: Reject H_o if z > 2.3263

$$z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{10.2-10}{0.6/\sqrt{100}} = 3.33$$

Since z value of 3.33 falls on the rejection region we ,therefore, reject the null hypothesis.

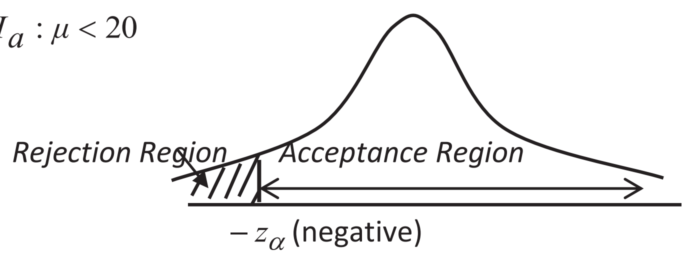
Example 2 A researcher claims that Fijian families use an average of 20 coconuts per month with standard deviation of 6. Test the **null hypothesis** $H_0 : \mu = 20$ against the **alternative hypothesis** $H_a : \mu < 20$ if a random sample of 64 Fijian families is found to consume an average of 19 coconuts per month. Use a 0.05 level of significance and state your conclusion.

Answer

$\bar{x}$ = *19 coconuts* ; σ =*6* ; n = *64* ; α = *0.05;*

$H_0 : \mu = 20$

$H_a : \mu < 20$



$Z_\alpha = -1.6449$

Critical region: Reject H_o if z < -1.6449

$$Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$$

$$= \frac{19-20}{6/\sqrt{64}}$$

$$= -1.33$$

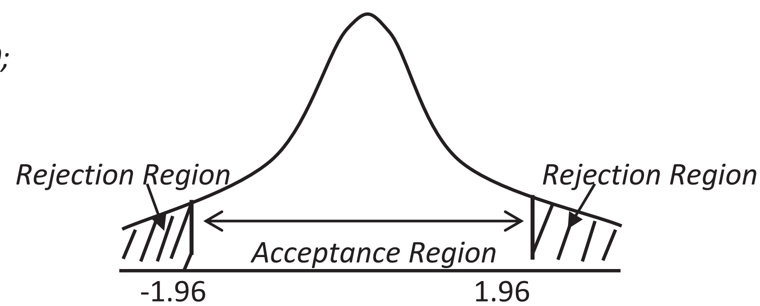
Since z value of -1.33 falls on the **acceptance region** we therefore, accept the null hypothesis

Example 3 The mean lifetime of a sample of 150 torch bulbs produced by a company is computed to be 400 hours with a standard deviation of 80 hours. If μ is the mean lifetime of all the bulbs produced by the company, **test** the hypothesis μ = 450 hours against $\mu \neq 450$ hours at 5% level of significance.

Answer

$\bar{x}$ = *400 hours* ; σ *=80* ; n = *150* ; α *= 5% = 0.05;* $\frac{\alpha}{2} = 0.025$

$H_o : \mu = 450;$
$H_A : \mu \neq 450;$



Using the table $z_{\sigma/2}$ = 1.96 (Using the inverse normal table with P= 0.475 $(1-\frac{\alpha}{2})$

Critical region: Reject H_0 if $z < -1.96$ or $z > 1.96$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{400 - 450}{80/\sqrt{150}} = -7.65$$

Since z value of - 7.65 falls in the rejection region hence we reject H_o .

Exercise 7.5

1. Sometime ago, surveys of very large tuna fish caught showed a mean weight of 40 kg with a standard deviation of 2.1 kg. Fishermen claimed that because of pollution, the net weight of tuna caught has decreased. A sample of 80 tuna fish were weighed.
   Construct a test in terms of the weight (kg) at a 1% level of significance to determine whether the null hypothesis $H_0 : \mu = 40$ kg can be accepted given the alternative hypothesis, $H_A : \mu < 40$ kg. What conclusion would be reached if the sample mean is 32 kg?

2. A farmer claims that each bundle of dalo has an average weight of 8kg with a standard deviation of 0.5kg. A sample of 50 bundles is chosen. It is found that the average weight is 7.8kg.

   Construct a hypothesis test at a 5% level of significance to confirm whether the mean is less than 8kg ($H_0 : \mu = 8$, $H_a : \mu < 8$).

3. A researcher claims that Fijian families use an average of 20 coconuts per month with standard deviation of 7.
   Test the **null hypothesis** $H_0 : \mu = 20$ against the **alternative hypothesis** $H_a : \mu > 20$ if a random sample of 64 Fijian families is found to consume an average of 21 coconuts per month. Use a 0.03 level of significance and state your conclusion clearly.

4. A farmer supplies 500 bundles of long beans to the market every week. He claims that each bundle has an average of 20 long beans with a standard deviation of 4 long beans. A sample of 50 bundles is chosen and the beans counted. It is found that the average number of long bean is 21.
   Test the null hypothesis, $H_0 : \mu = 20$ long beans against the alternative hypothesis, $H_A : \mu > 20$ long beans at a 1% significance level and state your conclusion.

5. A company has developed a fishing line that it claims has a mean breaking strength of 9 kg with a standard deviation of 0.6 kg. A random sample of 50 lines is tested and found to have a mean breaking strength of 8.9 kg. Construct a hypothesis test at 5% level of significance to determine whether the null hypothesis $H_0 : \mu = 9$ kg can be accepted given the alternative hypothesis, $H_A : \mu \neq 9$ kg and state your conclusion.

6. A chicken farmer weighed a random sample of 55 chicken from his farm. He wanted to test the claim that the mean weight of chicken at 6 weeks of age in a chicken farm is 1.95 kg with a standard deviation of 0.4 kg.

   Construct a test at 1% significance level to determine whether the null hypothesis $H_0 : \mu = 1.95$ kg can be accepted given the alternative hypothesis $H_A : \mu \neq 1.95$ kg. If the sample mean is 2.3 kg, what is your conclusion?

DID YOU KNOW?

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality . ~ Einstein
(https://www.stat.berkeley.edu/~aldous/Real-World/cover.html)

Review Exercise 7

1. Calculate the **minimum sample size** that is required to estimate the mean weight of a population, with standard deviation of 0.9kg, if we want the estimate to be **within 0.2kg** of its true value with **99% confidence.**

2. The probability of Ruci winning a game is **0.45**. If she plays **12 games**, what is the probability that she wins **10 games**?

3. An experiment consists of rolling 2 fair dice. Event **A** is "the number on the **first die** is 5" and event **B** is "the **sum** of the numbers is 9".

   a) Are events **A** and **B** mutually exclusive? Give a reason for your answer.

   b) What is the probability that event **A or B** occurs?

4. A normal random variable *X* has **standard deviation** of 4.
   Given that $P(X > 150) = 0.025$, find the **mean**.

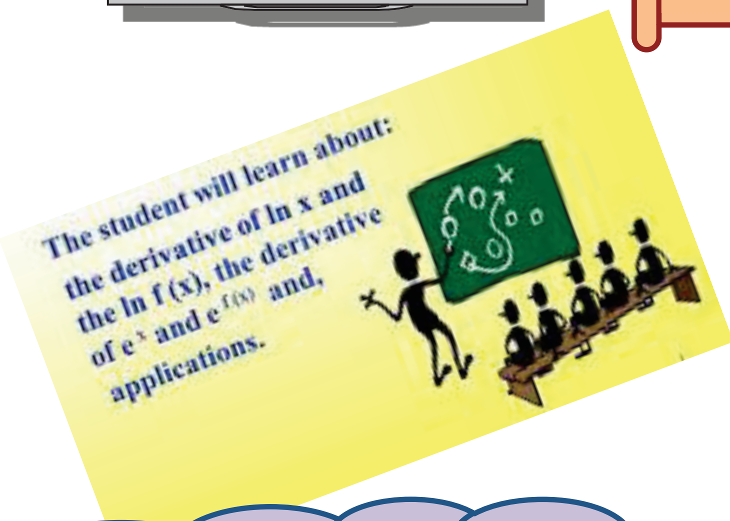
5. A manufacturer of sports equipment has developed a new fishing line that has an average breaking strength of 8 kg with a standard deviation of 0.6kg.
   Test the null hypothesis, $H_0 : \mu = 8kg$ against the alternative hypothesis, $H_A : \mu < 8kg$ if a random sample of 100 lines is tested and found to have a mean breaking strength of 7.8 kg. Use 2% significance level and state your conclusion clearly.

STRAND EIGHT

DIFFERENTIATION

SUB – STRAND 8.1

Derivatives of Functions



$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

$$\frac{d}{dx}(a^u) = (\ln a)a^u \frac{du}{dx}$$

LEARNING OUTCOMES

8.1.1 Find derivatives using
- Product rule.
- Quotient rule.
- Chain rule.
- Logarithms.
- Implicit differentiation.

8.1.1 Common Derivatives

- To differentiate, $y = kx^n$, multiply the coefficient k by the power n and then reduce the power by one.

$$y = x^n$$
$$y' \text{ or } \frac{dy}{dx} = n.x^{n-1}$$

$$f(x) = ax^n$$
$$f'(x) = (n \times a)x^{n-1}$$

- Derivative of a constant is equal to zero.

- $y=e^x$ has a special property that its derivative is the function itself.

$$y = e^x$$
$$\frac{dy}{dx} = e^x$$

- **Derivatives of trigonometric and logarithmic functions**

$y=f(x)$	$\frac{dy}{dx}$ or $f'(x)$
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Example 1 Find the derivative of $f(x) = x^2 - 1$

Answer

$$f(x) = x^2 - 1$$
$$f'(x) = 2x^{2-1} - 0$$
$$= 2x$$

Example 2 Differentiate $g(x) = 3x^{2} + \sqrt[3]{x}$

Answer

$$g(x) = 3x^{2} + x^{\frac{1}{3}}$$

$$g'(x) = 3\times 2\, x^{2-1} + \frac{1}{3}x^{\frac{1}{3}-1}$$

$$= 6\,x + \frac{1}{3}x^{-\frac{2}{3}}$$

Example 3 Find the derivative of $y = 4x^{2} + 3e^{x}$

Answer

Differentiate term by term

Power rule exponential

$$y = 4x^{2} + 3e^{x}$$

$$y' = 4\times 2x^{2-1} + 3e^{x}$$

$$= 8x + 3e^{x}$$

Example 4 Find the derivative of $y = \sin x + \ln x$

Answer

Differentiate term by term

$$y = \sin x + \ln x$$

$$y' = \cos x + \frac{1}{x}$$

Exercise 8.1.1

Find the derivative of the following:

a) $y = \dfrac{1}{x^{2}} + 3\sqrt{x} - 20$

b) $g(x) = \dfrac{1}{3x^{3}} - 5\cos x$

c) $f(x) = 3x^{2} + e^{x} - 42$

d) $h(x) = \tan x + \ln x$

DID YOU KNOW?

