(7 pages) **Reg. No. :** ....

Code No. : 10291 E **Sub. Code : AMMA 41**

B.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Fourth Semester

Mathematics – Core

ABSTRACT ALGEBRA

(For those who joined in July 2020 only)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. In $(\mathbb{Z}_7 - \{0\}, \odot)$, the inverse of 3 is ____________.

 (a) 2 (b) 3

 (c) 4 (d) 5

2. The order of $-i$ in $(c^*, -)$ is ____________.

 (a) 2 (b) infinite

 (c) 1 (d) 4

3. For group $(\mathbb{Z}_{12}, \oplus)$, the number of generators is ____________.

 (a) 4 (b) 3

 (c) 2 (d) 5

4. Choose the correct statement from the following statements.

 (a) Every cyclic group is abelian

 (b) Every abelian group is cyclic

 (c) Every element of a cyclic group is a generator of the group

 (d) $(\mathbb{Q}, +, \cdot)$ is a cyclic group

5. The kernel of the homomorphism $f : (\mathbb{Z}, +) \to \{1, -1\}$ defined by $f(n) = \begin{cases} 1 & if \quad n \text{ is even} \\ -1 & if \quad n \text{ is odd} \end{cases}$, is ____________.

 (a) $2\mathbb{Z}$ (b) $\mathbb{Z}$

 (c) $\{0\}$ (d) $\{1, -1\}$

6. The number of automorphisms of a cyclic group of order 'n' is ____________.

 (a) n (b) $\varphi(n)$

 (c) n^2 (d) 1

7. If R is a commutative ring, then $\forall a, b \in R$, ____________.

 (a) $(a-b)^2 = a^2 - b^2$

 (b) $(a+b)^2 = a^2 + b^2$

 (c) $(a+b)^2 = a^2 + 2ab + b^2$

 (d) $(a+b)^2 \neq 0$

8. An example of an infinite commutative ring without identity is ____________.

 (a) $(\mathbb{Z}, +, \cdot)$ (b) $(\mathbb{Z}_n, \oplus, \otimes)$

 (c) $(2\mathbb{Z}, +, \cdot)$ (d) $M_2(R)$

9. The map $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2 + 3$ is ____________.

 (a) a ring homomorphism

 (b) not a ring homomorphism

 (c) a ring isomorphism

 (d) a group homomorphism

10. Let $f : \mathbf{C} \to \mathbf{C}$ defined by $f(z) = \bar{z}$. Then $Kerf$ is ____________.

 (a) φ (b) $\{0\}$

 (c) $\{1\}$ (d) $\{i\}$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let H be a subgroup of a group G. Prove that

 (i) the identity element of H is the same as that of G.

 (ii) for each $a \in H$, the inverse of 'a' in H is the same as the inverse of 'a' in G.

 Or

 (b) Let G be a group and $H = \{a / a \in G$ and $ax = xa \, \forall x \in G\}$. Show that H is a subgroup of G.

12. (a) State and prove Lagrange's theorem.

 Or

 (b) Let H be a subgroup of G. Prove that the number of left cosets of H is the same as the number of right cosets of H.

13. (a) Let M and N be normal subgroups of a group G such that $M \cap N = \{e\}$. Show that every element of M commutes with every element of N.

 Or

 (b) Let $f : G \to G'$ be a homomorphism. Prove that the Kernel K of f is a normal subgroup of G.

[P.T.O.]

14. (a) If R is a ring such that $a^2 = a \forall a \in R$. Prove the following

(i) $a + a = 0$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) $ab = ba$.

Or

(b) Show that $\mathbb{Z}_n$ is an integral domain iff n is prime.

15. (a) Show that $R[x]$ is an integral domain iff R is an integral domain.

Or

(b) If $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = r$ where $x = qn + r$ and $0 \le r \le n$, prove that f is a homomorphism.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let G be the set of all real numbers except -1. Define $*$ on G by $a * b = a + b + ab$. Show that $(G, *)$ is a group.

Or

(b) Let A and B be two subgroup of a group G. Show that AB is a subgroup of G if and only if $AB = BA$.

17. (a) Show that a subgroup of a cyclic group is cyclic.

Or

(b) Prove that a group G has no proper subgroups if it is a cyclic group of prime order.

18. (a) State and prove Cayley's theorem.

Or

(b) Let N be a subgroup of a group G. Prove that the following are equivalent

(i) N is a normal subgroup of G

(ii) $aNa^{-1} = N \ \forall a \in G$

(iii) $aNa^{-1} \le N \ \forall a \in G$

(iv) $ana^{-1} \in N \ \forall n \in N$ and $a \in G$.

19. (a) Let R be a commutative ring with identity. Show that R is a field iff R has no proper ideals.

Or

(b) Let R be a commutative ring with identity. Prove that an ideal M of R is maximal iff R/M is a field.

20. (a) State and prove division algorithm.

Or

(b) State and prove fundamental theorem of homomorphism on rings.