(7 pages) Reg. No. : ....

Code No. : 10292 E **Sub. Code : AMMA 51**

B.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023

Fifth Semester

Mathematics — Core

LINEAR ALGEBRA

(For those who joined in July 2020 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Which of the following is a vector space under usual addition and scalar multiplication?

   (a) $V = \{a + b\sqrt{2} + c\sqrt{3} / a, b, c \in Q\}$ over Q

   (b) Z over Q

   (c) $Q[x]$ over R

   (d) Z over Z_5

2. The Kernel of the linear transformation $T : V_3(R) \to V_3(R)$ defined by $T(a,b,c) = (a,b,0)$ is ————

   (a) $\{(0,0,0)\}$ (b) $\{(0,0,c)/c \in R\}$

   (c) $\{(c,0,0)/c \in R\}$ (d) $\{(0,c,0)/c \in R\}$

3. If $S = \{(1,0,0),(2,0,0),(3,0,0)\}$ in $V_3(R)$ then $L(S) =$ ————

   (a) $\{(0,x,0)/x \in R\}$ (b) $\{(0,0,x)/x \in R\}$

   (c) $\{(x,0,0)/x \in R\}$ (d) $\{(0,0,0)\}$

4. $\dim V_n(R) =$ ————

   (a) $\frac{n-1}{2}$ (b) $n+1$

   (c) $n-1$ (d) n

5. The matrix of the linear transformation $T : V_3(R) \to V_2(R)$ given by $T(a,b,c) = (a+b, 2c-a)$ with respect to the standard basis is ————

   (a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

6. The norm of the vector in $V_3(R)$ with standard inner product (1, 2, 3) is ————

   (a) 5 (b) $\sqrt{15}$

   (c) $\sqrt{14}$ (d) $3\sqrt{38}$

7. The inverse of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is ————

   (a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| =$ ————

   (a) $ad - bc$ (b) $ab - cd$

   (c) $ac - bd$ (d) $ab - dc$

9. The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is ————

   (a) x^2 (b) $x^2 - 1$

   (c) $1 - x^2$ (d) $x - 1$

10. The eigen values of $\begin{pmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix}$ are

    (a) 1, 1, 2 (b) 3, 5, 3

    (c) 3, 4, 1 (d) 3, 0, 0

PART B — (5 × 5 = 25 marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) If A and B are subspaces of V prove that $A + B = \{v \in V / v = a + b, a \in A, b \in B\}$ is a subspace of V. Further show that $A + B$ is the smallest subspace containing A and B.

    Or

    (b) Show that $T : R^2 \to R^2$ defined by $T(a,b) = (2a - 3b, a + 4b)$ is a linear transformation.

12. (a) Prove that $S = \{(1,0,,0),(0,1,0),(1,1,1),(1,1,0)\}$ spans the vector space $V_3(R)$ but is not a basis.

    Or

    (b) Prove that any two bases of a finite dimensional vector space V have the same number of elements.

13. (a) Find the set of all unit vectors in $V_3(R)$ with standard norm.

Or

(b) Let W_1 and W_2 be subspaces of a finite dimensional inner product space. Show that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

14. (a) Find the characteristic equation of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Or

(b) Show that a square matrix A is involutory iff $A = A^{-1}$.

15. (a) Let f be the bilinear form defined on $V_2(R)$ by $f(x,y) = x_1y_1 + x_2y_2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Find the matrix of f with respect to the basis $\{(1,1), (1,2)\}$.

Or

(b) State and prove Cayley Hamilton theorem.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).
Each answer should not exceed 600 words.

16. (a) Let V be a vector space over F and W a subspace of V. Let $V/W = \{W + V / v \in V\}$. Show that V/W is a vector space over F under the following operations.

(i) $(W + V_1)(W + V_2) = W + V_1 + V_2$

(ii) $\alpha(W + V_1) = W + \alpha V_1$.

Or

(b) Let V be a vector space over a field F. Let A and B be subspaces of V. Show that $\frac{A+B}{A} \cong \frac{B}{A \cap B}$.

17. (a) Prove that any vector space of dimension n over a field F is isomorphic to $V_n(F)$.

Or

(b) Let V be a vectors space over a field F and S be a non-empty subset of V prove that

(i) $L(S)$ is a subspace of V

(ii) $S \subseteq L(S)$

(iii) If W is any subspace of V such that $S \subseteq W$, then $L(S) \subseteq W$ (i.e). $L(S)$ is the smallest subspace of V containing S.

18. (a) Let V be a finite dimensional inner product space. Let W be a subspace of V. Prove that $V = W \oplus W^{\perp}$.

Or

(b) Apply Gram - Schmidt process to construct an orthonormal basis for $V_3(R)$ with the standard inner product for the basis $\{V_1, V_2, V_3\}$ where $V_1 = (1,0,1)$, $V_2 = (1,3,1)$, $V_3 = (3,2,1)$.

19. (a) P.T. any square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

Or

(b) Find the inverse of the matrix $\begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ using Cayley - Hamilton theorem.

20. (a) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Or

(b) Reduce the quadratic form $x_1^2 + 4x_1x_2 + 4x_1x_3 + 4x_2^2 + 16x_2x_3 + 4x_3^2$ to the diagonal form using Lagrange's method.

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