Many aspects of civil engineering require calculus. Derivation of the basic fluid mechanics equations. All hydraulic analysis program that aids in the design of storm drain and open channel systems, uses calculus numerical methods to obtain the results ~ Michael Ocampo

8.1.2 Product and Quotient Rule

- **Product rule**:

$$y = f.g$$
$$\frac{dy}{dx} = f'g + g'f$$

- **Quotient rule**:

$$y = \frac{f}{g}$$
$$\frac{dy}{dx} = \frac{gf' - fg'}{g^2}$$

Example 1 Differentiate

a) $y = (x^2 - 1)(x + 2)$

b) $y = (\sqrt{x} - 1)(x^2 + 4)$

Answers

Use the product rule

$y = (x^2 - 1)(x + 2)$

$y = \quad f \times g$

Compute the derivatives of f and g

$f = x^2 - 1 \qquad g = x + 2$

$f' = 2x \qquad g' = 1$

Substitute in the formula:

$$\frac{dy}{dx} = f'g + g'f$$
$$= 2x(x + 2) + 1(x^2 - 1)$$
$$= 2x^2 + 4x + x^2 - 1$$
$$= 3x^2 + 4x - 1$$

$y = (\sqrt{x} - 1)(x^2 + 4)$

$y = \quad f \times g$

Compute the derivatives of f and g

$f = \sqrt{x} - 1 \qquad g = x^2 + 4$

$f' = \frac{1}{2} x^{-\frac{1}{2}} \qquad g' = 2x$

Substitute in the formula:

$$\frac{dy}{dx} = f'g + g'f$$
$$= \frac{1}{2} x^{-\frac{1}{2}}(x^2 + 4) + 2x(\sqrt{x} - 1) or$$
$$= \frac{(x^2 + 4)}{2\sqrt{x}} + 2x(\sqrt{x} - 1)$$

Example 2 Differentiate $f(x)=\dfrac{-3x+2}{x+1}$

Answer

$$f(x)=\frac{-3x+2}{x+1} \quad =\frac{f}{g}$$

Compute the derivatives of f and g

$$f=-3x+2 \qquad g=x+1$$
$$f'=-3 \qquad g'=1$$

Substitute in the formula:

$$\frac{dy}{dx}=\frac{gf'-fg'}{g^2}$$

$$f'(x)=\frac{(x+1)(-3)-(-3x+2)(1)}{(x+1)^2}$$

$$f'(x)=\frac{-5}{(x+1)^2}$$

Exercise 8.1.2

1. Differentiate the following:

a) $y=x^2(4x+1)$

b) $g(x)=2x.\sqrt{x+1}$

c) $f(x)=3x^2.\sqrt[3]{x}$

d) $y=3e^x.\sqrt{x}$

e) $f(x)=\sin x\cos x$

f) $y=\ln x\sin x$.

2. Find the derivative of the following:

a) $y=\dfrac{2x+7}{3x-5}$

b) $g(x)=\dfrac{\sqrt{x}+1}{3x^3}$

c) $f(x)=\dfrac{e^x}{\sqrt{x}}$

d) $y=\dfrac{\sin x}{\cos x}$

e) $f(x)=\dfrac{e^x}{\ln x}$

f) $y=\dfrac{e^x}{\cos x}$

8.1.3 Chain rule

This method is used only for **composite functions**. Suppose that y is the function consisting of variable u and u is the function consisting of variable x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Left side is exactly what results if we cancel the du's on the right side

The following steps can be used to find the derivative:

- Let inside function be u and differentiate to get $\frac{du}{dx}$
- Write y in terms of u and differentiate to get $\frac{dy}{du}$
- Multiply $\frac{dy}{du}$ with $\frac{du}{dx}$ to get $\frac{dy}{dx}$
- Rewrite back in terms of x.

- **Short Cut method**

$$y = f(g(x))$$
$$\frac{dy}{dx} = f'(g(x)) \times g'(x)$$

Find the derivative of the **outside function** evaluated at the inside function times the derivative of the **inside function**.

$$y = [f(x)]^n$$
$$\frac{dy}{dx} = n \times [f(x)]^{n-1} \times f'(x)$$

Example 1 Find the derivative of $y = (x^2 + 2x)^4$

Answer

Let function inside the bracket to be u and differentiate

$$u = (x^2 + 2x)$$
$$\frac{du}{dx} = 2x + 2$$

Write y in terms of u and differentiate

$$y = u^4$$
$$\frac{dy}{du} = 4u^3$$

Multiply and write back in terms of x.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3 \times 2x + 2$$
$$= 4(x^2 + 2x)^3 \times (2x + 2)$$

Example 2 Differentiate $y=e^{2x^2+1}$

Answer

$$\frac{dy}{dx}=e^{2x^2+1}\times 4x$$

$$=4xe^{2x^2+1}$$

$$e^{f(x)} \Rightarrow chain\ \ rule:$$
$$derivative\ \ of\ \ outer\ \ e^{f(x)}\times derivative\ \ of\ \ inner\ \ 2x^2+1$$

Example 3 Differentiate $y=\cos(2x^2)$

Answer

$$\frac{dy}{dx}=-\sin(2x^2)\times 4x$$
$$=-4x\sin(2x^2)$$

Derivative of outer function (cos) $\times$ derivative of inner

Finding Derivatives Using Combination of Rules

Example 1 Find the derivative of $y=x\ln 2x$

Answer Product rule

$y=\quad f\bullet g$

$y=x \qquad \ln 2x$

Compute the derivatives of f and g

$$f=x \qquad g=\ln 2x$$
$$f'=1 \qquad g'=\frac{1}{2x}\times 2$$
$$=\frac{1}{x}$$

Substitute in the formula:

$$\frac{dy}{dx}=f'g+g'f$$

$$=1\times\ln 2x+\frac{1}{x}\times x$$

$$=\ln 2x+1$$

Example 2 Find the derivative of $y = x^2 e^{-2x}$

Answer Product rule

$$y = \quad f \bullet g$$

$$y = x^2 \qquad e^{-2x}$$

Compute the derivatives of f and g

$$f = x^2 \qquad g = e^{-2x}$$

$$f' = 2x \qquad g' = e^{-2x} \times -2$$

$$= -2e^{-2x}$$

Substitute in the formula:

$$\frac{dy}{dx} = f'g + g'f$$

$$= 2xe^{-2x} + x^2\left(-2e^{-2x}\right) or$$

$$= 2xe^{-2x}(1 - x)$$

Example 3 Differentiate $y = \dfrac{4e^{-2x}}{\cos 2x}$

Answer using quotient rule:

$$y = \frac{4e^{-2x}}{\cos 2x} = \frac{f}{g}$$

Compute the derivatives of f and g

$$f = 4e^{-2x} \qquad g = \cos 2x$$

$$f' = 4e^{-2x} \times -2 \qquad g' = -\sin 2x \times 2$$

$$= -8e^{-2x} \qquad = -2\sin 2x$$

Substitute in the formula:

$$\frac{dy}{dx} = \frac{gf' - fg'}{g^2}$$

$$f'(x) = \frac{-8e^{-2x}(\cos 2x) + 8e^{-2x}\sin 2x}{(\cos 2x)^2}$$

Exercise 8.1.3

1. Differentiate the following:

   a) $y = (2-3x)^{-1}$ b) $y = (3-4x)^{\frac{3}{2}}$

   c) $y = 4(3-2x)^{-2}$ d) $y = 3(x^2-5x)^2$

2. Differentiate the following with respect to x.

   a) $y = -3\cos\dfrac{x}{2}$ b) $f(x) = \ln x \ \sin^2(x-3)$

   c) $g(x) = \tan\sqrt{x}$ d) $y = \dfrac{\cos 2x}{\sin x^3}$

   e) $y = \dfrac{x+1}{e^{3x}}$ f) $y = \sin^2(e^{2x})$

3. Given that $y = e^x \sin\sqrt{x^2+1}$, find $\dfrac{dy}{dx}$.

4. Given that $y = \dfrac{\sin x}{\cos x}$, use the quotient rule to show that $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$.

5. Differentiate the following with respect to x.

   a) $f(x) = e^{x^3} + 5x$ b) $g(x) = \sqrt{e^x} + \dfrac{1}{5x}$ c) $y = \dfrac{x^2}{e^{2x}}$

6. Differentiate the following with respect to x.

   a) $y = \sqrt{x}\ln x$ b) $f(x) = e^{-3x^2}\ln x$

DID YOU KNOW?

In business, Calculus can help by providing an accurate and measurable way to record changes in variables using numbers and mathematics. Derivatives are used to determine the maximum profit, minimum cost, rate of change of cost and how to maximize profit or minimize cost and production. ~ Michael Ocampo

8.1.4 Logarithmic differentiation

- Logarithmic function is the inverse function of $y = e^x$. Note that $y = \log_e x$ is often written as $y = \ln x$.

If
$$\boxed{\begin{aligned} y &= \ln x \\ \frac{dy}{dx} &= \frac{1}{x} \end{aligned}} \qquad \boxed{\begin{aligned} y &= \ln f(x) \\ \frac{dy}{dx} &= \frac{1}{f(x)} \times f'(x) \end{aligned}}$$

- Logarithmic differentiation: differentiation by taking logarithms is a method used to differentiate functions by employing the logarithmic derivative of a function *f*, the technique is often performed in cases where it is easier to differentiate the logarithm of a function rather than the function itself.

