

Section-A

Choose the correct answer from the given four options in each of the Questions 1 to 3.

If $*$ is a binary operation given by $*: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, $a * b = a + b^2$, then $-2 * 5$ is

- (A) -52 (B) 23 (C) 64 (D) 13

V.V.T Q. 1. Let $*$ be a binary operation on \mathbf{Z}^+ defined by $a * b = a^b$. Find the value of $4 * 2$.

Soln. : Given, $a * b = a^b$

$$\therefore 4 * 2 = 4^2 = 4 \times 4 = 16.$$

V.V.T Q. 1 Let $*$ be the binary operation on \mathbf{N} defined by $a * b = \text{LCM of } a \text{ and } b$. Find $5 * 7$

Soln. :

$$a * b = \text{LCM of } a \text{ and } b$$

$$\therefore 5 * 7 = \text{LCM of } 5 \text{ and } 7 = 35$$

V.V.T Or, Let $*$ be the binary operation on \mathbf{N} defined by

$$a * b = \text{LCM of } a \text{ and } b. \text{ Find } 18 * 24$$

Soln. : $a * b = \text{LCM}(a \text{ and } b)$

$$\therefore 18 * 24 = \text{LCM}(18, 24) = 72$$

Or, Let $*$ be the binary operation on \mathbf{N} defined by $a * b = \text{LCM of } a \text{ and } b$ for all $a, b \in \mathbf{N}$. Prove that $(20 * 16) * 8 = 20 * (16 * 8)$

Soln. : $a * b = \text{LCM of } a \text{ and } b$

$$\therefore (20 * 16) * 8 = (\text{LCM of } 20 \text{ and } 16) * 8$$

$$= 80 * 8$$

$$= \text{LCM of } 80 \text{ and } 8 = 80$$

.... (i)

$$20 * (16 * 8) = 20 * (\text{LCM of } 16 \text{ and } 8)$$

$$= 20 * 16$$

$$= (\text{LCM of } 20 \text{ and } 16 = 80$$

.... (ii)

(i) = (ii) hence proved.

V.V.T Q. 1. If $*$ is a binary composition defined on the set \mathbf{Z} of all integers by

$$a * b = 2a + 3b^2; a, b \in \mathbf{Z}. \text{ Find } (-1) * 7.$$

Soln. : $\because a * b = 2a + 3b^2$

$$(-1) * 7 = 2(-1) + 3(7)^2$$

$$= -2 + 3 \times 49$$

$$= -2 + 147$$

$$= 145$$

6 Let $*$ be a binary operation on \mathbf{Z} defined by $a * b = 2a + 3b$. Find the value of $2 * 5$.

V.T Sol. $\because a * b = 2a + 3b$

$$\therefore 2 * 5 = 2 \times 2 + 3 \times 5$$

$$= 4 + 15 = 19 \text{ Ans.}$$

38. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 3x^2 - 5$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ by $g(x) = \frac{x}{x^2 + 1}$.

Then $g \circ f$ is

(A) $\frac{3x^2 - 5}{9x^4 - 30x^2 + 26}$

(B) $\frac{3x^2 - 5}{9x^4 - 6x^2 + 26}$

(C) $\frac{3x^2}{x^4 + 2x^2 - 4}$

(D) $\frac{3x^2}{9x^4 + 30x^2 - 2}$

40. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is

(A) $(x+5)^{\frac{1}{3}}$

(B) $(x-5)^{\frac{1}{3}}$

(C) $(5-x)^{\frac{1}{3}}$

(D) $5 - x$

2. If $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is a function, then value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is

- (A) $-\frac{\pi}{6}$ (B) $-\frac{\pi}{6}$ (C) $\frac{5\pi}{6}$ (D) $\frac{7\pi}{6}$

 Evaluate $\tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right)$.

Solution $\tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right) = \tan^{-1}\left(-\sin\left(\frac{\pi}{2}\right)\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

 Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos(2\pi + \frac{\pi}{6})\right) = \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

 Evaluate: $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

Solution $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right] = \sin^{-1}\left[\cos\left(\frac{\pi}{3}\right)\right] = \sin^{-1}\left[\frac{1}{2}\right] = \frac{\pi}{6}$.

 The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

- (A) $-\frac{2\pi}{3}$ (B) $-\frac{\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{5\pi}{3}$

Solution (B) is the correct answer.

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{3}\right) = -\sin^{-1}\left(\sin\frac{\pi}{3}\right) = -\frac{\pi}{3}.$$

 The value of $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right)$ is

- (A) $\frac{19}{8}$ (B) $\frac{8}{19}$ (C) $\frac{19}{12}$ (D) $\frac{3}{4}$

Solution (A) is the correct answer. $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right) = \tan\left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{4}\right)$

$$= \tan\tan^{-1}\left(\frac{\frac{4}{3} + \frac{1}{4}}{1 - \frac{4}{3} \times \frac{1}{4}}\right) = \tan\tan^{-1}\left(\frac{19}{8}\right) = \frac{19}{8}.$$

Given that $\begin{pmatrix} 9 & 6 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$. Applying elementary row transformation

$R_1 \rightarrow R_1 - 2R_2$ on both sides, we get

$$(A) \begin{pmatrix} 3 & 6 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 1 & 2 \end{pmatrix}$$

$$(B) \begin{pmatrix} 3 & 6 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

$$(C) \begin{pmatrix} -3 & 6 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -3 & 2 \end{pmatrix}$$

$$(D) \begin{pmatrix} -3 & 6 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

4. If A is a square matrix of order 3 and $|A| = 5$, then what is the value of $|\text{Adj. } A|?$
Ans :- 25

If $\begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, then (x, y) is

$$(A) (1, 1) \quad (B) (1, -1)$$

$$(C) (-1, 1) \quad (D) (-1, -1)$$

V.V.3 Q. 3. Construct a (2×2) matrix. $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{(i+j)^2}{2}$

$$\text{Soln. : } a_{11} = \frac{(1+1)^2}{2} = 2$$

$$a_{21} = \frac{(2+1)^2}{2} = 9/2$$

$$a_{12} = \frac{(1+2)^2}{2} = 9/2$$

$$a_{22} = \frac{(2+2)^2}{2} = 8.$$

$$\therefore \text{Required Matrix, } A = \begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$$

V.V.4 Or, Construct a matrix of order 2×2 whose (i, j) element is given by

$$a_{ij} = \frac{|3i - 2j|}{3}$$

Soln. : A 2×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{ij} = \frac{|3i - 2j|}{3}$$

$$a_{11} = \frac{|3 \times 1 - 2 \times 1|}{3} = \frac{1}{3} \times 1 = \frac{1}{3}$$

$$a_{12} = \frac{|3 \times 1 - 2 \times 2|}{3} = \frac{1}{3} |-1| = \frac{1}{3}$$

$$a_{21} = \frac{|3 \times 2 - 2 \times 1|}{3} = \frac{1}{3} \times |4| = \frac{4}{3}$$

$$a_{22} = \frac{|3 \times 2 - 2 \times 2|}{3} = \frac{1}{3} |2| = \frac{2}{3}$$

$$\text{Hence } A = \begin{bmatrix} 1/3 & 1/3 \\ 4/3 & 1/3 \end{bmatrix}$$

~~V.V.Y~~ Construct a 2×4 matrix whose elements are given by

$$a_{ij} = \frac{(i+2j)^2}{2}$$

Soln. :

$$a_{11} = \frac{(1+2 \times 1)^2}{2} = \frac{9}{2}$$

$$a_{21} = \frac{(2+2 \times 1)^2}{2} = \frac{16}{2} = 8$$

$$a_{12} = \frac{(1+2 \times 2)^2}{2} = \frac{25}{2}$$

$$a_{22} = \frac{(2+2 \times 2)^2}{2} = \frac{36}{2} = 18$$

$$a_{13} = \frac{(1+2 \times 3)^2}{2} = \frac{49}{2}$$

$$a_{23} = \frac{(2+2 \times 3)^2}{2} = \frac{64}{2} = 32$$

$$a_{14} = \frac{(1+2 \times 4)^2}{2} = \frac{81}{2}$$

$$a_{24} = \frac{(2+2 \times 4)^2}{2} = \frac{100}{2} = 50$$

∴ Required Matrix is,

$$\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} & \frac{81}{2} \\ 8 & 18 & 32 & 50 \end{bmatrix}_{2 \times 4}$$

Q. Find the values of x, y and z from the following equation.

$$\begin{bmatrix} x+y & z \\ 5+z & y \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -5 & 4 \end{bmatrix}$$

Soln : $x+y=6$

..... (i)

$$5+z=5 \Rightarrow z=5-5=0$$

$$y=4$$

Putting the value of y in (i)

$$x+4=6 \Rightarrow x=6-4=2$$

$$\therefore x=2, y=4 \text{ and } z=0.$$

~~V.V.Y~~ Q. If $A = \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$ then find a matrix X such

such that $2A + B - X = 0$

Soln. : $2A + B - X = 0$

$$\Rightarrow 2A + B = X$$

Or,

$$X = 2A + B$$

$$= 2 \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 4 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 7 \\ -1 & 6 \end{bmatrix}$$

~~Q.~~ Construct a 3×2 matrix whose (i, j) th element

$$a_{ij} = \frac{1}{2} |i - 3j|$$

Soln. : Given, $a_{ij} = \frac{1}{2} |i - 3j|$ where $i = 1, 2, 3$ and $j = 1, 2$.

