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**Code No. : 20291 E** **Sub. Code : AMMA 41**

B.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Fourth Semester

Mathematics — Core

ABSTRACT ALGEBRA

(For those who joined in July 2020 only)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. The order of the element $-1$ in $(\mathbb{Z}, +)$ is ________

(a) 2

(b) infinite

(c) 1

(d) $-1$

2. If ________ $(ab)^2 = a^2b^2 \forall a, b \in G$ is true then

(a) $G$ is a cyclic group

(b) $G$ is non cyclic group

(c) $G$ is any group

(d) $G$ is finite

3. The set of all generators of the group $(\mathbb{Z}_{12}, \oplus)$ is ________

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 3, 6, 9\}$

(c) $\{1, 5, 7, 11\}$ (d) $\{2, 3, 5, 7, 11\}$

4. In an abelian group $G = \{1, i, -1, -i\}$, $\langle -1 \rangle =$ ________

(a) $\{1, i, -1, -i\}$ (b) $\{i, -i\}$

(c) $\{-1\}$ (d) $\{-1, 1\}$

5. If the map $f : (\mathbb{C}, +) \hookrightarrow (\mathbb{R}, +)$ defined by $f(x + iy) = y \forall x + iy \in \mathbb{C}$ is a homomorphism, then $\ker f =$ ________

(a) $\mathbb{R}$ (b) $\mathbb{R}^+$

(c) $\mathbb{C}$ (d) 0



6. The number of automorphisms of a cyclic group of order 7 is ________.

(a) 6 (b) 7

(c) 49 (d) 1

7. Let $R$ be a ring with identity. They $\forall a, b \in R$, ________

(a) $(a + b)^2 = a^2 + ab + ba + b^2$

(b) $(a - b)^2 = a^2 - 2ab + b^2$

(c) $(a + b)(a - b) = a^2 - b^2$

(d) $(a + b)^2 = a^2 + 2ab + b^2$

8. Which is not a ring?

(a) $(\mathbb{Z}, +, .)$ (b) $(Q, +, .)$

(c) $(\mathbb{R}+, .)$ (d) $(\mathbb{R}, ., +)$

9. Let $f(x), g(x) \in \mathbb{Z}_4[x]$ be defined as $f(x) = x^2 + 3x + 1$ and $g(x) = 2x^2 + x$. Then degree of $f(x)g(x) =$ ________

(a) 4 (b) 3

(c) 2 (d) 1



10. The product of the polynomials $2x + 4$ and $4x^2 + 3x + 1$ in $\mathbb{Z}_5[x]$ is

(a) $8x^3 + 2x^2 + 4x + 4$

(b) $3x^3 + 2x^2 + 4x + 4$

(c) $8x^3 + 22x^2 + 14x + 4$

(d) $3x^3 + 2x^2 + 3x + 4$

PART B — (5 × 5 = 25 marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) Let $G$ be a group and let '$a$' be a fixed element of G. Let $H_a = \{x \in G / ax = xa\}$. Show that $H_a$ is a subgroup of $G$.

Or

(b) Let $G$ be a group in which $(ab)^m = a^m b^m$ for three consecutive integers and for all $a, b \in G$. Show that $G$ is abelian.

12. (a) Let $H$ and $K$ be two finite subgroups of a group $G$. Show that $|HK| = \frac{|H||K|}{|H \cap K|}$.

Or

(b) State and Prove Euler's theorem.

13. (a) Prove that the intersection of $H$ and $K$ two normal subgroups of a group $G$ is a normal subgroup of $G$.

Or

(b) Let $G$ and $G'$ be two groups and $f : G \to G'$ be a homomorphism. Prove that $f$ is $1-1$ if and only if $\ker f = \{e\}$.

14. (a) Prove that any finite integral domain is a field.

Or

(b) Let $R$ be an integral domain. Let $a$ and $b$ be two non-zero elements of $R$. PRove that $a$ and $b$ are associates iff $a = bu$ where $u$ is a unit in $R$.



15. (a) If $f : \mathbb{Z} \to \mathbb{Z}_n$ is defined by $f(x) = r$, $x = qn + r$, $0 \le r < n$, show that $f$ is a homomorphism.

Or

(b) Let $R$ be any ring and $f(x), g(x) \in R[x]$. Define the binary operations '+' and '·' by

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + .... + (a_n + b_n)x^n$$

and $f(x) \cdot g(x) = c_0 + c_1 x + ... + c_{m+n} x^{m+n}$

where $c_0 = a_0 b_0$; $c_1 = a_0 b_1 + a_1 b_0$;...$c_{m+n} = a_m b_n$;

Prove that $R[x]$ is a ring.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 600 words.

16. (a) Prove that $C^*$ is a group under usual multiplication $(C^*, \cdot)$ which is defined by $(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$.

Or

(b) Show that the union of two subgroups of a group $G$ is a subgroup iff one is contained in the other.



17. (a) Let $H$ be a subgroup of a group $G$. Prove that

(i) any two left cosets of $H$ are either identical or disjoint.

(ii) union of all the left cosets of $H$ is $G$

(iii) the number of elements in any left coset $aH$ is the same as the number of elements in $H$.

Or

(b) Let $H$ be a subgroup of a group $G$. Prove that following

(i) $a \in H \Leftrightarrow aH = H$

(ii) $aH = bH \Leftrightarrow a^{-1}b \in H$

(iii) $a \in bH \Leftrightarrow a^{-1} \in Hb^{-1}$

(iv) $a \in bH \Leftrightarrow aH = bH$

18. (a) Show that isomorphism is an equivalence relation among groups.

Or

(b) For any group $G$, prove that

(i) $Aut\, G$ is a group under the composition of functions

(ii) $I(G)$ is normal subgroup of $Aut\, G$.



19. (a) Let $R$ be a commutative ring with identity. Prove that every maximal ideal of $R$ is a prime ideal of $R$.

Or

(b) Let $R$ be a commutative ring with identity. Show that $R$ is a field iff $R$ has no proper ideals.

20. (a) State and prove division algorithm.

Or

(b) State and prove fundamental theorem of homomorphism on rings.