Example 1 Use **logarithmic differentiation** to differentiate

$$y = \frac{(x-1)}{(x+3)^2}$$

Answer

$$y = \frac{(x-1)}{(x+3)^2}$$

$$\ln y = \ln \frac{(x-1)}{(x+3)^2}$$

Using $\ln \frac{a}{b} = \ln a - \ln b$

$$\ln y = \ln (x-1) - \ln (x+3)^2$$

$$\ln y = \ln (x-1) - 2\ln (x+3)$$

Using $\ln a^n = n \ln a$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1} - 2 \times \frac{1}{x+3}$$

$$\frac{dy}{dx} = \left(\frac{1}{x-1} - \frac{2}{x+3}\right) \times y \Rightarrow substitute\ \ y$$

$$\frac{dy}{dx} = \left(\frac{1}{x-1} - \frac{2}{x+3}\right) \times \frac{(x-1)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{-x+5}{(x-1)(x+3)} \times \frac{(x-1)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{-x+5}{(x+3)^3}$$

Simplify

$$\frac{1}{x-1} - \frac{2}{x+3} = \frac{(x+3)-2(x-1)}{(x-1)(x+3)} = \frac{x+3-2x+2}{(x-1)(x+3)} = \frac{-x+5}{(x-1)(x+3)}$$

Example 2 Use **logarithmic differentiation** to differentiate
$y=(x-1)^3(x+3)^2$

Answer

$$y=(x-1)^3(x+3)^2$$

$$\ln y=\ln\left[(x-1)^3(x+3)^2\right]$$

$$\ln y=\ln(x-1)^3+\ln(x+3)^2$$ Using $\ln ab=\ln a+\ln b$

$$\ln y=3\ln(x-1)+2\ln(x+3)$$ Using $\ln a^n=n\ln a$

$$\frac{1}{y}\frac{dy}{dx}=3\times\frac{1}{x-1}+2\times\frac{1}{x+3}$$

$$\frac{dy}{dx}=\left(\frac{3}{x-1}+\frac{2}{x+3}\right)\times y\Rightarrow substitute\ \ y$$

Simplify

$$\frac{3}{x-1}+\frac{2}{x+3}$$
$$=\frac{3(x+3)+2(x-1)}{(x-1)(x+3)}$$
$$=\frac{3x+9+2x-2}{(x-1)(x+3)}$$
$$=\frac{5x+7}{(x-1)(x+3)}$$

$$\frac{dy}{dx}=\left(\frac{3}{x-1}+\frac{2}{x+3}\right)\times(x-1)^3(x+3)^2$$

$$\frac{dy}{dx}=\frac{5x+7}{(x-1)(x+3)}\times(x-1)^3(x+3)^2$$

$$\frac{dy}{dx}=(5x+7)(x-1)^{3-1}(x+3)^{2-1}$$

$$\frac{dy}{dx}=(5x+7)(x-1)^2(x+3)$$

Example 4 Use **logarithmic differentiation** to differentiate $y=(x+1)^{\sin x}$

Answer

$$y=(x+1)^{\sin x}$$

$$\ln y=\ln(x+1)^{\sin x} \qquad \Rightarrow take\ \ \ln\ \ on\ \ both\ \ sides$$

$$\ln y=\sin x\ \ln(x+1) \qquad \Rightarrow \ln a^n=n\ln a$$

Use the product rule on right hand side

$$f=\sin x \qquad g=\ln(x+1)$$

$$f'=\cos x \qquad g'=\frac{1}{x+1}$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\ \ln(x+1)+\sin x\times\frac{1}{x+1}$$

$$\frac{dy}{dx}=\left(\cos x\ \ln(x+1)+\sin x\times\frac{1}{x+1}\right)\times y\Rightarrow substitute\ \ y$$

$$\frac{dy}{dx}=\left(\cos x\ \ln(x+1)+\frac{\sin x}{x+1}\right)\times(x+1)^{\sin x}$$

Example 5 Differentiate $y = \log_3 x$

Answer

$$y = \log_3 x$$

$$3^y = x \qquad \text{[Write in base index form]}$$

$$\ln 3^y = \ln x \qquad \text{[Take ln on both sides]}$$

$$y \ln 3 = \ln x$$

$$\ln 3 \frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{(\ln 3)x}$$

Exercise 8.1.4

Use **logarithmic differentiation** to find the derivatives of the following.

a) $y = \sqrt{(x-1)}(x+3)^2$

b) $y = 3^{-x}$

c) $y = \dfrac{(x-1)^2}{(x+3)}$

d) $y = x^x$

e) $y = (x^3 - 2x)^{\ln x}$

f) $y = \pi^{\sin xy}$

g) $y = (\cos\ x)^x$

h) $y = 2^x$

i) $y = \log_3 x$

j) $y = \log x$

k) $y = \log_2 x$

l) $y = \log 2x$

8.1.5 Implicit differentiation

This type of questions will include both x and y on both or one side of the equation.

- When differentiating with respect to the variable y attach $\frac{dy}{dx}$
- Make $\frac{dy}{dx}$ the subject by collecting all $\frac{dy}{dx}$ on one side of the equation.

Example 1 Use implicit differentiation to find $\frac{dy}{dx}$ given that $3x^2+5y^4=x$

Answer

When differentiating the variable y, attach $\frac{dy}{dx}$.

$$3x^2 + 5y^4 = x$$

$$6x + 20y^3\frac{dy}{dx} = 1$$

$$20y^3\frac{dy}{dx} = 1-6x$$

$$\frac{dy}{dx} = \frac{1-6x}{20y^3}$$

Example 2 Use implicit differentiation to find $\frac{dy}{dx}$ given that $x^2+y^2=xy$

Answer

For each term:

$$x^2 + y^2 = xy$$

When differentiating the variable y, attach $\frac{dy}{dx}$.

To differentiate xy use the product rule: $f=x$ and $g=y$

$$2x + 2y\frac{dy}{dx} = y+x\frac{dy}{dx}$$

$$2x+2y\frac{dy}{dx}=y+x\frac{dy}{dx}$$

$$2x-y=x\frac{dy}{dx}-2y\frac{dy}{dx}$$

$$2x-y=\frac{dy}{dx}(x-2y)$$

$$\frac{dy}{dx}=\frac{2x-y}{x-2y}$$

Example 3 Find $\frac{dy}{dx}$ if $\sin y + y^2e^x = x^3$.

Answer

For each term:

$\sin y \quad + \quad y^2e^x \quad = x^3$

When differentiating the variable y, attach $\frac{dy}{dx}$.

use the product rule:
$f = y^2$ and $g = e^x$
$f' = 2y\frac{dy}{dx}$ and $g' = e^x$

$$\cos y\frac{dy}{dx} + e^x\,2y\frac{dy}{dx} + y^2e^x = 3x^2$$

$$\cos y\frac{dy}{dx} + y^2e^x + 2ye^x\frac{dy}{dx} = 3x^2$$

Make $\frac{dy}{dx}$ the subject:

$$\cos y\frac{dy}{dx} + 2ye^x\frac{dy}{dx} = 3x^2 - y^2e^x$$

$$\frac{dy}{dx}\left(\cos y + 2ye^x\right) = 3x^2 - y^2e^x$$

$$\frac{dy}{dx} = \frac{3x^2 - y^2e^x}{\cos y + 2ye^x}$$

Exercise 8.1.5

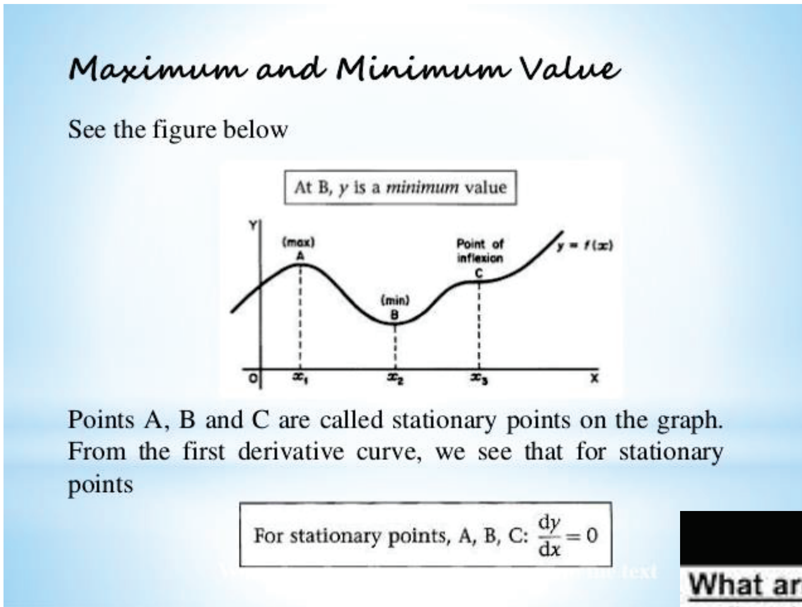
Find $\frac{dy}{dx}$ for the following:

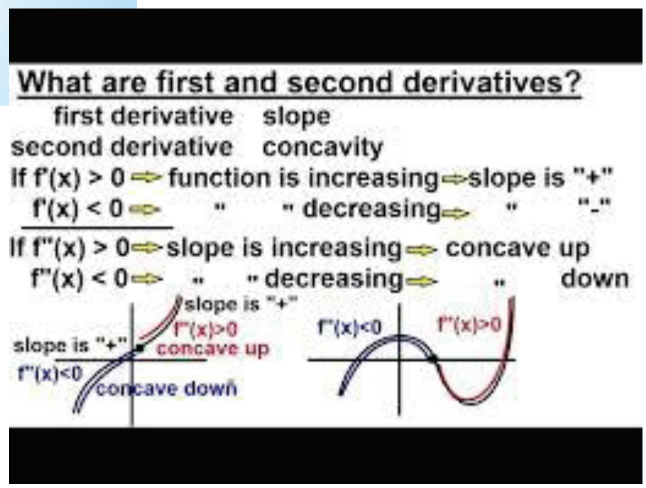
1. $y^3 - 3x = 12$
2. $\sin y - x + 1 = y$
3. $x^2e^y - 3x = 20$
4. $xy + 5 = \cos y$
5. $\sin(xy) - 5x = 1$
6. $\ln y - 3xy = 2e^x$
7. $x\ln y^2 - xe^x = 12y$
8. $x^2e^y - 3y + \cos x = \frac{2}{x}$

DID YOU KNOW? *Isaac Newton developed the use of calculus in his laws of motion and gravitation. Astronomical Science and space technology deeply depend on calculus. Also, calculus is used in building tracks ~* Michael Ocampo

SUB – STRAND 8.2

Applications of Differentiation



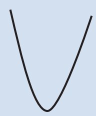


LEARNING OUTCOMES

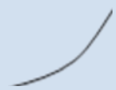
8.2.1 Find intervals of concavity.
Determine inflection and turning points.
8.2.2 Solve applied maximum and minimum problems.
8.2.3 Solve related rate problems.
8.2.4 Solve kinematics problems.

8.2.1 Concavity, Points of inflection and Turning Points

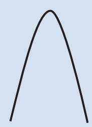
A piece of the graph of f is **concave upward** if all the tangents on the interval are below the curve. The curve is bent upward, like an upright bowl or cup.