∴ $a_{11} = \frac{1}{2} |1 - 3 \times 1| = \frac{1}{2} |-2| = \frac{1}{2} \times 2 = 1$

$$a_{12} = \frac{1}{2} |1 - 3 \times 2| = \frac{1}{2} |-5| = \frac{1}{2} \times 5 = 5/2$$

$$a_{21} = \frac{1}{2} |2 - 3 \times 1| = \frac{1}{2} |2 - 3| = \frac{1}{2} \times 1 = 1/2$$

$$a_{22} = \frac{1}{2} |2 - 3 \times 2| = \frac{1}{2} |-4| = \frac{1}{2} \times 4 = 2$$

$$a_{31} = \frac{1}{2} |3 - 3 \times 1| = \frac{1}{2} \times 0 = 0$$

$$a_{32} = \frac{1}{2} |3 - 3 \times 2| = \frac{1}{2} |-3| = \frac{1}{2} \times 3 = 3/2$$

∴ the required matrix is

$$\begin{bmatrix} 1 & 5/2 \\ 1/2 & 2 \\ 0 & 3/2 \end{bmatrix}$$

~~Q.~~ Write the value of λ for matrix A of order 2×2 when $|A| = 5$ and $|\lambda A| = 20$

Soln. : We know for order n

$$|KA| = K^n |A|$$

∴ $|\lambda A| = 20$

⇒ $\lambda^2 |A| = 20$

⇒ $\lambda^2 \times 5 = 20$

⇒ $\lambda^2 = 4$

∴ $\lambda = \pm 2$

~~Ques.~~ Find the value of determinant

$$\begin{vmatrix} \sin 70^\circ & -\cos 70^\circ \\ \sin 20^\circ & \cos 20^\circ \end{vmatrix}$$

Soln. : $\begin{vmatrix} \sin 70^\circ & -\cos 70^\circ \\ \sin 20^\circ & \cos 20^\circ \end{vmatrix}$

$$= \sin 70^\circ \cdot \cos 20^\circ + \cos 70^\circ \cdot \sin 20^\circ$$

$$= \sin (70^\circ + 20^\circ) = \sin 90^\circ = 1.$$

~~Ques.~~ x का मान ज्ञात कीजिए यदि,

$$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & x \\ 6 & 4 \end{vmatrix}$$

Soln. :

$$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & x \\ 6 & 4 \end{vmatrix}$$

$$\text{or, } 2 - 20 = 8x - 6x$$

$$\text{or, } 2x = -18$$

$$\text{or, } x = -9$$

Or, सारणिक का मान ज्ञात कीजिए -

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

~~(1)~~ Soln :

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta - (-\sin^2\theta)$$

$$= \cos^2\theta + \sin^2\theta = 1.$$

Q1 Find the value of derivative of $\tan^{-1}(e^x)$ w.r.t. x at the point $x = 0$.

A $\frac{1}{2}$ Ans

Q2, Find dy/dx , if $y = \frac{1 - \cos x}{1 + \cos x}$

Soln. :
$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \\ &= \frac{1 + \cos x (\sin x) - (1 - \cos x) (-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\sin x + \sin x \cdot \cos x + \sin x - \sin x \cdot \cos x}{(1 + \cos x)^2} = \frac{2\sin x}{(1 + \cos x)^2} \end{aligned}$$

Q3 If $\lim_{x \rightarrow 3^+} f(x) = 7$ and $f(x)$ is continuous, then find the value of $f(3)$

Soln. : Since $f(x)$ is continuous

$$\therefore \lim_{x \rightarrow 3^+} f(x) = f(3)$$

$$\text{or, } 7 = f(3)$$

$$\therefore f(3) = 7.$$

Q4, Find dy/dx if $y = \cos(\sin^{-1}x)$

Soln : $y = \cos(\sin^{-1}x)$

Putting $\sin^{-1}x = t$, we get

$$\therefore \frac{dy}{dx} = -\sin t \text{ and } \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dt} = \frac{dy}{dt} \times \frac{dt}{dx} = -\sin t \cdot \frac{1}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

3. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point
- (A) (1, 2) (B) (2, 1)
 (C) (1, -2) (D) (-1, 2)

Q. 5. Find the slope of tangent to the curve $y = x^3 - 3x + 2$ at $x = 2$.

Soln. : Given, $y = x^3 - 3x + 2$

$$\text{slope of tangent} = \frac{dy}{dx}$$

$$= \frac{d(x^3 - 3x + 2)}{dx}$$

$$= 3x^2 - 3$$

Given, $x = 2$

$$\therefore \frac{dy}{dx} = 3(2)^2 - 3 = 12 - 3 = 9.$$

Or, Find the slope of tangent to curve $y = 3x^3 - 4x$ at $x = 4$

$$\text{Soln : slope} = \left(\frac{dy}{dx} \right)$$

$$y = 3x^3 - 4x$$

$$\therefore \frac{dy}{dx} = \frac{d(3x^3 - 4x)}{dx} = 9x^2 - 4$$

Given $x = 4$

$$\therefore \text{slope} = \left(\frac{dy}{dx} \right)_x = 9 \times (4)^2 - 4 = 144 - 4 = 140$$

57. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

- (A) 0 (B) 12 (C) 16 (D) 32

Section-B

11. Let n be a fixed positive integer and R be the relation in \mathbf{Z} defined as $a R b$ if and only if $a - b$ is divisible by n , $\forall a, b \in \mathbf{Z}$. Show that R is an equivalence relation.

~~Ex 15~~ Let \mathbf{R} be the set of real numbers and $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 4x + 5$. Show that f is invertible and find f^{-1} .

Solution Here the function $f: \mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x) = 4x + 5 = y$ (say). Then

$$4x = y - 5 \quad \text{or} \quad x = \frac{y-5}{4}.$$

This leads to a function $g: \mathbf{R} \rightarrow \mathbf{R}$ defined as

$$g(y) = \frac{y-5}{4}.$$

Therefore, $(g \circ f)(x) = g(f(x)) = g(4x + 5)$

$$= \frac{4x+5-5}{4} = x$$

or

$$g \circ f = I_{\mathbf{R}}$$

Similarly

$$(f \circ g)(y) = f(g(y))$$

$$= f\left(\frac{y-5}{4}\right)$$

$$= 4\left(\frac{y-5}{4}\right) + 5 = y$$

or

$$f \circ g = I_{\mathbf{R}}.$$

Hence f is invertible and $f^{-1} = g$ which is given by

$$f^{-1}(x) = \frac{x-5}{4}$$

Let $*$ be a binary operation defined on \mathbf{Q} . Find which of the following binary operations are associative

- (i) $a * b = a - b$ for $a, b \in \mathbf{Q}$.
- (ii) $a * b = \frac{ab}{4}$ for $a, b \in \mathbf{Q}$.
- (iii) $a * b = a \rightarrow b + ab$ for $a, b \in \mathbf{Q}$.
- (iv) $a * b = ab^2$ for $a, b \in \mathbf{Q}$.

Solution

(i) $*$ is not associative for if we take $a = 1, b = 2$ and $c = 3$, then

$$(a * b) * c = (1 * 2) * 3 = (1 - 2) * 3 = -1 - 3 = -4 \text{ and}$$

$$a * (b * c) = 1 * (2 * 3) = 1 * (2 - 3) = 1 - (-1) = 2.$$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(ii) $*$ is associative since \mathbf{Q} is associative with respect to multiplication.

(iii) $*$ is not associative for if we take $a = 2, b = 3$ and $c = 4$, then

$$(a * b) * c = (2 * 3) * 4 = (2 - 3 + 6) * 4 = 5 * 4 = 5 - 4 + 20 = 21, \text{ and}$$

$$a * (b * c) = 2 * (3 * 4) = 2 * (3 - 4 + 12) = 2 * 11 = 2 - 11 + 22 = 13$$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(iv) $*$ is not associative for if we take $a = 1, b = 2$ and $c = 3$, then $(a * b) * c = (1 * 2) * 3 = 4 * 3 = 4 \times 9 = 36$ and $a * (b * c) = 1 * (2 * 3) = 1 * 18 = 1 \times 18^2 = 324$.

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

Q. 1 Find gof and fog , if $f(x) = |x|$ and $g(x) = |5x - 2|$.

Sol. $f(x) = |x|, g(x) = |5x - 2|$

$$gof(x) = g(f(x)) = g(|x|) = |5|x| - 2|$$

$$fog(x) = f(g(x)) = f(|5x - 2|) = ||5x - 2|| = |5x - 2|$$

Q. 2 Find gof and fog , if $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.

Sol. $f(x) = 8x^3, g(x) = x^{\frac{1}{3}}$

$$gof(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x.$$

$$fog(x) = f(g(x)) = f(x^{\frac{1}{3}}) = 8 \cdot (x^{\frac{1}{3}})^3 = 8x.$$

Q. 3 Let $f: \mathbf{R} \rightarrow \mathbf{R}: f(x) = (3 - x^3)^{\frac{1}{3}}$, then show that $(fof)(x) = x$.

Sol. We have, $(fof)(x) = f\{f(x)\} = f(3 - x^3)^{\frac{1}{3}}$

$$= f(y); \quad [\text{where, } y = (3 - x^3)^{\frac{1}{3}}]$$

$$= (3 - y^3)^{\frac{1}{3}} = [3 - (3 - x^3)]^{\frac{1}{3}} \quad [\because y^3 = (3 - x^3)]$$

$$= (x^3)^{\frac{1}{3}} = x$$

Hence, $(fof)(x) = x$

Show that the function $f: \mathbf{R} \rightarrow \mathbf{R} : f(x) = 3 - 4x$ is one-one onto and hence bijective.

Sol. We have,

$$\begin{aligned} f(x_1) &= f(x_2) \Rightarrow 3 - 4x_1 = 3 - 4x_2 \\ \Rightarrow -4x_1 &= -4x_2 \Rightarrow x_1 = x_2 \\ \therefore f &\text{ is one-one.} \end{aligned}$$

$$\text{Let, } y = 3 - 4x. \text{ Then, } x = \frac{(3-y)}{4}.$$

Thus, for each $y \in \mathbf{R}$ (condition of f), there exists $x = \frac{(3-y)}{4} \in \mathbf{R}$

$$\text{such that } f(x) = f\left(\frac{(3-y)}{4}\right) = \left\{3 - 4 \cdot \frac{(3-y)}{4}\right\} = 3 - (3 - y) = y$$

This shows that every element in co-domain of f has its pre-image in $\text{dom}(f)$.

