(7 pages) Reg. No. : ....

Code No. : 7383 **Sub. Code : ZMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Fourth Semester

Mathematics — Core

ADVANCED ALGEBRA — II

(For those who joined in July 2021-2022)

Time :Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. The dimension of the vector space K over the field F is ———— of K over F.
   (a) degree (b) dimension
   (c) basis (d) element

2. A complex number which is not algebraic is called ————
   (a) imaginary (b) real
   (c) transcendental (d) ideal

3. If $f(x) \in F[x]$, a finite extension E of F is called ———— over F if over E but not over any proper subfield of E, $f(x)$ can be factored as a product of linear factors.
   (a) normal (b) extension
   (c) splitting field (d) finite ring

4. The element $a \in K$ is a root of $P(x) \in F[x]$ of ———— if $(x - a)^m \mid p(x)$ whereas $(x - a)^{m+1} \nmid p(x)$
   (a) divisor (b) multiplicity m
   (c) root m (d) basis m

5. If G is a group of automorphisms of K, then the ———— of G is the set of all element $a \in k$ such that $\sigma(a) = a$ for all $a \in G$.
   (a) finite field (b) infinite
   (c) fixed field (d) root

6. The automorphism σ of k is in $G(K, F)$ if and only if $\sigma(\alpha) =$ ———— for every $\alpha \in F$.
   (a) 1 (b) α
   (c) 0 (d) ∞

7. Any two finite fields having the same number of elements are ————
   (a) equal (b) unequal
   (c) constant (d) isomorphic

8. The multiplicative group of nonzero elements of a finite field is ————
   (a) isomorphic (b) fixed
   (c) cyclic (d) unequal

9. For $x = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ in Q the ———— of x is defined by $x^* = \alpha_0 - \alpha_1 i + \alpha_2 j + \alpha_3 k$
   (a) parallel (b) perpendicular
   (c) adjoint (d) normal

10. If $x \in Q$ then the norm of x is defined by ————
    (a) x (b) xx^*
    (c) x^{-1} (d) $-x$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that if a, b in k are algebraic over F, then $a \pm b$, ab, a/b(if $b \neq 0$ re all algebraic over F.

    Or

    (b) Prove that if $a \in k$ is algebraic of degree n over F, then $[F(a) : F] = n$.

12. (a) Prove that the polynomial $f(x) \in F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a trivial common factor.

    Or

    (b) Prove that there is an isomorphism τ^{**} of $\frac{F(x)}{f(x)}$ onto $\frac{F'(t)}{f'(t)}$ with the property that for every

    $\alpha \in F \cdot \alpha\tau^{**} = \alpha^1, (x + (f(x))\tau^{**} = t + (f'(t))$.

13. (a) Prove that the fixed field of G is a subfield of K.

    Or

    (b) Let K be a normal extension of F and let H be a subgroup of $G(K, F)$ let $K_H = \{x \in K \mid \sigma(x) = 0 \text{ for all } \sigma \in H\}$ be the fixed field of H. Then prove that $[K : K_H] = O(H)$, and $H = G(K, KH)$.

[P.T.O.]

14. (a) Let D be a division ring of characteristic $P > 0$ with center Z and let $P = \{0, 1, 2 .... (p-1)\}$ be the subfield of Z isomorphic to J_p suppose that $a \in D$, $a \notin Z$ is such that $a^{p^n} = a$ for some $n \geq 1$. Then prove that there exists an $x \notin D$ such that

(i) $x a x^{-1} \neq a$

(ii) $x a x^{-1} \in P(a)$ the field obtained by adjoining a to P.

Or

(b) Let D be a division ring such that for every $a \in D$ there exists a positive integer $n(a) > 1$ such that $a^{n(a)} = a$. Then show that D is a commutative field.

15. (a) If C be the field of complex numbers and suppose that the division ring D is algebraic over C then prove that $D = C$.

Or

(b) State and prove Lagrange identity.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 600 words.

16. (a) Prove that if L is an algebraic extension of K and if K is an algebraic extension of F then L is an algebraic extension of F.

Or

(b) Show that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

17. (a) Prove that any splitting fields E and E' of the polynomials $f(x) \in F[x]$ and $f'(t) \in F'[t]$ respectively are isomorphic by an isomorphism ϕ with the property that $\alpha\phi = \alpha'$ for every $\alpha \in F$.

Or

(b) Prove that if F is of characteristic O and if a, b are algebraic over F, then there exists an element $C \in F(a, b)$ such that $F(a, b) = F(c)$.

18. (a) Prove that if K is a normal extension of F if and only if K is the splitting field of some polynomial over F.

Or

(b) If K is a finite extension of F, then prove that $G(K, F)$ is a finite group and its order $O(G(K, F))$ satisfies $O(G, (K, F)) \leq [K : F]$.

19. (a) Prove that a finite division ring is necessarily a commutative field.

Or

(b) Let G be a finite abelian group enjoying the property that the relating $x^n = e$ is satisfied by atmost n element of G for every integer n. Then prove that G is a cyclic group.

20. (a) Prove that every positive integer can be expressed as the sum of squares of four integers.

Or

(b) State and prove Frobenius theorem.

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