Or

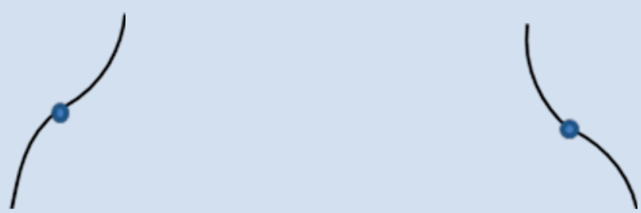


A piece of the graph of f is **concave downward** if all the tangents on the interval are above the curve. The curve is bent down, like an upside down bowl.



Or

A point at which the graph changes from being concave up to concave down, or vice versa, is called a **point of inflection**.



The concavity of a graph and the points of inflections can be determined by using the **second derivative**.

(a) If $f''(x) > 0$ for all x of an interval, then the graph of f is concave upward on the interval.

(b) If $f''(x) < 0$ for all x of an interval, then the graph of f is concave downward on the interval.

(c) If $f''(x) = 0$ **and** the concavity changes, there is a **point of inflection**.

To find **Turning Points**:

- Find $f'(x)$.
- Put $f'(x) = 0$ and solve for x.
- Find the y—values by substituting the x—values you got from solving $f'(x) = 0$ into $\boldsymbol{y = f(x)}$.

For cubic polynomials, the turning point with a bigger y -value is the local maximum point

👂 **Example 1:** Find intervals of concavity and inflection points if any of $y = 3x^2 - 9x + 9$

✍ **Answer**

First, the second derivative is just $f''(x) = 6$
Since this is never zero, there are *no* points of inflection. And the value of $f''(x)$ is always 6, so is always > 0 so the curve is entirely *concave upward*.
Since concavity does not change, there is no point of inflection.

👂 **Example 2:** Find turning points, intervals of concavity , inflection points of $f(x) = x^3 - 1.5x^2 - 6x + 5$ and sketch its graph.

✍ **Answer**

To find turning points

$f'(x) = 3x^2 - 3x - 6$

At turning point

$$
\begin{aligned}
&f'(x) = 0 \\
&3x^2 - 3x - 6 = 0 \\
&3(x^2 - x - 2) = 0 \\
&(x^2 - x - 2) = 0 \\
&(x+1)(x-2) = 0 \\
&x = -1,\ 2
\end{aligned}
$$

Substitute x values in $f(x) = x^3 - 1.5x^2 - 6x + 5$ *to get y values.*

The coordinates of the turning points are (-1, 8.5) and (2, -5)

Concave upward

$f''(x) = 6x - 3$

Solving $f''(x) > 0$

$6x - 3 > 0$

$6x > 3$

$x > 0.5$

Thus it is concave upward in the interval $x > 0.5$

Concave Downwards

Solving $f''(x) < 0$

$$6x - 3 < 0$$

$$6x < 3$$

$$x < 0.5$$

Thus it is concave downward in the interval $x < 0.5$.

Point of inflection

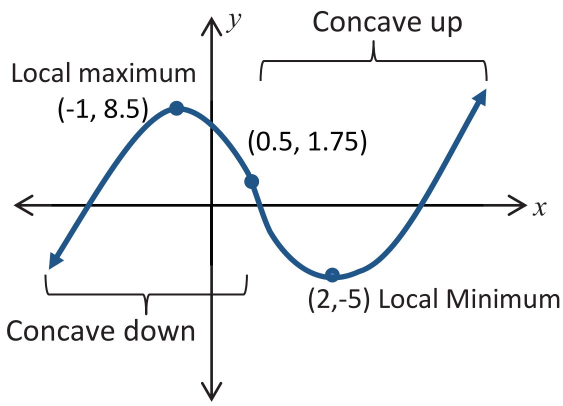
The graph has an inflection point at $x = 0.5$ since the concavity changes at this point.

$f(0.5) = 0.5^3 - 1.5 \times 0.5^2 - 6 \times 0.5 + 5$

$= 1.75$

The coordinates of point of inflection = (0.5, 1.75)

Graph showing key features



Exercise 8.2.1

1. Find turning points, intervals of concavity, coordinates of point of inflection, y-intercept and sketch the graph of the curves:

   a) $y = x^3 - 6x^2 - 15x + 20$ b) $y = x^3 - 9$

   c) $f(x) = x^3 - 3x^2$ d) $f(x) = (x-1)^3 + 2$

2. Find the x value of the point of inflection of the graph with the equation $y = 2x^{-1} - 2x^2$.

3. Find c and d so that $f(x) = cx^3 + dx^2 + 1$ has a point of inflection at (-1, 2).

8.2.2 Applied Maximum and Minimum problems

Applied problems in which we have to find the **maximum or minimum** are sometimes called **optimization problems.**

Steps to solve **optimization problems** :

- List the **information**: Read the question carefully and extract the relevant information.
- Develop a function that describes the situation algebraically: These are usually the **quantities** asked for in the problem. Draw appropriate diagrams where applicable.
- Find the derivative of the function, equate to zero and solve for the quantity.
- Determine whether it is a maximum or minimum and answer the question that has been asked.

Example 1 A closed cylindrical container with volume 1000 cm^3 is to be made out of a rectangular piece of aluminum sheet. What should the dimensions of the cylinder be (radius and height) such that **minimum** amount of aluminum is used?

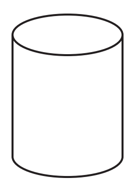
Answer

volume 1000 cm^3

$$Volume = \pi r^2 h$$

$$1000 = \pi r^2 h$$

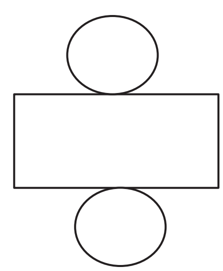
$$h = \frac{1000}{\pi r^2}$$



$$Surface\ \ Area = 2(\pi r^2) + 2\pi r h$$

$$S = 2(\pi r^2) + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S = 2\pi r^2 + 2000 r^{-1}$$



Find the derivative, equate to zero and solve for r:

$$S' = 0$$
$$S = 2\pi r^2 + 2000 r^{-1}$$

$$S' = 4\pi r - 2000 r^{-2} = 0$$

$$4\pi r = 2000 r^{-2}$$

$$4\pi r = \frac{2000}{r^2}$$

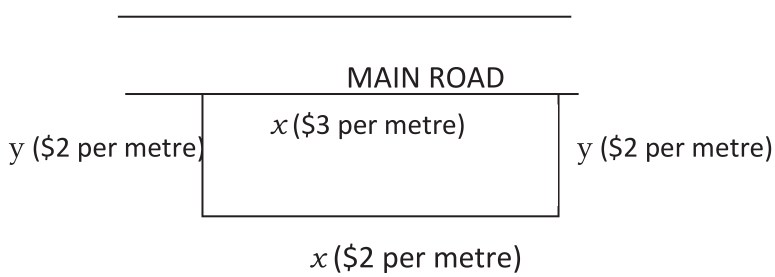
$$4\pi r^3 = 2000$$

$$r = 5.42$$

$$h = \frac{1000}{\pi r^2} = 10.84$$

Thus, radius and height are 5.42cm and 10.84cm

Example 2 A rectangular field is to be fenced for the storage area of a company. The field is to have an area of 2000m^2. The material for the front side costs \$3 per metre while the material for the other three sides costs \$2 per metre.



Find the dimensions for the field that will **minimize** the cost of fencing.

Answer

Area of 2000

$$Area = l \times w$$

$$2000 = xy$$

$$y = \frac{2000}{x}$$

The cost of material

$$C = 3x + 2y + 2y + 2x$$

$$C = 5x + 4y$$

$$C = 5x + 4\left(\frac{2000}{x}\right)$$

$$C = 5x + 8000x^{-1}$$

Find the derivative, equate to zero and solve for x:

$$C'(x) = 0$$

$$C(x) = 5x + 8000x^{-1}$$

$$C'(x) = 5 - 8000x^{-2} = 0$$

$$5 = \frac{8000}{x^2}$$

$$x^2 = \frac{8000}{5}$$

$$= 1600$$

$$x = 40m$$

$$y = \frac{2000}{x} = \frac{2000}{40} = 50m$$

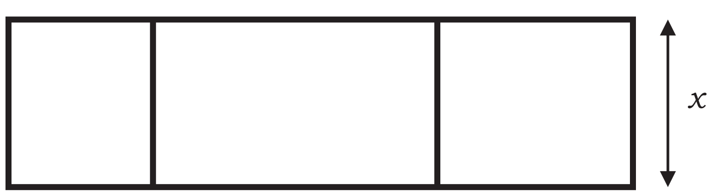
Thus, length and width are 40m and 50m respectively.

Exercise 6.3.2

1. A closed rectangular container with square base is to have a volume of 2000cm^3. The material for the top and bottom will cost \$2 per cm^2 and the material for the side will cost \$3 per cm^2. Find the dimensions of the container of **least cost**.

2. A box in the shape of a cuboid with a square base is to be made so that the sum of its dimensions is 20 cm. Find the **maximum volume**.

3. A field is shown below. It is to be divided into three paddocks and fenced using 1200 m of fence. Find the length and width of the field that will give a **maximum area**.



4. A farmer wishes to fence off a rectangular enclosure. Two existing hedges at right angles to each other will form two sides of the enclosure, so that he has to fence the other two sides. What is the **maximum area** that he can enclose using 100 m of fencing.

5. A packaging firm is designing boxes modeled by a cuboid. The cross-section of the box is to be a rectangle with length twice the width. If the volume of the box is to be $8\,m^3$,determine the **minimum Surface Area**.

6. A rectangular lot is bounded at the back by a river. No fence is needed along the river . If the fence along the front costs $1.50 per foot, along the sides $1 per foot, find the dimensions of the **largest** lot which can be fenced in for $300.

7. Find the **maximum volume** of a cone if the sum of its height and radius is 10 cm.

In medicine, calculus can be used to find the amount of blood pumped through the heart per unit time. Doctors often use calculus in the estimation of the progression of the illness. In biology, it is utilized to formulate rates such as birthand death rates. Also, calculus is used to find out the rate of change of the surface area for a rapidly growing adolescent. ~ Michael Ocampo

8.2.3 Related Rate problems

The usual context for rates of change is with **respect to t. [time]**

Steps

1. Identify the known rate of change and the rate of change to be found.
2. Write an equation that relates the quantities in step 1.
   To develop your equation, you will probably use:
   i. a simple geometric fact (like the relation between a circle's area and its radius, or the relation between the volume of a cone and its base-radius and height); or
   ii. a trigonometric function (like tan θ = opposite/adjacent); or
   iii. the Pythagoras theorem; or
   iv. similar triangles**.**
3. Take the derivative with respect to time of both sides of your equation.
4. Solve for unknown

Example 1 A stone is dropped into a pool of water. The radius of the circular ripple formed increases at 3 m/s. Calculate the rate at which the **area** of the ripple is increasing when radius is 8m.