$\therefore f$ is onto.

Hence, the given function is bijective.

Let $f: \mathbf{R} \rightarrow \mathbf{R} : f(x) = 4x + 3$ for all $x \in \mathbf{R}$. Show that f is invertible and find f^{-1} .

Sol. We have,

$$f(x_1) = f(x_2) \Rightarrow 4x_1 + 3 = 4x_2 + 3$$

$$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$$

$\therefore f$ is one-one.

$$\text{Again, } y = 4x + 3 \Rightarrow x = \frac{(y-3)}{4}$$

Now, if $y \in \mathbf{R}$ (co-domain of f), then there exists $x = \frac{(y-3)}{4} \in \mathbf{R}$

$$\text{such that } f(x) = f\left(\frac{(y-3)}{4}\right) = \left\{4 \cdot \frac{(y-3)}{4} + 3\right\} = y$$

$\therefore f$ is onto.

Thus, f is one-one onto and therefore invertible.

Now, $y = f(x)$

$$\Rightarrow y = 4x + 3$$

$$\Rightarrow x = \frac{(y-3)}{4}$$

$$\Rightarrow f^{-1}(y) = \frac{(y-3)}{4} \quad [\because f(x) = y \Rightarrow x = f^{-1}(y)]$$

Thus, we define $f^{-1}: \mathbf{R} \rightarrow \mathbf{R} : f^{-1}(y) = \frac{(y-3)}{4}$ for all $y \in \mathbf{R}$.

Let $f(x) = x + 7$ and $g(x) = x - 7$; $x \in \mathbf{R}$. Find the following—

- (a) $f \circ f(7)$, (b) $f \circ g(7)$, (c) $g \circ f(7)$, (d) $g \circ g(7)$.

$$\text{Sol. (a) } f \circ f(x) = f[f(x)] = f(x+7) = x+7+7 = x+14$$

$$\therefore f \circ f(7) = 7+14 = 21 \text{ Ans.}$$

$$(b) f \circ g(x) = f[g(x)] = f(x-7) = x-7+7 = x$$

$$\therefore f \circ g(7) = 7 \text{ Ans.}$$

$$(c) g \circ f(x) = g[f(x)] = f(x+7) = x+7-7 = x$$

$$\therefore g \circ f(7) = 7 \text{ Ans.}$$

$$(d) g \circ g(x) = g[g(x)] = f(x-7) = x-7-7 = x-14$$

$$\therefore g \circ g(7) = 7-14 = -7 \text{ Ans.}$$

V.V.3

Let $f: \mathbb{R} \rightarrow \left\{ \frac{2}{3} \right\} \rightarrow \mathbb{R}$

defined by $f(x) = \frac{4x+3}{6x-4}$ Show that $f \circ f(x) = x$

Soln. : $f \circ f(x) = f[f(x)] = f\left(\frac{4x+3}{6x-4}\right)$

$$= \frac{4 \times \frac{4x+3}{6x-4} + 3}{6 \times \frac{4x+3}{6x-4} - 4}$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x \quad \text{Proved.}$$

If $f(x) = \frac{3x+4}{5x-7}$; $x \neq \frac{7}{5}$ and $g(x) = \frac{7x+4}{5x-3}$; $x \neq \frac{3}{5}$

V.V.4 then show that $f \circ g(x) = x$

Soln. : $f \circ g(x) = f[g(x)] = \left(\frac{7x+4}{5x-3} \right)$

$$= \frac{3 \left(\frac{7x+4}{5x-3} \right) + 4}{5 \left(\frac{7x+4}{5x-3} \right) - 7} = \frac{41x}{41} = x.$$

If $f: R \rightarrow R$ be defined by $f(x) = (3-x^3)^{1/3}$, find $f \circ f(x)$.

V.V.4 Soln. : $f \circ f(x) = f[f(x)] = f[(3-x^3)^{1/3}]$

$= f(y)$, where $y = (3-x^3)^{1/3}$

$= (3-y)^{1/3} = [3-(3-x^3)]^{1/3}$

$= (x^3)^{1/3} = x^1 = x$.

Let $f(x) = x^2 - 11$ and $g(x) = |x|$; $x \in \mathbb{R}$. Find the following :

(i) $f \circ g(-3)$ (ii) $g \circ f(-3)$

(iii) $f \circ f(-3)$ (iv) $g \circ g(-3)$

V.V.4 Soln. : (i) $f \circ g(-3) = f\{g(-3)\} = f(|-3|) = f(3) = 9 - 11 = -2$

(ii) $g \circ f(-3) = g\{f(-3)\} = g(9 - 11) = g(-2) = |-2| = 2$

(iii) $f \circ f(-3) = f\{f(-3)\} = f(9 - 11) = f(-2) = 4 - 11 = -7$

(iv) $g \circ g(-3) = g\{g(-3)\} = g(3) = |3| = 3$

If $f(x) = \frac{\sin x - \cos x}{\sin x + \cos x}$ find $f(\pi/3)$

Soln. : $f(x) = \frac{\sin x - \cos x}{\sin x + \cos x}$

$$\therefore f(\pi/3) = f(60^\circ) = \frac{\sin 60^\circ - \cos 60^\circ}{\sin 60^\circ + \cos 60^\circ}$$

$$= \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{\sqrt{3}}{2} + \frac{1}{2}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1}{2+\sqrt{3}}$$

Prove that $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$.

OR

Solve the equation $\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}\frac{2}{3}$, $-\sqrt{3} > x > \sqrt{3}$.

Prove that $\cot^{-1} \left\{ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right\} = \frac{x}{2}$, $0 < x < \frac{\pi}{2}$

OR

Solve the equation for x if $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$, $x > 0$

4 Prove that

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$$

Solution We have

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} \quad (\text{since } \cot^{-1}x = \tan^{-1}\frac{1}{x}, \text{ if } x > 0)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}}\right) + \tan^{-1}\frac{1}{18} \quad (\text{since } x \cdot y = \frac{1}{7} \cdot \frac{1}{8} < 1)$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}}\right) \quad (\text{since } xy < 1)$$

$$= \tan^{-1}\frac{65}{195} = \tan^{-1}\frac{1}{3} = \cot^{-1}3$$

Prove that $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$.

OR

Solve the equation $\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}\frac{2}{3}$, $-\sqrt{3} > x > \sqrt{3}$.

Prove that $\cot^{-1} \left\{ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right\} = \frac{x}{2}$, $0 < x < \frac{\pi}{2}$

OR

Solve the equation for x if $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$, $x > 0$

4 Prove that

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$$

Solution We have

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} \quad (\text{since } \cot^{-1}x = \tan^{-1}\frac{1}{x}, \text{ if } x > 0)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}}\right) + \tan^{-1}\frac{1}{18} \quad (\text{since } x \cdot y = \frac{1}{7} \cdot \frac{1}{8} < 1)$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}}\right) \quad (\text{since } xy < 1)$$

$$= \tan^{-1}\frac{65}{195} = \tan^{-1}\frac{1}{3} = \cot^{-1}3$$

Show that

$$2 \tan^{-1} \left\{ \tan \frac{\alpha}{2} \cdot \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right\} = \tan^{-1} \frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta}$$

Solution L.H.S. = $\tan^{-1} \frac{2 \tan \frac{\alpha}{2} \cdot \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}$ (since $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$)

$$\begin{aligned} &= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \cdot \frac{1 - \tan \frac{\beta}{2}}{1 + \tan \frac{\beta}{2}}}{1 - \tan^2 \frac{\alpha}{2} \left(\frac{1 - \tan \frac{\beta}{2}}{1 + \tan \frac{\beta}{2}} \right)^2} \\ &= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \cdot \left(1 - \tan^2 \frac{\beta}{2} \right)}{\left(1 + \tan^2 \frac{\beta}{2} \right)^2 - \tan^2 \frac{\alpha}{2} \left(1 - \tan^2 \frac{\beta}{2} \right)} \\ &= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \left(1 - \tan^2 \frac{\beta}{2} \right)}{\left(1 + \tan^2 \frac{\beta}{2} \right) \left(1 - \tan^2 \frac{\alpha}{2} \right) + 2 \tan \frac{\beta}{2} \left(1 + \tan^2 \frac{\alpha}{2} \right)} \\ &= \tan^{-1} \frac{\frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \frac{1 - \tan^2 \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}}{\frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} + \frac{2 \tan \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}} \\ &= \tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta} \right) = \text{R.H.S.} \end{aligned}$$

Q. Prove that, $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Soln. : L. H. S. = $\left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} \right) + \left(\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} \right)$

$$= \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{5}}{1 - \frac{1}{3} \times \frac{1}{5}} \right) + \tan^{-1} \left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}} \right)$$

$$= \tan^{-1} \left(\frac{\frac{8}{15}}{\frac{14}{15}} \right) + \tan^{-1} \left(\frac{\frac{15}{56}}{\frac{55}{56}} \right)$$

$$= \tan^{-1} \frac{8}{14} + \tan^{-1} \frac{15}{55} = \tan^{-1} \frac{4}{7} + \tan^{-1} \frac{3}{11}$$

$$= \tan^{-1} \left(\frac{\frac{4}{7} + \frac{3}{11}}{1 - \frac{4}{7} \times \frac{3}{11}} \right) = \tan^{-1} \left(\frac{\frac{65}{77}}{\frac{65}{77}} \right)$$

$$= \tan^{-1} (1) = \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4} = \text{R.H.S.}$$

Q. If $\tan^{-1} \left(\frac{x-1}{x-2} \right) + \tan^{-1} \left(\frac{x+1}{x+2} \right) = \frac{\pi}{4}$

then find the value of x.

