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(8 pages)

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M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2021–2022)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. For a continuous function T satisfying for any closed unit sphere in N its image T (S) is a bounded set N' the norm is defined as ————

(a) $\sup\{\|T(x)\|:\|x\|\le 1\}$

(b) $\sup\{\|T(x)\|:\|x\|=1\}$

(c) $\inf\{\|T(x)\|:\|x\|\le 1\}$

(d) $\inf\{\|T(x)\|:\|x\|\le 1\}$

2. A complete normed linear space is called as ————

(a) Metric (b) Hilbert

(c) Empty (d) Banach

3. The ———— of the linear transformation T is the subset $B \times B'$ consists of all ordered pairs of the form $(x, T(x))$

(a) open (b) graph of T

(c) open map (d) closed map

4. The isometric isomorphism $x \to F_x$ is called the ———— of N into N^*

(a) bijective

(b) injective

(c) natural imbedding

(d) N into N^{**}

5. A complete Banach space whose norm arises from an inner product is said to be ———— space.

(a) Banach (b) Complete

(c) Hilbert (d) Hausdorff

6. Two vectors x and y in a Hilbert space H are said to be ———— of $\langle x, y\rangle = 0$.

(a) parallel (b) orthogonal

(c) equal (d) unequal

7. A non empty subset of a Hilbert space H which consists of mutually orthogonal unit vectors is called as ———— set.

(a) ortho-normal (b) empty

(c) whole set (d) power set

8. The conjugate operator T^* of T is given by $(T^* f)\ x =$ ————

(a) $T^* f\,(x)$ (b) $fT^*(x)$

(c) $f(Tx)$ (d) $T^*(f(x))$

9. An operator N on H is said to be ———— if it commutes with its adjoint.

(a) normal (b) unitary

(c) singular (d) orthogonal

10. An operator A on H satisfying the condition $A = A^*$ is called ————

(a) adjoint (b) self adjoint

(c) unitary (d) inverse

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) If M is a closed linear subspace of a normal linear space N and x_0 is a vector not in M, then prove that there exists a functional f_0 in N^* such that $f_0(M)=0$ and $f_0(x_0)\neq 0$.

Or

(b) If N and N' are normed linear spaces then prove that the set $B(N, N')$ of all continuous linear transformations of N into N' is itself a normed linear space with respect to the pointwise linear operations and the norm given by $\|T\| = \sup\{\|T(x)\|:\|x\|\le 1\}$

12. (a) Prove that if N is a normal linear space then the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak * topology.

Or

(b) State and prove closed graph theorem.

13. (a) Prove that if x and y are any two vectors in a Hilbert space then $|(x, y)| \le \|x\|\|y\|$.

Or

(b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then prove that the linear subspace $M + N$ is also closed.

14. (a) Let $\{e_1, e_2 \cdots e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then prove that

$\sum |(x, e_i)|^2 \leq \|x\|^2$; $\quad x - \Sigma(x, e_i) e_i \perp e_j$ for each j

Or

(b) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Prove that the following conditions are all equivalent to one another.

(i) $\{e_i\}$ is complete

(ii) $x \perp \{e_i\} \Rightarrow x = 0$

(iii) if x is an arbitrary vector in H, then $x = \Sigma(x, e_i) e_i$

(iv) if x is an arbitrary vector in H, then $\|x\|^2 = \Sigma |(x, e_i)|^2$

15. (a) If P and Q are the projections on closed linear subspaces M and N of H, then prove that $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$.

Or

(b) Prove that if T is an operator on H, then T is normal $\Leftrightarrow$ its real and imaginary parts commute.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 600 words.

16. (a) State and prove Hahn-Banach theorem.

Or

(b) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M is defined by $\|x + M\| = \inf\{\|x + m\| : m \in M\}$ then prove that N/M is a normed linear space. Also prove that if N is a Banach space that N/M is also so.

17. (a) State and prove open mapping theorem.

Or

(b) Prove that if B is a Banach space, then B is reflexive $\Leftrightarrow B^*$ is reflexive. If N is finite dimensional normed linear space of dimension n show that N^* also has dimension N. Prove that N is reflexive.

18. (a) State and prove uniform boundedness theorem.

Or

(b) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non zero vector Z_0 in H such that $Z_0 \perp M$.

19. (a) Prove that the adjoint operation $T \to T^*$ on $\mathcal{B}(H)$ has the following properties.

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $(\alpha T)^* = \bar{\alpha} T^*$

(iii) $(T_1 T_2)^* = T_2^* T_1^*$

(iv) $\|T^* T\| = \|T\|^2$

Or

(b) If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in H, then prove that $x - \Sigma(x, e_i) e_i \perp e_j$ for each j.

20. (a) Prove that if $P_1, P_2 \cdots P_n$ are the projections on closed linear subspaces $M_1, M_2, \cdots M_n$ of H then $P = P_1 + P_2 \cdots + P_n$ is a projection $\Leftrightarrow$ the P_i S are pairwise orthogonal and P is a projection on $M = M_1 + M_2 + \cdots + M_n$.

Or

(b) If N_1 and N_2 are normal operators on H with either commute with the adjoint of the other then prove that $N_1 + N_2$ and $N_1 N_2$ are normal.