Answer

Radius increases at 3 $m/s \Rightarrow \frac{dr}{dt} = 3m/s$

$$\frac{dA}{dt} = ?, \text{ when } r = 8$$

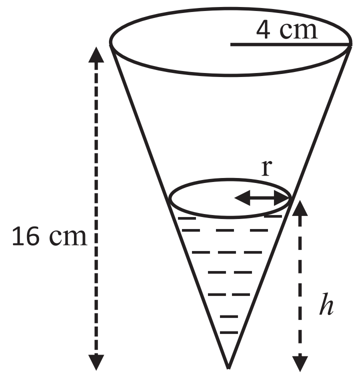
$A = \pi r^2$

Find the derivative of the function **with respect to t**

$$\begin{aligned}\frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\\ &= (2\pi r)\times 3\\ &= 6\pi r\\ &= 6\pi(8)\\ &= 48\pi \quad m^2/s\end{aligned}$$

Substitute r = 8m

Example 2 The radius of a conical filter is 4 cm and its height is 16 cm. Initially it is full of liquid.



As the liquid flows out, the volume decreases at a **constant rate**, $\frac{dV}{dt} = -2\ \text{cm}^3$ per second. At what **rate** will the depth of the liquid change when h = 8 cm deep?

Answer

$$\frac{dV}{dt} = -2\ cm^3/s$$

$$\frac{dh}{dt} = ?,\ \text{when } h = 8$$

Finding the expression for *r* using **similar triangles**.

$$\frac{r}{h} = \frac{4}{16} \Rightarrow r = \frac{4}{16}h$$

$$r = \frac{h}{4}$$

Substitute $r = \frac{h}{4}$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{4}\right)^2 \times h$$

$$V = \frac{\pi h^3}{48}$$

Find the derivative **with respect to** h.

$$V = \frac{\pi}{48} h^3$$

$$\frac{dV}{dh} = \frac{\pi}{48}\left(3h^2\right) = \frac{3\pi h^2}{48}$$

$$\frac{dh}{dt} = ?, \text{ when } h = 8$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=8} = \frac{48}{3\pi h^2} \times -2$$

$$= \frac{48}{3\pi\left(8^2\right)} \times -2$$

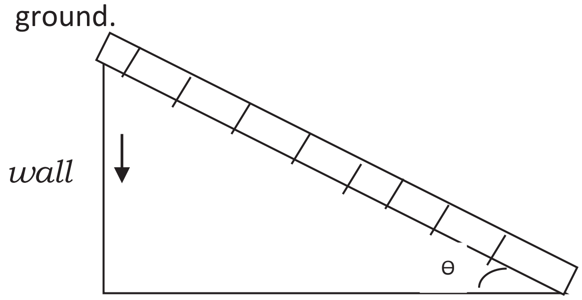
Substitute $h = 8$cm

$$= \frac{-1}{2\pi}\, cm/s$$

Negative sign indicates that h is decreasing.

Exercise 8.2.3

1. A 5m long ladder is learning against a wall, making an angle Θ with the ground.



If the top of the ladder slides down the wall at a rate of 0.5 m/s , at what rate:

a) will the angle between the ladder and the ground change when the top is 4m above the ground?

b) will the foot of the ladder move along the ground when the top of the ladder is 3m above the ground?

2. The surface area of a large spherical balloon is increasing at a constant rate of $\pi\, m^2/s$. At what rate is the diameter increasing when radius is 2.0 *m*?

3. A manufacturer finds that the profit (in dollars) received from making and selling x quantities of her product is given by $P(x)=(x^2+x)^2$. If the rate of production is kept at 5 units per month, what is the rate of change of profit when 20 units have been made?

4. An object is dropped in water and generates a circular wave. If the area enclosed by the wave increases at a rate of 5 m^2 per second, find the rate of change of the radius of the wave when the area is 15 m^2.

5. Sand is poured onto the ground and forms a conical pile whose altitude is twice the radius of the base. If sand is being poured at a 2m^3/minute, find the rate at which the radius is increasing when the radius is 1m.

6. The radius of the base of an inverted cone is 10cm and its altitude is 20 cm. if water is poured into the cone at a rate of 100 cm^3 per second, find the rate at which the water is rising when it is 15 cm deep.

7. A boy who is 1.25m tall is walking away from a lamp post which is 3m high at a speed of 1.6m/s. At what speed is the tip of his shadow moving when he is 10m from the pole.

DID YOU KNOW?

In Economics, calculus can be used when working on concepts like margins. Calculus are used to determine the most opportune time to buy or sell goods or when considering the effects of price on how much consumers purchase. Economic research uses calculus functional relationships, relation of income, market prediction and so on. Differentiation is used to find optimum solutions of economics. Credit card companies use calculus for monthly payments ~ Michael Ocampo

8.2.4 Kinematics using Differentiation

- **Displacement**(s) refers to straight line distance in a particular direction.
- **Velocity**(v) is the rate of change of displacement. It is a speed in a specified direction.
- **Acceleration**(a) describes the rate of change of velocity.

s = displacement	measured in m
v = velocity	measured in m/s
a = acceleration	measured in m/s^2
t = time	measured in s

displacement $\xrightarrow{\textit{differentiate}}$ velocity $\xrightarrow{\textit{differentiate}}$ acceleration

- **Common terms**

The term initially means when the time $t = 0$

If the velocity $v = 0$ the object is stationary or at rest. It is not moving.

Example 1 The displacement in meters of a particle after t seconds is given by $s = 3t^2 - 6t + 7$.

a) Find the expression for velocity.
b) Find the initial velocity.
c) Calculate the velocity after 3s.
d) When is the particle at rest?
e) Find the acceleration.

Answer

a) Expression for the velocity: Differentiate displacement equation

$$s = 3t^2 - 6t + 7$$

$$\frac{ds}{dt} = (3\times2)t^{2-1} - (6\times1)t^{1-1} + 0$$

$$v = 6t - 6$$

b) Initial velocity means velocity at $t = 0$

$\Rightarrow v = 6\times0 - 6$

$v = -6$ m/s

c) Velocity after 3s:

At $t = 3 \Rightarrow v = 6t - 6$

$v = 6\times3 - 6$

$= 12$ m/s

d) At **rest** means the particle is not moving so velocity = 0

$$v = 6t - 6$$
$$6t - 6 = 0$$
$$t = 1s$$

e) Find the acceleration: differentiate velocity expression

$$\Rightarrow \; v = 6t - 6$$
$$a = 6 \; m/s^2$$

Exercise 8.2.4

1. The displacement, s , in meters of a particle is given by $s = 80t - 12t^2$.
   a) Find the expression for velocity.
   b) Find the initial velocity.
   c) When is the particle stationary?

2. The velocity, v , in m/s of a particle after t seconds is given by $v=7-3t^2+6t^3$.
   a) What is the initial velocity?
   b) Calculate the velocity after 5s.
   c) Find an expression for acceleration.
   d) Calculate the acceleration after 3s.

3. The displacement, s , in meters of a particle after t seconds is given by $s=\frac{3}{t^3}-6t^2+7$.
   a) Find an expression for velocity.
   b) Find an expression for acceleration.
   c) Calculate the acceleration after 3s.

4. Complete:
   Initially means when ________ = 0.
   "At rest" means ________ = 0.

DID YOU KNOW?

In business, Calculus can help by providing an accurate and measurable way to record changes in variables using numbers and mathematics. Derivatives are used to determine the maximum profit, minimum cost, rate of change of cost and how to maximize profit or minimize cost and production. ~ Michael Ocampo

Review Exercise 8

1. Find $\frac{dy}{dx}$ given $y = \frac{2e^{3-x}}{3\sin x}$

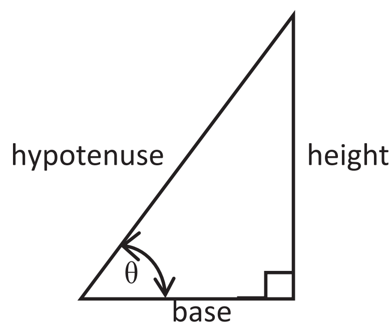
2. Given $y = 2x(ln\ x - 1) + 7e^3$, find $\frac{dy}{dx}$

3. Use **implicit differentiation** to find $\frac{dy}{dx}$:

   a) $3x^2 + 5y^4 = x + 3$ b) $2x^4 - x^2 \cos y + 3y^2 = -4$

4. The slant edge of a right circular cone is 6 cm in length. Find the height of the cone when the volume is a **maximum**.

5. A triangle has hypotenuse of fixed length 61 cm. The length of the base and height is variable.



The angle, θ, between the hypotenuse and the base is increasing at a **constant** rate. Initially θ is 0.09 radians. After 5 seconds, θ increases to 0.14 radians.

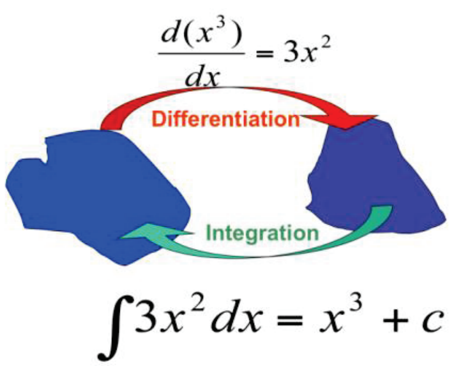
Find the **rate** at which the height of the triangle is increasing after 10 seconds. Assume that at this instant the triangle is right-angled.