Soln. : We have

$$\tan^{-1} \left(\frac{x-1}{x-2} \right) + \tan^{-1} \left(\frac{x+1}{x+2} \right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{\left\{ \frac{x-1}{x-2} + \frac{x+1}{x+2} \right\}}{\left\{ 1 - \frac{(x-1)}{(x-2)} \cdot \frac{(x+1)}{(x+2)} \right\}} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{(x-1)(x+2) + (x+1)(x-2)}{(x^2 - 4) - (x^2 - 1)} \right\} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{(x^2 + x - 2) + (x^2 - x - 2)}{-3} \right\} = \frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2 - 4}{-3} = \tan \frac{\pi}{4} = 1$$

$$\Rightarrow 2x^2 - 4 = -3 \Rightarrow 2x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}}$$

~~Q.~~ Solve : $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$

Soln. : $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$

$$\Rightarrow \sin^{-1}2x = \frac{\pi}{3} - \sin^{-1}x$$

$$\Rightarrow \sin(\sin^{-1}2x) = \sin(\pi/3 - \sin^{-1}x)$$

$$\Rightarrow 2x = \sin \frac{\pi}{3} \cos(\sin^{-1}x) - \cos \frac{\pi}{3} \sin(\sin^{-1}x)$$

$$\Rightarrow 2x = \frac{\sqrt{3}}{2} \cos(\sin^{-1}x) - \frac{1}{2} x \quad [\because \sin(\sin^{-1}x) = x]$$

$$\Rightarrow 2x + \frac{x}{2} = \frac{\sqrt{3}}{2} \cos(\sin^{-1}x)$$

$$\Rightarrow \frac{5x}{2} = \frac{\sqrt{3}}{2} \cos(\sin^{-1}x) \quad \dots (i)$$

$$\text{let } \sin^{-1}x = \theta \quad \therefore x = \sin\theta$$

$$\cos\theta = \sqrt{1 - \sin^{-1}x} \Rightarrow \theta = \sqrt{1 - x^2}$$

$$\Rightarrow \cos(\sin^{-1}x) = \sqrt{1 - x^2}$$

$$\text{From (i), } \frac{5x}{2} = \frac{\sqrt{3}}{2} \sqrt{1 - x^2}$$

$$\Rightarrow 5x = \sqrt{3} \sqrt{1 - x^2}$$

$$25x^2 = 3(1 - x^2), \text{ squaring both sides}$$

$$28x^2 = 3$$

$$x = \sqrt{\frac{3}{28}} \text{ Ans.}$$

~~Q.~~ Prove that— $\tan^{-1}\sqrt{x} = \frac{1}{2} \cos^{-1}\left(\frac{1-x}{1+x}\right)$.

Sol. Putting, $x = \tan^2\theta$

$$\text{We get, RHS} = \frac{1}{2} \cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{1}{2} \cos^{-1}(\cos 2\theta)$$

$$= \left(\frac{1}{2} \times 2\theta\right) = \theta = \tan^{-1}\sqrt{x} = \text{LHS}$$

 Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Ans

12. Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Using $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Using $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = 2 \begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

Using $C_1 \rightarrow C_1 + C_2 + C_3$ and taking (-1) common from both C_2 and C_3

$$\Delta = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

~~Ques.~~ Solve for x , $\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$

LHS

13. Given, $\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$

Using $R_2 \rightarrow R_2 - R_1$, we get $\begin{vmatrix} x+2 & x+6 & x-1 \\ 4 & -7 & 3 \\ -3 & -4 & 7 \end{vmatrix} = 0$ 1½

Using $C_2 \rightarrow C_2 - C_1$, we get $\begin{vmatrix} x+2 & 4 & -3 \\ 4 & -11 & -1 \\ -3 & -1 & 10 \end{vmatrix} = 0$ 1½

Therefore, $(x+2)(-111) - 4(37) - 3(-37) = 0$

which on solving gives $x = -\frac{7}{3}$

1

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & -3 \end{pmatrix}$, verify that $(AB)' = B' A'$.

$\therefore AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 7 & 3 & -4 \\ 15 & 5 & -6 \end{pmatrix}$

Therefore, $LHS = (AB)' = \begin{pmatrix} 7 & 15 \\ 3 & 5 \\ -4 & -6 \end{pmatrix}$

$RHS = B' A' = \begin{pmatrix} 1 & 3 \\ -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 3 & 5 \\ -4 & -6 \end{pmatrix}$ and hence $LHS = RHS$

(i) $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

(ii) $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we have:

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)^3 \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3 , we have:

$$\Delta = 2(x+y+z)^3 (1)(1-0) = 2(x+y+z)^3$$

Hence, the given result is proved.

✓

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \\ &= (1+x+x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \end{aligned}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$\begin{aligned} \Delta &= (1+x+x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix} \\ &= (1+x+x^2)(1-x)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix} \\ &= (1-x^3)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix} \end{aligned}$$

Expanding along R_1 , we have:

$$\begin{aligned} \Delta &= (1-x^3)(1-x)(1) \begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix} \\ &= (1-x^3)(1-x)(1+x+x^2) \\ &= (1-x^3)(1-x^3) \\ &= (1-x^3)^2 \end{aligned}$$

Hence, the given result is proved.

By using properties of determinants, show that

$$\begin{vmatrix} a^2 + 1 & ab & ac \\ ab & b^2 + 1 & bc \\ ca & cb & c^2 + 1 \end{vmatrix} = (1 + a^2 + b^2 + c^2)$$

Sol. LHS $= \Delta = \begin{vmatrix} a^2 + 1 & ab & ac \\ ab & b^2 + 1 & bc \\ ca & cb & c^2 + 1 \end{vmatrix}$

$$\Rightarrow \Delta = \begin{vmatrix} a\left(a + \frac{1}{a}\right) & ab & ac \\ ab & b\left(b + \frac{1}{b}\right) & bc \\ ca & cb & c\left(c + \frac{1}{c}\right) \end{vmatrix}$$

Taking a, b, c common from C₁, C₂, C₃ respectively.

$$\Delta = (abc) \begin{vmatrix} a + \frac{1}{a} & a & a \\ b & b + \frac{1}{b} & b \\ c & c & c + \frac{1}{c} \end{vmatrix}$$

Applying R₁ $\rightarrow aR_1$, R₂ $\rightarrow bR_2$, R₃ $\rightarrow cR_3$ and dividing whole determinant by abc.

$$\Delta = \frac{(abc)}{(abc)} \cdot \begin{vmatrix} a^2 + 1 & a^2 & a^2 \\ b^2 & b^2 + 1 & b^2 \\ c^2 & c^2 & c^2 + 1 \end{vmatrix}$$

Applying R₁ $\rightarrow R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 1 + a^2 + b^2 + c^2 & 1 + a^2 + b^2 + c^2 & 1 + a^2 + b^2 + c^2 \\ b^2 & b^2 + 1 & b^2 \\ c^2 & c^2 & c^2 + 1 \end{vmatrix}$$

Taking out (1 + a² + b² + c²) common from R₁

$$\Delta = (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2 + 1 & b^2 \\ c^2 & c^2 & c^2 + 1 \end{vmatrix}$$

Applying C₂ $\rightarrow C_2 - C_1$ and C₃ $\rightarrow C_3 - C_1$

$$\Delta = (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$$\Delta = (1 + a^2 + b^2 + c^2) \cdot 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\Delta = (1 + a^2 + b^2 + c^2) \cdot 1 \cdot (1 - 0) = (1 + a^2 + b^2 + c^2) = \text{RHS.}$$

By using properties of determinants, show that

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Sol. L.H.S. = $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$,

$$\Delta = \begin{vmatrix} 2(x+y+2z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Delta = 2(x+y+z) \begin{vmatrix} 0 & -(x+y+z) & y-z \\ 0 & x+y+z & -(x+2y) \\ 1 & x & z+x+2y \end{vmatrix}$$

$$\Delta = 2(x+y+z) \begin{vmatrix} -(x+y+z) & (y-z) \\ x+y+z & -(x+2y) \end{vmatrix}$$

$$\Delta = 2(x+y+z)^2 \begin{vmatrix} -1 & (y-z) \\ 1 & -(x+2y) \end{vmatrix}$$

$$\Delta = 2(x+y+z)^2 [(x+2y) - (y-z)]$$

$$\Delta = 2(x+y+z)^2 (x+y+z)$$

$$\Delta = 2(x+y+z)^3 = \text{R.H.S.}$$

V.V. ~~Ex~~

Prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Soln. : Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad (\text{Taking } a, b, c \text{ common from } R_1, R_2 \text{ and } R_3 \text{ respectively.})$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad (\text{Taking } a, b, c \text{ common from } C_1, C_2 \text{ and } C_3 \text{ respectively.})$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} \quad (C_2 \rightarrow C_2 + C_1; C_3 \rightarrow C_3 + C_1)$$

$$= a^2b^2c^2 \times (-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} \quad (\text{Expanding along } R_1)$$

$$= a^2b^2c^2 \cdot (-1) \cdot (0 - 4) = 4a^2b^2c^2 \quad \text{Proved.}$$

Q. V. I. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

Soln. : Let, $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

(4) Taking a common from C_1 we get,

$$= \begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix} [C_2 \rightarrow C_2 - bC_1; C_3 \rightarrow C_3 - cC_1]$$

Taking a common from C_2 we get

$$= a^2 \begin{vmatrix} 1 & 1 & a+b \\ 2 & 3 & 4a+3b \\ 3 & 6 & 10a+6b \end{vmatrix} [C_3 \rightarrow C_3 - bC_2]$$

Taking a common from C_3 we get

$$= a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} [C_2 \rightarrow C_2 - C_1; C_3 \rightarrow C_3 - C_1]$$

Expanding along R_1 we get

$a^3 \times 1 = a^3$ Proved.