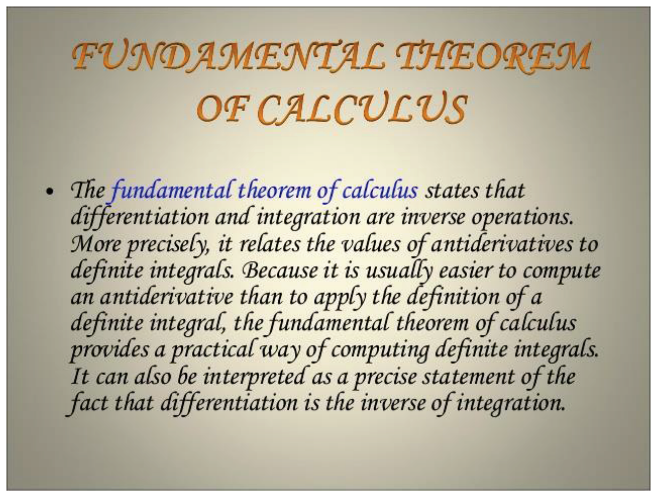
STRAND NINE

INTEGRATION

SUB – STRAND 9.1

Anti - derivatives of Functions





LEARNING OUTCOMES

9.1.1 Evaluate integrals using u-substitution.

9.1.2 Evaluate integrals using partial fractions.

9.1.3 Evaluate integrals using trigonometric identities.

9.1.1 Algebraic Substitution

9.1.1.1 Type I and Type II

- Integration by u Substitution: an u substitution transforms the given integrals into easier ones.

Steps:

1. Make an appropriate choice for u. Usually we take u to be an expression whose derivative appears as a factor of the integrand.

2. Compute $\frac{du}{dx}$.

3. Make dx the subject.

4. Substitute u and dx from 3 . Check that the integral is now in terms of u. This new integral should be easier than the initial one.
   If not, then your u Substitution is incorrect. Go back to step 2 and come up with another substitution.

5. Evaluate the resulting integral.

6. Do not forget that the answer is a function of x. You should substitute back the initial **variable** x.

For some special cases (Type II), there is a need to convert the integrand to an expression that can be easily integrated.

All the steps are similar to u Substitution but **with one change**:

- Need to make x the subject from the "u" equation

Also, note if $f(x) = x^n$, $\int f(x)\,dx = \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C}$

The *definite integral* of $f(x)$ between a (lower limit) and b (upper limit) can be defined as follows:

$$\begin{aligned}\int_a^b f(x)dx &= [\mathrm{F}(x)]_a^b \\ &= \mathrm{F}(b) - \mathrm{F}(a)\end{aligned}$$

Example 1 Find $\int \frac{2x}{x^2-3}\,dx$

Answer: Type I Substitution

Take u to be the denominator since its derivative appears in the numerator.

Let $u = x^2 - 3$

$$\frac{du}{dx} = 2x$$

$$du = 2xdx$$

$$\frac{du}{2x} = dx$$

Write integral in terms of u.

$$\int \frac{2x}{x^2+3}\,dx = \int \frac{2x}{u}\,\frac{du}{2x}$$

$$= \int \frac{1}{u}\,du$$

Evaluate the resulting integral

Substitute $u = x^2 + 3$

$$\int \frac{1}{u}\,du \quad = \ln|u| + C$$

$$\therefore \int \frac{2x}{x^2-3}\,dx = \ln\left|x^2-3\right| + C$$

Short cut: If the derivative of the denominator = numerator then take the *ln* of the absolute value of denominator.

For example $\int \frac{3x^2}{x^3-2}\,dx = \ln\left|x^3-2\right| + C$

Example 2 Find $\int \frac{x^2}{\sqrt{x^3+9}}\,dx$

Answer: Type I Substitution

Take u to be x^3+9 since its derivative appears in the numerator.

$$u = x^3 + 9$$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\,dx \qquad \Rightarrow \frac{du}{3x^2} = dx$$

Write integral in terms of u.

$$\int \frac{x^2}{\sqrt{x^3+9}}\,dx \qquad = \int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2}$$

$$= \int \frac{du}{3\sqrt{u}}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$$

Evaluate the resulting integral.

$$\frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{3}\int u^{-\frac{1}{2}}\quad du$$

Simplify

$$= \frac{1}{3}\left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) = \frac{1}{3}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}\sqrt{u} + C$$

substitute for u

$$\therefore \quad \int \frac{x^2}{\sqrt{x^3+9}}\quad dx = \frac{2}{3}\sqrt{x^3+9} + C$$

Example 3 Find $\int x(3x+1)^9\,dx$

Answer: Type II Substitution

Take u to be the expression in the brackets.

$$u = 3x+1$$

$$\frac{du}{dx} = 3$$

$$\frac{du}{3} = dx$$

Make "dx" the subject.

Substitute

$$\int x(3x+1)^9\ dx = \int xu^9\frac{du}{3}$$

$$= \frac{1}{3}\int xu^9\,du$$

In this case x do not cancel as in examples 1 and 2.

To remove x, make x the subject from the "u" equation

$$x = \frac{u-1}{3}$$

Substitute x to write integral in terms of u.

Simplify

$$\int x(3x+1)^9.dx = \int \underbrace{\frac{u-1}{3}.u^9\frac{1}{3}}du$$

$$= \int \frac{u^{10}-u^9}{9}du$$

$$= \frac{1}{9}\int u^{10}-u^9.du$$

Evaluate the resulting integral

$$\frac{1}{9}\int u^{10}-u^9.du = \frac{1}{9}\left(\frac{u^{11}}{11}-\frac{u^{10}}{10}\right)+C$$

$$= \frac{u^{11}}{99}-\frac{u^{10}}{90}+C$$

Substitute for u

$$\therefore \int x(3x+1)^9\ dx = \frac{(3x+1)^{11}}{99}-\frac{(3x+1)^{10}}{90}+C$$

Example 4 Evaluate $\int_0^2 x(x^2+1)^3 dx$

Answer

Let $u = x^2+1$

$$\frac{du}{dx} = 2x$$

$$\Rightarrow \frac{du}{2x} = dx$$

$$\int x\left(x^2+1\right)^3 dx = \int x\ u^3\ \frac{du}{2x}$$

$$= \int \frac{u^3}{2}du$$

$$= \frac{u^4}{8}+\text{C}$$

Method 1: Find the limits of integration in terms of u

When $x = 0$	When $x = 2$
$u = x^2+1$	$u = x^2+1$
$u = 0^2+1$	$u = 2^2+1$
$=1$	$= 5$

$$\left[\frac{u^4}{8}\right]_1^5 = \frac{5^4}{8} - \frac{1^4}{8}$$

$$= 78$$

Method 2: Evaluate in terms of x

$$\frac{u^4}{8} = \frac{(x^2+1)^4}{8}$$

$$\left[\frac{(x^2+1)^4}{8}\right]_0^2 = \frac{(2^2+1)^4}{8} - \frac{(0^2+1)^4}{8}$$

$$= 78$$

Exercise 9.1.1.1

1. Evaluate the following:

a) $\int \frac{3}{2x+1}dx$ b) $\int_2^{10} \frac{1}{x+2}dx$ c) $\int \frac{1}{2x-1}dx$

d) $\int \frac{3x}{x^2-2}dx$ e) $\int \frac{3x^2+2}{x^3+2x}dx$ f) $\int_1^2 \frac{2x}{x^2+1}dx$

2. Evaluate the following:

a) $\int_1^9 x(x^2+1)^5\,dx$ b) $\int 18x^5(x^6+2)^2\,dx$ c) $\int x^2\sqrt{x^3+1}\,dx$

d) $\int \frac{x}{\sqrt{5+x}}dx$ e) $\int x\sqrt{x-4}\,dx$ f) $\int \frac{x}{(x+3)^5}dx$

DID YOU KNOW?

Calculus was initially developed for better navigation system. Engineers use calculus for building skyscrapers and bridges. In robotics, calculus is used on how robotic parts will work on given commands. Also, calculus is used to improve safety of vehicles. ~ Michael Ocampo

9.1.1.2 Trigonometric, Exponential and Hyperbolic Function

- The derivative and integration are opposite process.

- Hyperbola

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

- The exponential function

$$\int e^{x}\ dx = e^{x} + C \Rightarrow \int e^{ax}\ dx = \frac{e^{ax}}{a} + C$$

- Trigonometric function

$$\int \cos x\ dx = \sin x + C$$

$$\int -\sin x\ dx = \cos x + C$$

$$\int \sin x = -\cos x + C$$

$$\int \sec^{2} x = \tan x + C$$

Example 1 Find $\int \frac{1}{5x} dx$

Answer

$$\int \frac{1}{5}\left(\frac{1}{x}\right)\ dx = \frac{1}{5}\int \frac{1}{x} dx$$

$$= \frac{1}{5}\ln|x| + C$$

Example 2 Find $\int \sin 3x \, dx$

Answer

$$u=3x, \quad \frac{du}{dx}=3, \quad \frac{du}{3}=dx$$

$$\int \sin 3x \, dx = \int \sin u \, \frac{du}{3}$$

$$= \frac{1}{3}\int \sin u \, du$$

$$= \frac{1}{3}(-\cos u)+C$$

Substitute for u

$$\therefore \int \sin 3x \, dx = \frac{-\cos 3x}{3} + C \quad or \quad -\frac{1}{3}\cos 3x + C$$

Short cut: integrate the outer function (evaluated at inner) divide by the derivative of inner linear function

Example 3 Find $\int e^{2x+1} \, dx$

Answer

Method 1: Integration by u Substitution

Take u to be the power (exponent)

Let u = $2x$ +1

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$\frac{du}{2} = dx$$

Write integral in terms of u.

$$\int e^{2x+1} dx = \int e^{u} \frac{du}{2}$$

Evaluate the resulting integral.

$$\frac{1}{2}\int e^{u} \, du = \frac{1}{2}e^{u} + C$$

Substitute for u

$$\therefore \int e^{2x+1} dx = \frac{e^{2x+1}}{2} + C$$

Method 2: integrate the outer function (evaluated at inner) divide by the derivative of inner linear function.

$$f(x) = 2x + 1, \qquad f'(x) = 2$$

$$\therefore \int e^{f(x)} \, dx = \frac{e^{f(x)}}{f'(x)} + C$$

$$= \frac{e^{2x+1}}{2} + C$$

Example 4 Evaluate $\int_{1}^{e+1} \frac{1}{x-1}\,dx$

Answer

$$\int_{1}^{e+1} \frac{1}{x-1}dx = \Big[\ln|x-1|\Big]_{1}^{e+1}$$

$$= \ln|e+1-1| - \ln|1|$$

$$= \ln e - \ln 1$$

$$= 1$$

> **Short cut**: If the derivative of the denominator = numerator then take the *ln* of the absolute value of denominator.

Example 5 Find $\int \cos x . e^{\sin x} . dx$

Answer

Take u to be the power.