~~V.V~~ Show that the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation

$$A^3 - A^2 - 3A - I = 0. \text{ Hence find } A^{-1}.$$

Q

Soln. : $A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \times 1 + 0 \times (-2) + (-2) \times 3 & 1 \times 0 + 0 \times (-1) + (-2) \times 4 & 1 \times (-2) + 0 \times 2 + (-2) \times 1 \\ -2 \times 1 + (-1) \times (-2) + 2 \times 3 & -2 \times 0 + (-1) \times (-1) + 2 \times 4 & (-2) \times (-2) + (-1) \times 2 + 2 \times 1 \\ 3 \times 1 + 4 \times (-2) + 1 \times 3 & 3 \times 0 + 4 \times (-1) + 1 \times 4 & 3 \times (-2) + 4 \times 2 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0 - 6 & 0 + 0 - 8 & -2 + 0 - 2 \\ -2 + 2 + 6 & 0 + 1 + 8 & 4 - 2 + 2 \\ 3 - 8 + 3 & 0 - 4 + 4 & -6 + 8 + 1 \end{bmatrix} = -5 \begin{bmatrix} -8 & -4 \\ 9 & 4 \\ -2 & 0 & 3 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \times 1 + (-8)(-2) + (-4) \times 3 & -5 \times 0 + (-8)(-1) + (-4) \times 4 & (-5)(-2) + (-8) \times 2 + (-4) \times 1 \\ 6 \times 1 + 9(-2) + 4 \times 3 & 6 \times 0 + 9(-1) + 4 \times 4 & 6 \times (-2) + 9 \times 2 + 4 \times 1 \\ -2 \times 1 + 0 \times (-2) + 3 \times 3 & -2 \times 0 + 0 \times (-1) + 3 \times 4 & (-2)(-2) + 0 \times 2 + 3 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 + 16 - 12 & 0 + 8 - 16 & 10 - 16 - 4 \\ 6 - 18 + 12 & 0 - 9 + 16 & -12 + 18 + 4 \\ -2 + 0 + 9 & 0 + 0 + 12 & 4 + 0 + 3 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & 10 \\ 7 & 12 & 7 \end{bmatrix}$$

$$\therefore A^3 - A^2 - 3A - I = \begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & 10 \\ 7 & 12 & 7 \end{bmatrix} - \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -6 \\ -6 & -3 & 6 \\ 9 & 12 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore A^3 - A^2 - 3A - I = 0$$

$$\Rightarrow A^{-1} (A^3 - A^2 - 3A - I) = 0$$

$$\Rightarrow A^2 - A - 3I - A^{-1} = 0$$

$$\therefore A^{-1} = A^2 - A - 3I = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -8 & -2 \\ 8 & 10 & 2 \\ -5 & -4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

16. $f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$

Answer

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2) = -2$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < -1$

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (-2) = -2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2x) = 2 \times (-1) = -2$$

$$\therefore \lim_{x \rightarrow -1} f(x) = f(-1)$$

Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x) = 2c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(-1, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x) = 2 \times 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2 = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(c)$$

Therefore, f is continuous at $x = 1$

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2) = 2$

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

Answer

The given function is $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$

If f is continuous at $x = 0$, then

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \lambda(x^2 - 2x) = \lim_{x \rightarrow 0^+} (4x + 1) = \lambda(0^2 - 2 \times 0)$$

$$\Rightarrow \lambda(0^2 - 2 \times 0) = 4 \times 0 + 1 = 0$$

$$\Rightarrow 0 = 1 = 0, \text{ which is not possible}$$

Therefore, there is no value of λ for which f is continuous at $x = 0$

At $x = 1$,

$$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$$

$$\lim_{x \rightarrow 1} (4x + 1) = 4 \times 1 + 1 = 5$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, for any values of λ , f is continuous at $x = 1$

23. Find all points of discontinuity of f , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = \frac{\sin c}{c} \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(\frac{\sin x}{x} \right) = \frac{\sin c}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } f(c) = c + 1 \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 1) = c + 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 0$

Case III:

$$\text{If } c = 0, \text{ then } f(c) = f(0) = 0 + 1 = 1$$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 1) = 1$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Answer

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c) = c^2 \sin \frac{1}{c}$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(x^2 \sin \frac{1}{x} \right) = \left(\lim_{x \rightarrow c} x^2 \right) \left(\lim_{x \rightarrow c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

$$\text{If } c = 0, \text{ then } f(0) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right)$$

$$\text{It is known that, } -1 \leq \sin \frac{1}{x} \leq 1, \quad x \neq 0$$

$$\Rightarrow -x^2 \leq \sin \frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(-x^2 \right) \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq \lim_{x \rightarrow 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = 0$$

$$\text{Similarly, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

$$26. f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$$

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f

at $x = \frac{\pi}{2}$ equals the limit of f at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

$$\text{Put } x = \frac{\pi}{2} + h$$

$$\text{Then, } x \rightarrow \frac{\pi}{2} \Rightarrow h \rightarrow 0$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{2}} f(x) &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} \\ &= k \lim_{h \rightarrow 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2} \end{aligned}$$

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the required value of k is 6.

$$28. \quad f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

Answer

$$\text{The given function is } f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

The given function f is continuous at $x = p$, if f is defined at $x = p$ and if the value of f at $x = p$ equals the limit of f at $x = p$

It is evident that f is defined at $x = p$ and $f(\pi) = k\pi + 1$

$$\begin{aligned} \lim_{x \rightarrow \pi^-} f(x) &= \lim_{x \rightarrow \pi^+} f(x) = f(\pi) \\ \Rightarrow \lim_{x \rightarrow \pi^-} (kx+1) &= \lim_{x \rightarrow \pi^+} \cos x = k\pi + 1 \end{aligned}$$

$$\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

$$29. \quad f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

Answer

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of f at $x = 5$ equals the limit of f at $x = 5$

It is evident that f is defined at $x = 5$ and $f(5) = kx+1 = 5k+1$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x \rightarrow 5^-} (kx+1) = \lim_{x \rightarrow 5^+} (3x-5) = 5k+1$$

$$\Rightarrow 5k+1 = 15-5 = 5k+1$$

$$\Rightarrow 5k+1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Ques. 1. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in $[-4, 2]$ and is differentiable in $(-4, 2)$.

$$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$$

$$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$$

$$\therefore f(-4) = f(2) = 0$$

⇒ The value of $f(x)$ at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$

$$\Rightarrow f'(x) = 2x + 2$$

$$\therefore f'(c) = 0$$

$$\Rightarrow 2c + 2 = 0$$

$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

Ques. 4. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer

The given function is $f(x) = x^2 - 4x - 3$

f , being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$$f(1) = 1^2 - 4 \times 1 - 3 = -6, f(4) = 4^2 - 4 \times 4 - 3 = -3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$$f'(c) = 1$$

$$\Rightarrow 2c - 4 = 1$$

$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)$$

Hence, Mean Value Theorem is verified for the given function.

✓ 5. Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Answer

The given function f is $f(x) = x^3 - 5x^2 - 3x$

f , being a polynomial function, is continuous in $[1, 3]$ and is differentiable in $(1, 3)$ whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7, f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 - (-7)}{3 - 1} = -10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$$f'(c) = -10$$

$$\Rightarrow 3c^2 - 10c - 3 = 10$$

$$\Rightarrow 3c^2 - 10c + 7 = 0$$

$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$

$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$

$$\Rightarrow (c-1)(3c-7) = 0$$

$$\Rightarrow c = 1, \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1, 3)$ is the only point for which $f'(c) = 0$

Find
 A^{-1} 9.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

6 marks

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

We have,

$$|A| = 1(10 - 0) - 2(0 - 0) + 3(0 - 0) = 10$$

Now,

$$A_{11} = 10 - 0 = 10, A_{12} = -(0 - 0) = 0, A_{13} = 0 - 0 = 0$$

$$A_{21} = -(10 - 0) = -10, A_{22} = 5 - 0 = 5, A_{23} = -(0 - 0) = 0$$

$$A_{31} = 8 - 6 = 2, A_{32} = -(4 - 0) = -4, A_{33} = 2 - 0 = 2$$

$$\therefore adj A = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adj A = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

6

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Show that $A^3 - 6A^2 + 5A + 11I = O$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, $A^3 - 6A^2 + 5A + 11I = O$.

Now,

$$A^3 - 6A^2 + 5A + 11I = O$$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0 \quad [\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0]$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(IA^{-1})$$

$$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I) \quad \dots (1)$$

Now,

$$A^2 - 6A + 5I$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6 \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 12 & -6 & 18 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} . Using A^{-1} solve the system of equations

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Answer

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\therefore |A| = 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1 \neq 0$$

$$\text{Now, } A_{11} = 0, A_{12} = 2, A_{13} = 1$$

$$A_{21} = -1, A_{22} = -9, A_{23} = -5$$

$$A_{31} = 2, A_{32} = 23, A_{33} = 13$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj } A) = -\frac{1}{1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \quad \dots(1)$$

Now, the given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}.$$

The solution of the system of equations is given by $X = A^{-1}B$.