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$\Rightarrow \frac{du}{\cos x} = dx$$

Write integral in terms of u.

$$\int \cos x . e^{\sin x} . dx = \int \cos x \ e^{u} \ \frac{du}{\cos x}$$

$$= \int e^{u} du$$

Evaluate the resulting integral.

$$\int e^{u} du = e^{u} + C$$

> Substitute for u

$$\therefore \int \cos x . e^{\sin x} dx = e^{\sin x} + C$$

Example 6 Evaluate $\int_{0}^{\ln 2} 3e^{x}\,dx$

Answer

$$\int_{0}^{\ln 2} 3e^{x}\,dx = \Big[3e^{x}\Big]_{0}^{\ln 2}$$

$$= 3e^{\ln 2} - 3e^{0}$$

$$= 3 \times 2 - 3$$

$$= 3$$

Example 7 Evaluate $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$

Answer

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$\Rightarrow \frac{du}{\cos x} = dx$$

$$\int \cos x \sin^2 x . dx = \int \cos x \; u^2 \frac{du}{\cos x}$$

$$= \int u^2 \; du$$

$$= \frac{u^3}{3} + C$$

Method 1: Find the limits of integration in terms of u

When $x = 0$ $u = \sin x$ $u = sin\ 0$ $= 0$	When $x = \frac{\pi}{2}$ $u = \sin x \Rightarrow u = \sin \frac{\pi}{2}$ $= 1$

$$\left[\frac{u^3}{3}\right]_0^1 = \frac{1^3}{3} - 0$$

$$= \frac{1}{3}$$

Method 2: Evaluate in terms of x

$$\frac{u^3}{3} = \frac{\sin^3 x}{3}$$

$$\left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}} = \frac{\sin^3\left(\frac{\pi}{2}\right)}{3} - \frac{\sin^3 0}{3}$$

$$= \frac{1}{3}$$

Exercise 9.1.1.2

1. Evaluate the following:

   a) $\int \sin 5x\, dx$ b) $\int e^{-2x} dx$

   c) $\int \cos 3x dx$ d) $\int \sin(3x+2)\, dx$

2. Find the anti-derivatives of the following:

   a) $\int x e^{x^2} dx$ b) $\int 2x^2 e^{x^3} \; dx$ c) $\int \frac{e^x+2}{e^x+2x} \; dx$

   d) $\int \frac{e^x}{e^x+1} dx$ e) $\int \frac{\ln x}{2x} dx$ f) $\int \frac{\cos x}{e^{\sin x}} dx$

3. Evaluate the following:

   a) $\int_0^{\pi} \cos(2x) \; dx$ b) $\int_0^1 e^{5x} \; dx$ c) $\int_1^3 1+\frac{1}{x} \; dx$

   d) $\int_1^e \frac{1}{x} \, dx$ e) $\int_0^{\frac{\pi}{2}} \sin x \, e^{\cos x} \; dx$ f) $\int_e^{e^2} \frac{\ln x}{x} dx$

9.1.2 Use of Partial Fractions

If the integrand (the expression after the integral sign) is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in **partial fractions** before integration takes place.

Example 1 Find $\int \frac{3x+3}{x^2+x-2}dx$

Answer

Step 1: Write as partial fractions

$$\frac{3x+3}{x^2+x-2}=\frac{3x+3}{(x+2)(x-1)}$$

$$=\frac{A}{(x+2)}+\frac{B}{(x-1)}$$

$$=\frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$3x+3=A(x-1)+B(x+2)$$

Let x =1

3 +3 =3B → B= 2

Let x = -2

3(-2) +3 = -3A → A = 1

$$\therefore \frac{3x+3}{x^2+x-2}=\frac{1}{(x+2)}+\frac{2}{(x-1)}$$

Step 2: Integrate

$$\int \frac{3x+3}{x^2+x-2}dx = \int \frac{1}{(x+2)}dx + \int \frac{2}{(x-1)}dx$$

$$= \ln|x+2| + \ln|x-1| + C$$

Example 2 Find $\int \frac{x+4}{(x+2)^2(x+3)}dx$

$$given \ \frac{x+4}{(x+2)^2(x+3)} = \frac{1}{(x+3)} - \frac{1}{(x+2)} + \frac{2}{(x+2)^2}$$

Answer

$$\int \frac{x+4}{(x+2)^2(x+3)}dx = \int \frac{1}{(x+3)}dx - \int \frac{1}{(x+2)}dx + \int \frac{2}{(x+2)^2}dx$$

Let $u = x + 2$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int \frac{2}{u^2}du = \int 2u^{-2}du = 2\int \frac{u^{-2+1}}{-1}du$$

$$= -2u^{-1} = \frac{-2}{u} = \frac{-2}{x+2}$$

Integrate each term separately:

$$\therefore \int \frac{x+4}{(x+2)^2(x+3)}dx = \ln|x+3| - \ln|x+2| - \frac{2}{x+2} + C$$

Example 3 Find $\int \frac{x^3+2x-4}{x^3-4x}dx$

$$given \ \frac{x^3+2x-4}{x^3-4x} = 1 + \frac{1}{x} - \frac{2}{(x+2)} + \frac{1}{(x-2)}$$

Answer

Integrate each term separately:

$$\int \frac{x^3+2x-4}{x^3-4x}dx = \int 1 \ dx + \int \frac{1}{x} \ dx - \int \frac{2}{(x+2)} \ dx + \int \frac{1}{(x-2)} \ dx$$

$$= x + \ln|x| - 2\ln|x+2| + \ln|x-2| + C$$

Exercise 9.1.2

Evaluate the following:

a) $\int \frac{2x+1}{x^2+x-2} \ dx$ b) $\int_2^3 \frac{2}{x^2-1} \ dx$ c) $\int_4^8 \frac{4}{x^2-2x-3} \ dx$

d) $\int \frac{15-4x-x^2}{(x+1)(x-2)^2} \ dx$ e) $\int \frac{3x+5}{x^2+x-12} \ dx$ f) $\int \frac{10x+24}{(x-3)(x^2+9)} \ dx$

9.1.3 Integrating using trigonometric identities

- Integrating product of sine and cosine: The following formulae will be used to simplify the integration.
  - $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
  - $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
  - $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

- The following identities can be used to integrate $\cos^2 x$ and $\sin^2 x$
  - $\cos 2A = 2\cos^2 A - 1$
    $= 1 - 2\sin^2 A$

Example 1 Find $\int \sin 5x \cos 2x \, dx$

Answer

Divide each side by 2 in the first formula

- $2\sin A\cos B = \sin(A+B) + \sin(A-B) \Rightarrow \sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$\int \sin 5x \cos 2x \, dx = \int \frac{1}{2}[\sin(5x+2x) + \sin(5x-2x)] \, dx$$

$$= \frac{1}{2}\int [\sin 7x + \sin 3x] \, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 7x}{7} - \frac{\cos 3x}{3}\right] + C$$

$$= \frac{-\cos 7x}{14} - \frac{\cos 3x}{6} + C$$

Short cut: integrate the outer function (evaluated at inner) and divide by the derivative of inner linear function.

Example 2 Find $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$

Answer

We have to use the identity

- $\cos 2A = 1 - 2\sin^2 A$

Change the subject

$\Rightarrow 2\sin^2 A = 1 - \cos 2A$

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\cos 2x\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin\pi}{2} - (0 - \frac{\sin 0}{2})\right]$$

$$= \frac{\pi}{4}$$

Exercise 9.1.3

1. Find the anti-derivatives of the following:

   $a)\ \int 2\cos 4x \cos x \quad dx$ $\qquad b)\ \int \sin 3x \sin 2x \quad dx$

   $c)\ \int_0^{\pi} \sin 10x \cos 4x\, dx$ $\qquad d)\ \int_{\pi}^{\frac{\pi}{2}} 2\cos 3x \cos(-x)\, dx$

2. Find the anti-derivatives of the following:

   $a)\ \int \cos^2 x\, dx$ $\qquad b)\ \int \left(4\sin^2 x + 3\right) dx$ $\qquad c)\ \int 3\sin^2 x\, dx$

SUB – STRAND 9.2

Applications of Integration

Indefinite Integral

General:

$$\int f(x)dx = F(x) + C$$

This represents a *family* of functions

Integral Sign

Integrand

The Indefinite Integral

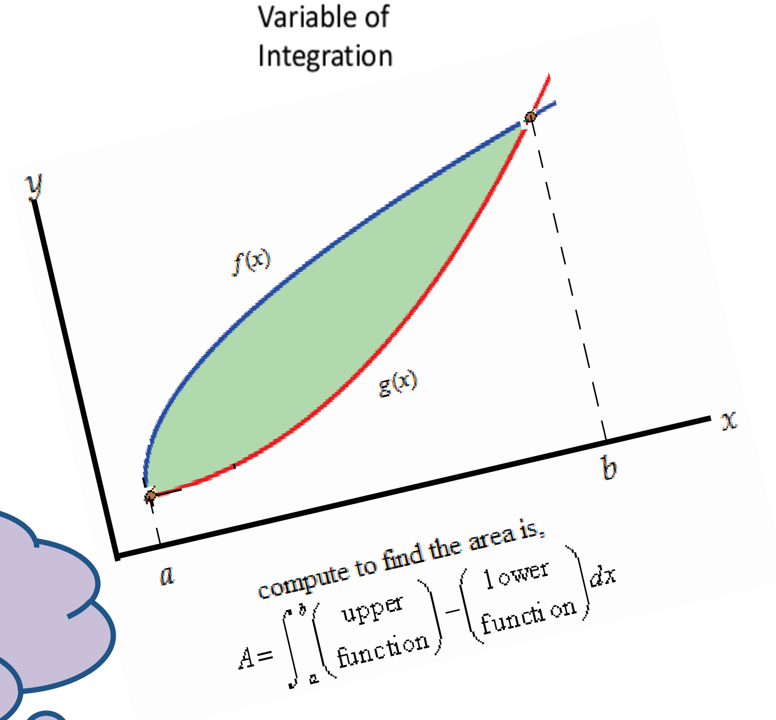
The constant of Integration

Variable of Integration

definite Integral

This gives you a *number*

Definite: $\int_a^b f(x)dx = F(b) - F(a)$



LEARNING OUTCOMES

9.2.1 Calculate the area between two graphs.

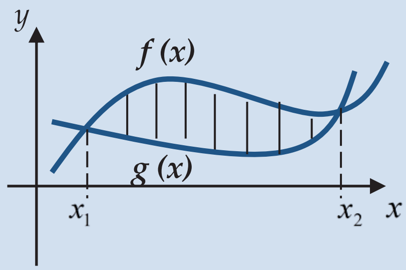
9.2.2 Solve kinematics problems.