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \quad [\text{Using (1)}]$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

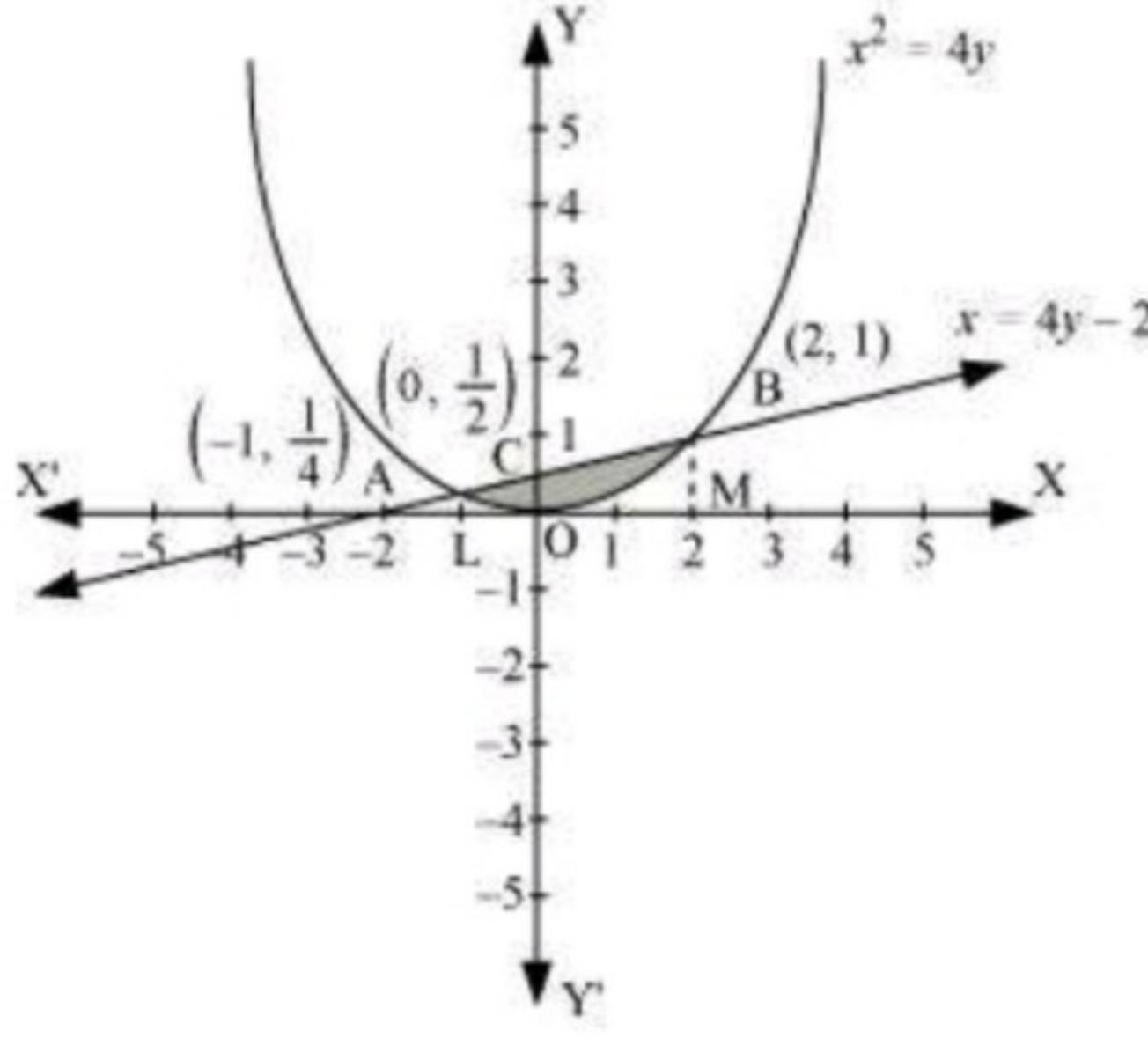
$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2, \text{ and } z = 3$.

Q. 10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Answer

The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area OBAO.



Let A and B be the points of intersection of the line and parabola.

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$

Coordinates of point B are $(2, 1)$.

We draw AL and BM perpendicular to x-axis.

It can be observed that,

$$\text{Area OBAO} = \text{Area OBCO} + \text{Area OACO} \dots (1)$$

$$\text{Then, Area OBCO} = \text{Area OMBC} - \text{Area OMBO}$$

$$\begin{aligned} &= \int_0^2 \frac{x+2}{4} dx - \int_0^2 \frac{x^2}{4} dx \\ &= \frac{1}{4} \left[\frac{x^2}{2} + 2x \right]_0^2 - \frac{1}{4} \left[\frac{x^3}{3} \right]_0^2 \\ &= \frac{1}{4} [2+4] - \frac{1}{4} \left[\frac{8}{3} \right] \\ &= \frac{3}{2} - \frac{2}{3} \\ &= \frac{5}{6} \end{aligned}$$

$$\text{Similarly, Area OACO} = \text{Area OLAC} - \text{Area OLAO}$$

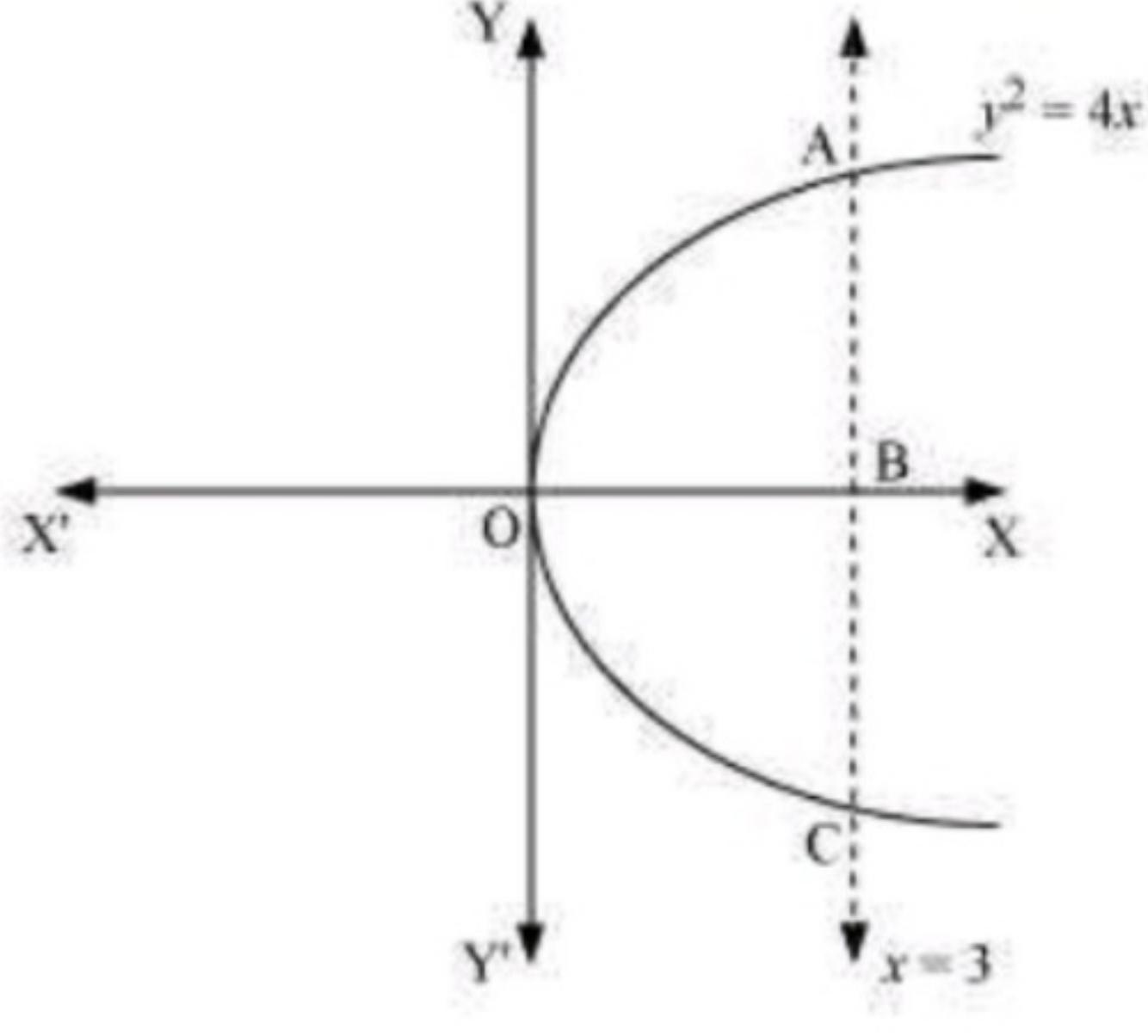
$$\begin{aligned} &= \int_{-1}^0 \frac{x+2}{4} dx - \int_{-1}^0 \frac{x^2}{4} dx \\ &= \frac{1}{4} \left[\frac{x^2}{2} + 2x \right]_{-1}^0 - \frac{1}{4} \left[\frac{x^3}{3} \right]_{-1}^0 \\ &= -\frac{1}{4} \left[\frac{(-1)^2}{2} + 2(-1) \right] - \left[-\frac{1}{4} \left(\frac{(-1)^3}{3} \right) \right] \\ &= -\frac{1}{4} \left[\frac{1}{2} - 2 \right] - \frac{1}{12} \\ &= \frac{1}{2} - \frac{1}{8} - \frac{1}{12} \\ &= \frac{7}{24} \end{aligned}$$

$$\text{Therefore, required area} = \left(\frac{5}{6} + \frac{7}{24} \right) = \frac{9}{8} \text{ units}$$

11 ✓ Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Answer

The region bounded by the parabola, $y^2 = 4x$, and the line, $x = 3$, is the area OACO.



The area OACO is symmetrical about x -axis.

$$\therefore \text{Area of OACO} = 2 \text{ (Area of OAB)}$$

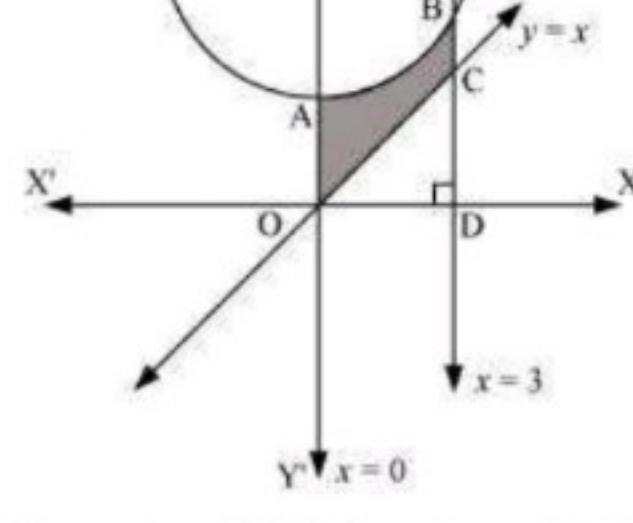
$$\begin{aligned}\text{Area OACO} &= 2 \left[\int_0^3 y \, dx \right] \\ &= 2 \int_0^3 2\sqrt{x} \, dx \\ &= 4 \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^3 \\ &= \frac{8}{3} \left[(3)^{\frac{3}{2}} \right] \\ &= 8\sqrt{3}\end{aligned}$$

Therefore, the required area is $8\sqrt{3}$ units.