9.2.1 Area Between Two Graphs

If the upper curve or line on the top is $f(x)$ and the lower curve or line at the bottom is $g(x)$, then the area between two curves or lines from a to b is given by:

$$Area = \int_{Lower\ Limit}^{Upper\ Limit} (Upper\ \ Curve\ - \ Lower\ \ Curve)$$

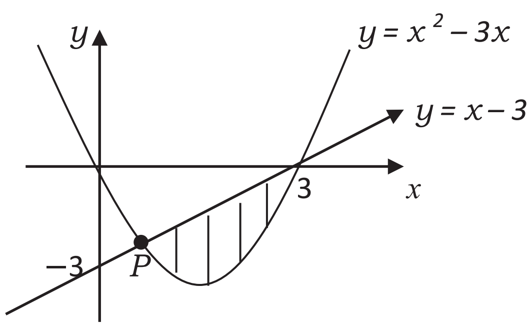
Consider the diagram given below:



$$\boxed{Area = \int_{x_1}^{x_2} (f(x) - g(x))dx,}$$

where x_1 and x_2 are x coordinates of the points of intersection

Example The diagram given below shows the sketches of the functions $y = x - 3$ and $y = x^2 - 3x$



a) Show that the x – coordinate of point P is 1.
b) Find the area of the shaded region.

✍ **Answers**

a) Point of intersection:

$$y_1 = y_2$$

$$x^2 - 3x = x - 3$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$x = 1,\ 3$$

$$\therefore x\text{-coordinate of point } P = 1$$

b) Area of the shaded region:

$$Area = \int_a^b (f(x) - g(x))dx$$

$$= \int_1^3 ((x-3) - (x^2 - 3x))dx$$

$$= \int_1^3 (x - 3 - x^2 + 3x)dx$$

$$= \int_1^3 (-x^2 + 4x - 3)dx$$

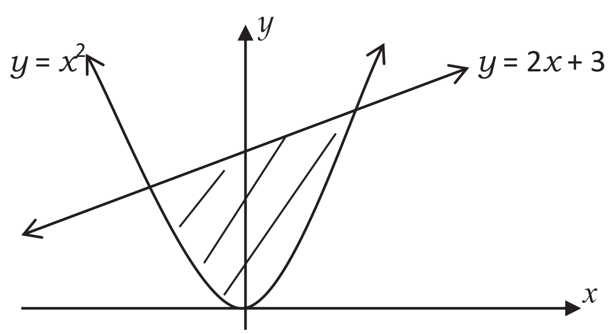
$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

$$= \left[\left(-\frac{3^3}{3} + 2(3^2) - 3(3)\right) - \left(-\frac{1^3}{3} + 2(1^{2)} - 3(1)\right)\right]$$

$$= \frac{4}{3} \textit{square units}$$

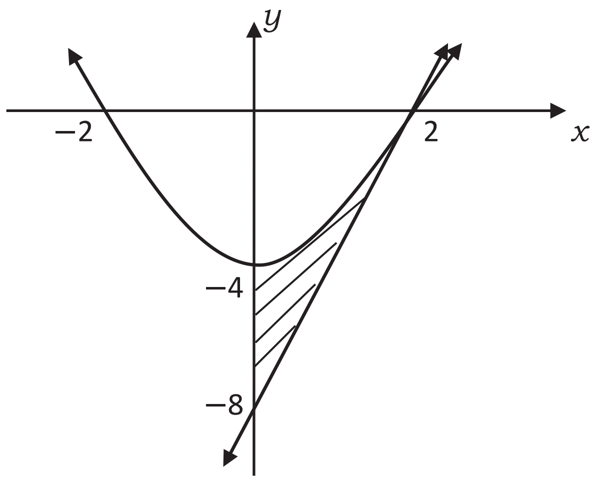
Exercise 9.2.1

1. Two functions are shown below.

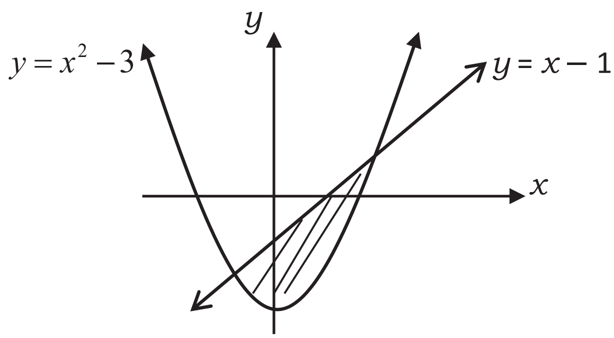


Calculate the area of the shaded region.

2. The diagram below shows the graph of the quadratic function $y = x^2 - 4$ and the straight line $y = 4x - 8$. Find the area of the shaded region.



3. The figure shown below shows the curve $y = x^2 - 3$ and the straight line $y = x - 1$



   a) Calculate the x – coordinates of the point of intersection of the two graphs.

   b) Calculate the area of the shaded region.

4. Find the area bounded by the functions $y = \sqrt{x}$, $y = x^2 - 2x$ and x=2.

5. Find the area bounded by the curve $y=x^2+2x-3$ and the line y = $2x$+1.

6. Find the area bounded by $y = \dfrac{1}{x}$, $y = -8$, x = 2 and x = 5.

7. Find the area bounded by $y = e^x$, $y = -\dfrac{1}{x}$ and the lines x = 1 and x = 2.

9.2.2 Kinematics using Integration

- **Displacement**(s) refers to straight line distance in a particular direction.
- **Velocity**(v) is the rate of change of displacement. It is a speed in a specified direction.
- **Acceleration**(a) is the rate of change of velocity.

$s =$ displacement	measured in m
$v =$ velocity	measured in m/s
$a =$ acceleration	measured in m/s^2
$t =$ time	measured in s

$$\begin{tikzcd} \text{displacement} \arrow[r, bend left, "\textit{differentiate}"] & \text{velocity} \arrow[l, bend left, "\textit{integrate}"] \arrow[r, bend left, "\textit{differentiate}"] & \text{acceleration} \arrow[l, bend left, "\textit{integrate}"] \end{tikzcd}$$

➢ **Common terms**

The term **initially** means when the time t = 0

If the velocity v = 0 the object is **stationary** or **at rest**. It is not moving.

Example A particle moves in a straight line so that its acceleration after t seconds is given by $a = 2t + 7$.

a) Find a formula for the velocity of the particle at time t given that initial velocity = 0.
b) Calculate the velocity at t = 3 s.
c) Find a formula for the displacement of the particle at time t given that initial displacement = 0.
d) Calculate the displacement after 9 seconds.

Answers

a) Acceleration is given we want the formula for velocity, v. So we need to integrate.

$$v = \int 2t + 7 \; dt$$

$$= \frac{2t^2}{2} + 7t + c$$

$$= t^2 + 7t + c$$

Find c by noting that when t =0, v = 0 $\Rightarrow$ c =0

Thus, the formula for the velocity of the particle at time t is $v = t^2 + 7t$

b) Substitute $t=3$

$$v = t^2 + 7t$$
$$= 3^2 + 7(3)$$
$$= 30\, m/s$$

c) Integrate velocity

$$s = \int t^2 + 7t\, dt$$
$$= \frac{t^3}{3} + \frac{7t^2}{2} + c$$

Find c by noting that when $t=0$, $v = 0$ $\Rightarrow$ c =0

$$\therefore s = \frac{t^3}{3} + \frac{7t^2}{2}$$

d) Substitute $t = 9$

$$s = \frac{9^3}{3} + \frac{7 \times 9^2}{2}$$
$$= 526\frac{1}{2} m$$

Exercise 9.2.2

1. A body moves with a velocity of $5t - 2$ m/s, where t is the time in seconds. Find the distance the body moves in the first 5 seconds.

2. The acceleration of a body from a fixed point P is given by $a = 4t - 11$. The body is initially 2m from P and has the velocity of 11m/s after 6s.
   a) Find the formula for velocity.
   b) Find the velocity of the body after 5 seconds.
   c) When is the body at rest?
   d) Find the formula for distance travelled from point P.
   e) How far from P is the body after 10s?

3. A rock is thrown vertically upwards with an initial velocity of 40m/s from a point 5 m above the ground level. The velocity of the rock after t seconds is given by $v = 40 - 10t$.
   a) Find a formula for the height of the rock above the ground at time t.
   b) Find the maximum height reached by the rock.
   c) Calculate the distance travelled in the **third** second.

DID YOU? KNOW

The Petronas Towers in Kuala Lumpur experience high forces due to winds. Integration was used to design the building for strength. ~ Chayan Sengar

Review Exercise 9

1. Using the substitution $u = x^2$, find $\int 2xe^{x^2} \; dx$

2. Evaluate the following:

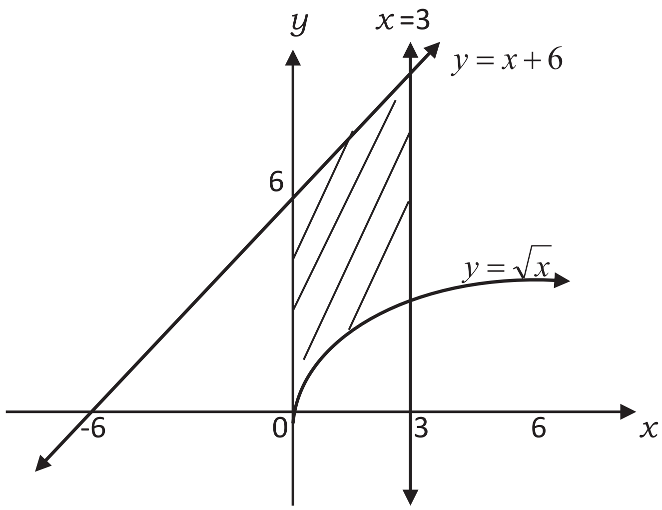
   $a) \int 4x^2 \, Sin(x^3 - 1) \; dx$ $\qquad$ $b) \int \sin^3 x \cos x \, dx$

3. Find $\int x^2 \sqrt{2x-1}\, dx$

4. Find $\int \frac{10x+30}{(x-2)(x^2-9)} \; dx$

5. Evaluate $\int_0^{\pi} 2\sin 3x \sin(-x) \; dx$

6. The functions $y = \sqrt{x}$ and $y = x + 6$ and the straight line x =3 are shown below. Find the shaded area.



7. The acceleration of a body (in square metres per second) is defined by a = 6t + 15. Find the velocity after 2 seconds if it has an initial velocity of 2m/s.

DID YOU KNOW?

The Sydney Opera House is a very unusual design based on slices out of a ball. Many differential equations (one type of integration) were solved in the design of this building.~ Chayan Sengar

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