3 ✓ Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

Answer

The area bounded by the curves, $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 3$, is represented by the shaded area OCBAO as



$$\text{Then, Area OCBAO} = \text{Area ODBAO} - \text{Area ODCO}$$

$$\begin{aligned}&= \int_0^3 (x^2 + 2) \, dx - \int_0^3 x \, dx \\ &= \left[\frac{x^3}{3} + 2x \right]_0^3 - \left[\frac{x^2}{2} \right]_0^3 \\ &= [9 + 6] - \left[\frac{9}{2} \right] \\ &= 15 - \frac{9}{2} \\ &= \frac{21}{2} \text{ units}\end{aligned}$$

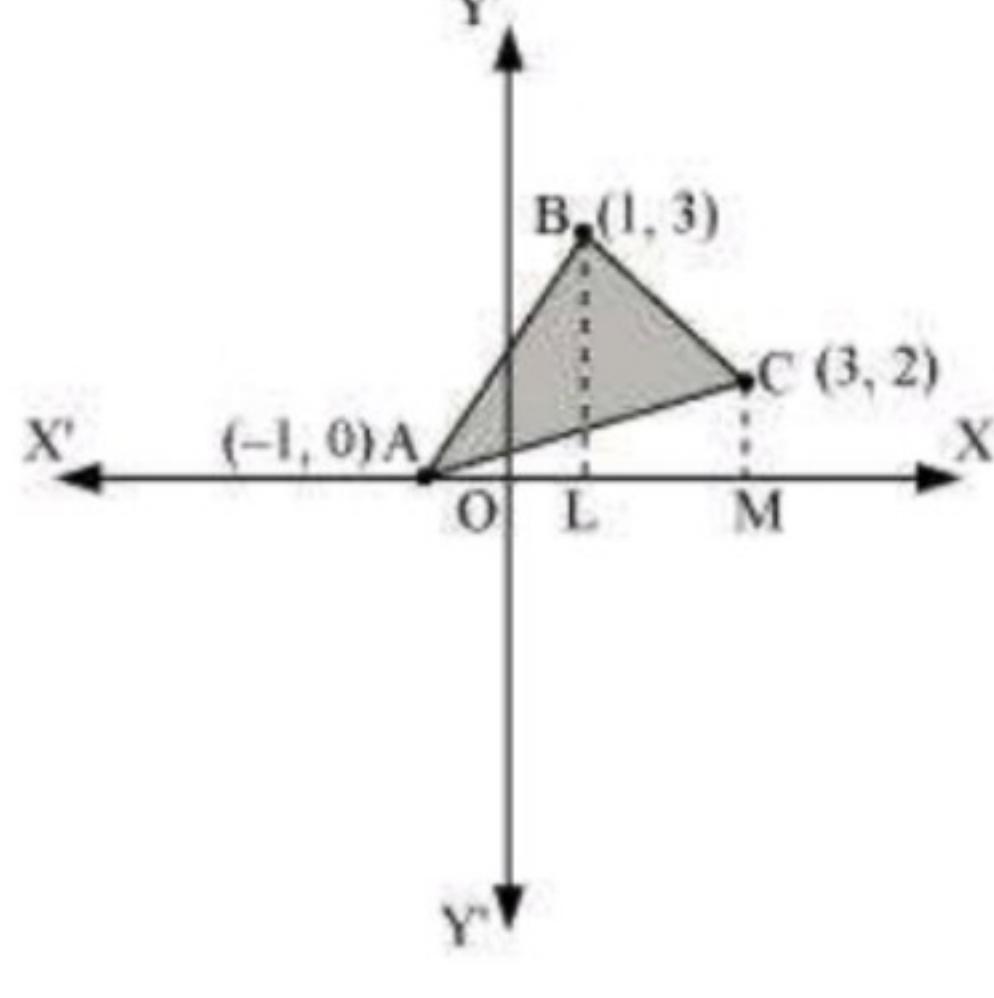
6 ✓ Using integration find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.

Answer

BL and CM are drawn perpendicular to x -axis.

It can be observed in the following figure that,

$$\text{Area } (\Delta ABC) = \text{Area } (ALBA) + \text{Area } (BLMCB) - \text{Area } (AMCA) \dots (1)$$



Equation of line segment AB is

$$y - 0 = \frac{3-0}{1+1}(x+1)$$

$$y = \frac{3}{2}(x+1)$$

$$\therefore \text{Area } (ALBA) = \int_{-1}^1 \frac{3}{2}(x+1) dx = \frac{3}{2} \left[\frac{x^2}{2} + x \right]_{-1}^1 = \frac{3}{2} \left[\frac{1}{2} + 1 - \frac{1}{2} - 1 \right] = 3 \text{ units}$$

Equation of line segment BC is

$$y - 3 = \frac{2-3}{3-1}(x-1)$$

$$y = \frac{1}{2}(-x+7)$$

$$\therefore \text{Area } (BLMCB) = \int_1^3 \frac{1}{2}(-x+7) dx = \frac{1}{2} \left[-\frac{x^2}{2} + 7x \right]_1^3 = \frac{1}{2} \left[-\frac{9}{2} + 21 + \frac{1}{2} - 7 \right] = 5 \text{ units}$$

Equation of line segment AC is

$$y - 0 = \frac{2-0}{3+1}(x+1)$$

$$y = \frac{1}{2}(x+1)$$

$$\therefore \text{Area } (AMCA) = \frac{1}{2} \int_{-1}^3 (x+1) dx = \frac{1}{2} \left[\frac{x^2}{2} + x \right]_{-1}^3 = \frac{1}{2} \left[\frac{9}{2} + 3 - \frac{1}{2} - 1 \right] = 4 \text{ units}$$

Therefore, from equation (1), we obtain

$$\text{Area } (\Delta ABC) = (3 + 5 - 4) = 4 \text{ units}$$

Question 5:

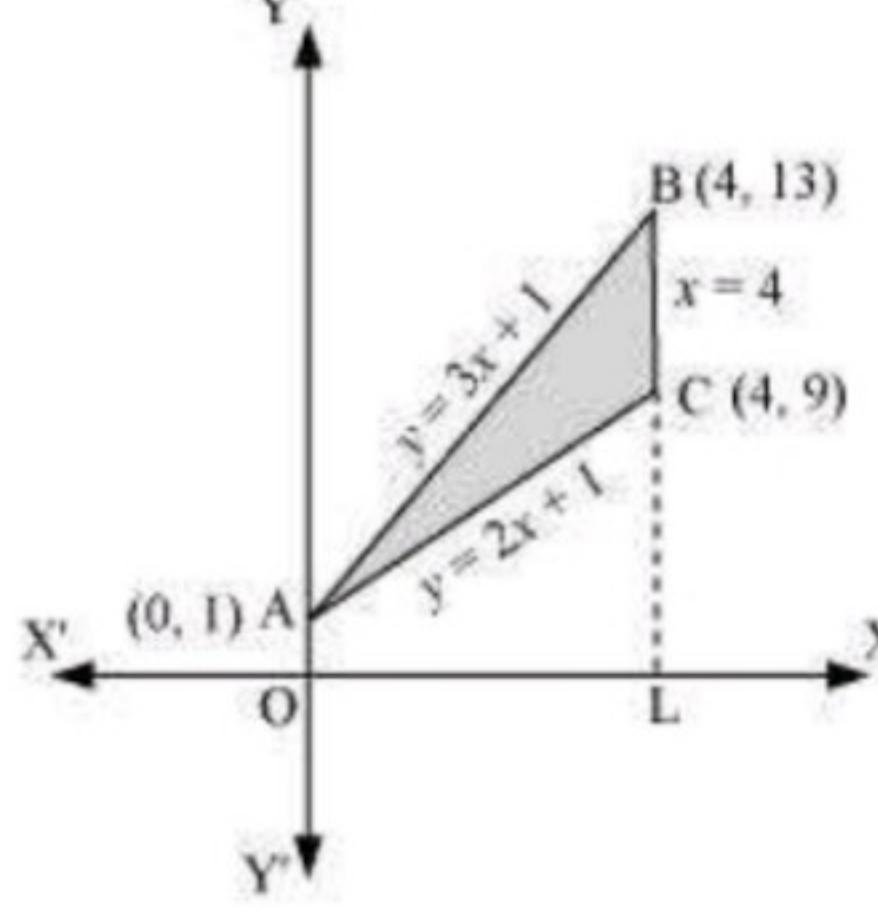
cont.

- 6✓ Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Answer

The equations of sides of the triangle are $y = 2x + 1$, $y = 3x + 1$, and $x = 4$.

On solving these equations, we obtain the vertices of triangle as A(0, 1), B(4, 13), and C(4, 9).



It can be observed that,

$$\text{Area } (\Delta ACB) = \text{Area } (\text{OLBAO}) - \text{Area } (\text{OLCAO})$$

$$\begin{aligned}
 &= \int_0^4 (3x+1) dx - \int_0^4 (2x+1) dx \\
 &= \left[\frac{3x^2}{2} + x \right]_0^4 - \left[\frac{2x^2}{2} + x \right]_0^4 \\
 &= (24+4) - (16+4) \\
 &= 28 - 20 \\
 &= 8 \text{ units}
 \end{aligned}$$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx \dots (1)$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^7 x$ is an odd function.

It is known that, if $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx$ is

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot dx$

It is known that if $f(x)$ is an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

if $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$I = 0 + 0 + 0 + 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx$

$= 2[x]_0^{\frac{\pi}{2}}$

$= \frac{2\pi}{2}$

$$25. \quad \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$$

$$\begin{aligned} I &= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx \\ &= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx \\ &= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\operatorname{cosec}^2 \frac{x}{2}}{2} - \cot \frac{x}{2} \right) dx \end{aligned}$$

$$\text{Let } f(x) = -\cot \frac{x}{2}$$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}\right) = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(f(x) + f'(x) \right) dx$$

$$= \left[e^x \cdot f(x) \right]_{\frac{\pi}{2}}^{\pi}$$

$$= - \left[e^x \cdot \cot \frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$= - \left[e^{\pi} \times \cot \frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot \frac{\pi}{4} \right]$$

$$= - \left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1 \right]$$

$$= e^{\frac{\pi}{2}}$$

18 Evaluate—

$$\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x}{\cos x + \sin x} dx$$

$$\text{Sol. Let } I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x}{\cos x + \sin x} dx = \frac{1}{2} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x + \sin x + \cos x - \sin x}{\cos x + \sin x} dx$$

$$= \frac{1}{2} \left[\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x + \sin x}{\cos x + \sin x} dx + \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x - \sin x}{\cos x + \sin x} dx \right] = \frac{1}{2} \left[x \right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} + \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

$$\text{Put, } \cos x + \sin x = z$$

$$\Rightarrow (-\sin x + \cos x) dx = dz$$

$$\Rightarrow (\cos x - \sin x) dx = dz$$

$$\therefore I = \frac{1}{2} \left[\left[x \right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} + \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{dz}{z} \right] = \frac{1}{2} \left[\frac{3\pi}{8} - \frac{\pi}{8} + [\log \cos x + \sin x] \right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$= \frac{1}{2} \left[\frac{3\pi}{8} - \frac{\pi}{8} + \log \cos \frac{3\pi}{8} + \sin \frac{3\pi}{8} - \log \cos \frac{\pi}{8} + \sin \frac{\pi}{8} \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} + \log \frac{\cos \frac{3\pi}{8} + \sin \frac{3\pi}{8}}{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}} \right]$$

Ans.

$$13 \text{ Evaluate } \int_0^\pi \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx$$

$$\text{Sol. Let } I = \int_0^\pi \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx \quad \dots \dots \text{(i)}$$

$$\begin{aligned} &= \int_0^\pi \frac{(\pi - x)}{[a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)]} dx \\ &= \int_0^\pi \frac{(\pi - x)}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx \end{aligned} \quad \dots \dots \text{(ii)}$$

Adding equation (i) and (ii), we get

$$2I = \int_0^\pi \frac{(x + \pi - x)}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx = \pi \int_0^\pi \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)}$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)}$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(a^2 + b^2 \tan^2 x)} \quad [\text{dividing num. and denom. by } \cos^2 x]$$

$$= 2\pi \int_0^{\infty} \frac{dt}{(a^2 + b^2 t^2)} \quad [\text{where } \tan x = t]$$

$$= \frac{2\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a^2}{b^2} + t^2\right)}$$

$$= \left[\frac{2\pi}{b^2} \times \frac{b}{a} \tan^{-1} \left(\frac{bt}{a} \right) \right]_0^\infty = \frac{2\pi}{ab} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{2\pi}{ab} \left(\frac{\pi}{2} - 0 \right)$$

$$2I = \frac{\pi^2}{ab} \quad \text{or,} \quad I = \frac{\pi^2}{2ab}$$

$$\text{Hence } \int_0^\pi \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx = \frac{\pi^2}{2ab}$$

25 Evaluate— $\int \frac{3x-2}{(x+1)^2(x+3)} dx$

Sol. $\frac{3x-2}{(x+1)^2(x+3)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$
 $\Rightarrow 3x-2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$
 $= A(x^2 + 4x + 3) + B(x+3) + C(x^2 + 2x + 3)$

Comparing coefficients of x^2 , x and of constant term of both sides,
we get $A + C = 0$, $4A + B + 2C = 3$ and $3A + 3B + C = -2$.

Solving these equations, we get

$$A = \frac{11}{4}, B = \frac{-5}{2} \text{ and } C = \frac{-11}{4}$$

Thus the integrand is given by

$$\begin{aligned}\frac{3x-2}{(x+1)^2(x+3)} &= \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} + \frac{11}{4(x+3)} \\ \therefore \int \frac{3x-2}{(x+1)^2(x+3)} dx &= \frac{11}{4} \int \frac{dx}{x+1} - \frac{5}{2} \int \frac{dx}{(x+1)^2} - \frac{11}{4} \int \frac{dx}{x+3} \\ &= \frac{11}{4} \log|x+1| + \frac{5}{2(x+1)} - \frac{11}{4} \log|x+3| + C \\ &= \frac{11}{4} \log \left| \frac{x+1}{x+3} \right| + \frac{5}{2(x+1)} + C\end{aligned}$$

16 Evaluate— $\int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx$

Sol. The integrand is a proper rational function. Decompose the rational function into partial fraction.

$$\frac{x^2 + x + 1}{(x+2)(x^2+1)} \equiv \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$\therefore x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2)$$

Equating the coefficients of x^2 , x and of constant term of both sides, we get $A + B = 1$, $2B + C = 1$ and $A + 2C = 1$.

Solving these equations, we get

$$A = \frac{3}{5}, B = \frac{2}{5} \text{ and } C = \frac{1}{5}.$$

Thus the integrand is given by

$$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)} + \frac{1}{5} \left(\frac{2x+1}{x^2+1} \right)$$

$$\begin{aligned} \therefore \int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx &= \frac{3}{5} \int \frac{dx}{x+2} + \frac{1}{5} \int \frac{2x}{x^2+1} dx + \frac{1}{5} \int \frac{1}{x^2+1} dx \\ &= \frac{3}{5} \log|x+2| + \frac{1}{5} \log|x^2+1| + \frac{1}{5} \tan^{-1}x + C \end{aligned}$$

5 Find the following integrals—

(a) $\int e^x \frac{(x-1)}{(x+1)^3} dx$

Sol. $I = \int e^x \frac{(x-1)}{(x+1)^3} dx = \int e^x \frac{x+1-2}{(x+1)^3} dx = \int e^x \left\{ \frac{x+1}{(x+1)^3} - \frac{2}{(x+1)^3} \right\} dx$
 $= \int e^x \left\{ \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right\} dx = \int e^x [f(x) + f'(x)] dx, \text{ where } f(x) = \frac{1}{(x+1)^2}$
 $= e^x \quad f(x) + C = \frac{e^x}{(x+1)^2} + C \text{ Ans.}$

(b) $\int \frac{1}{1+\tan x} dx$

Sol. $\int \frac{dx}{1+\tan x} = \int \frac{\cos x dx}{\cos x + \sin x} = \frac{1}{2} \int \frac{\cos x + \sin x + \cos x - \sin x dx}{\cos x + \sin x}$
 $= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$
 $= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2} \int \frac{\cos x - \sin x}{\cos x + \sin x} dx \quad \dots\dots(i)$

Now, consider $I = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x) dx = dt$

Therefore, $I = \int \frac{dt}{t} \log |t| + C_2 = \log |\cos x + \sin x| + C_2$

Putting it in (i) we get

$$\begin{aligned} \int \frac{dx}{1+\tan x} &= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2} \log |\cos x + \sin x| + \frac{C_2}{2} \\ &= \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2} \\ &= \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| + C, \quad \left(C = \frac{C_1 + C_2}{2} \right) \end{aligned}$$

4 Evaluate— $\int \cos^5 x \, dx$

Sol. $\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx$

$$= \int (1 - \sin^2 x)^2 \cdot \cos x \, dx = \int (1 - t^2)^2 dt, \text{ where } \sin x = t$$

$$= \int (1 + t^4 - 2t^2) dt = \int dt + \int t^4 dt - 2 \int t^2 dt$$

$$= t + \frac{t^5}{5} - \frac{2t^3}{3} + C = \sin x + \frac{1}{5} \sin^5 x - \frac{2}{3} \sin^3 x + C$$

1 Find the integrals of the following-

(a) $\frac{1-\cos x}{1+\cos x}$

Sol.
$$\int \frac{1-\cos x}{1+\cos x} dx = \int \frac{\frac{2\sin^2 \frac{x}{2}}{2}}{\frac{2\cos^2 \frac{x}{2}}{2}} dx = \int \tan^2 \frac{x}{2} dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$
$$= \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + C = 2 \tan \frac{x}{2} - x + C$$

Q. 24. Find the local maxima or local minima of $f(x) = x^3 - 6x^2 + 9x + 15$.
Also, find the local maximum or local minimum values as the case may be.

Soln. : Here, $f(x) = x^3 - 6x^2 + 9x + 15 \Rightarrow f'(x) = 3x^2 - 12x + 9$.

(b) For a local maxima or minima, we must have $f'(x) = 0$.

$$\begin{aligned} \text{Now, } f'(x) &= 0 \Rightarrow 3(x^2 - 4x + 3) = 0 \\ &\Rightarrow 3(x - 3)(x - 1) = 0 \Rightarrow x = 3 \text{ or } x = 1. \end{aligned}$$

Case 1 When $x = 3$

In this case, when x is slightly less than 3 then $f'(x) = 3(x - 3)(x - 1)$ is negative and when x is slightly more than 3 then $f'(x)$ is positive.

Or, If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then show that $A^{-1} = A^2$

V.V.T.

(6)

$$\text{Soln. : } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\therefore |A| = 1(0-0) + 1(0-0) + 1(0+1) = 1 \neq 0$$

$$A_{11} = + \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} = 0, A_{12} = \begin{vmatrix} -2 & 0 \\ 1 & 0 \end{vmatrix} = 0, A_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 0+1=1$$

$$A_{21} = + \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0, A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1, A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = (0+1)=1$$

$$A_{31} = - \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1, A_{32} = - \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -(0-2)=2, A_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1+2=1$$

$$\therefore \text{Adj. } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj. } A = \frac{1}{1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\text{Now, } A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{Hence } A^{-1} = A^2$$