(8 pages) Reg. No. : ....

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M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Third Semester

Mathematics – Core

MEASURE AND INTEGRATION

(For those who joined in July 2021 – 2022)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. The outer measure of A, $m^{a}(A)$ is defined by

(a) $\inf\left\{\sum_{k=1}^{\infty} l(I_k) \Big/ \bigcup_{k=1}^{\infty} I_k \subseteq A\right\}$

(b) $\sup\left\{\sum_{k=1}^{\infty} l(I_k) \Big/ A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$

(c) $\inf\left\{\sum_{k=1}^{\infty} l(I_k) \Big/ A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$

(d) $\inf\left\{l(I_k) \Big/ A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$

2. Which one of the following is not true?

(a) Outer measure is defined for all sets of real number

(b) the outer measure of an interval is its length

(c) outer measure is translation invariant

(d) outer measure is countably additive

3. $\{x \in E / f(x) > c\}$ is the same as

(a) $\bigcap_{k=1}^{\infty}\left\{x \in E \Big/ f(x) > c - \frac{1}{k}\right\}$

(b) $\bigcup_{k=1}^{\infty}\left\{x \in E \Big/ f(x) \geq c + \frac{1}{k}\right\}$

(c) $\bigcap_{k=1}^{\infty}\{x \in E / f(x) > k\}$

(d) $\bigcap_{k=1}^{\infty}\left\{x \in E \Big/ f(x) \geq c + \frac{1}{k}\right\}$

4. For any C, we have $\{x \in E / \max\{f_1, f_2, \cdots, f_n\}(x) > c\}$ is

(a) $\bigcup_{k=1}^{n}\{x \in E / f_k(x) > c\}$

(b) $\bigcap_{k=1}^{n}\{x \in E / f_k(x) > c\}$

(c) $\bigcup_{k=1}^{n}\{x \in E / (f_1 + f_2 + . + f_k)(x) > c\}$

(d) $\{x \in E / f_k(x) > c\}$ for some k

5. The Dirichlet's function f is

(a) Both Riemann integrable and integrable over [0,1]

(b) Riemann integrable but not integrable over [0,1]

(c) Integrable but not Riemann integrable over [0,1]

(d) Neither Riemann integrable nor integrable over [0,1]

6. Let $E = [0, 1]$. Define $f_n = n \cdot \chi\left(0, \frac{1}{n}\right)$. Then $\lim_{n\to\infty} \int_E f_n$ is

(a) 0 (b) 1

(c) ∞ (d) n

7. $[-5\, f]^-$ is

(a) $5f^+$ (b) $-5f^-$

(c) $-5f^+$ (d) $5f^-$

8. Define f on $[0,1]$ by $f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & if \quad 0 < x \leq 1 \\ 0 & if \quad x = 0 \end{cases}$. Take $P_n = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right\}$. Then $V(f, P_n)$ is

(a) 0 (b) $1 + \frac{1}{2} + \frac{1}{3} + . + \frac{1}{2n}$

(c) $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ (d) 1

9. A decomposition of X into the union of two disjoint sets A and B for which A is positive for γ and B negative is called a ____________ for γ.

(a) Jordan decomposition

(b) Hahn decomposition

(c) Lebesgue decomposition

(d) Royden decomposition

10. The measure $|\gamma|$ is defined on $\mathfrak{M}$ by $|\gamma|(E) =$

(a) $\gamma^+(E) - \gamma^- / E)$ (b) $\gamma^-(E) + \gamma^+(E)$

(c) $\gamma(E \cap X)$ (d) $|\gamma^+(E) - \gamma^-(E)|$

[P.T.O.]

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define the outer measure m^* and prove that a countable set has outer measure zero.

Or

(b) Show that the union of two measurable sets is measurable.

12. (a) Let the function f be defined on a measurable set E. Prove that f is measurable if and only if f_n each open set 0, $f^{-1}(0)$ is measurable.

Or

(b) Let f be a measurable real-valued function on E. Assume f is founded on E. Prove that for each $\varepsilon > 0$, there are simple functions ϕ_ε and Ψ_ε defined on E which have the following approximation properties $\phi_\varepsilon \le f \le \Psi_\varepsilon$ and $0 \le \Psi_\varepsilon - \phi_\varepsilon < \varepsilon$ on E.

13. (a) Let f be a founded measurable function on a set of finite measure E. Prove that f is integrable over E.

Or

(b) Let f be a nonnegative measurable function on E. Prove that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

14. (a) Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Prove that $\int_{A\cup B} f = \int_A f + \int_B f$.

Or

(b) Let f be an increasing function on the closed, bounded interval $[a,b]$. Prove that f' is integrable over $[a,b]$ and $\int_a^b f' \le f(b) \le f(a)$.

15. (a) Prove that a function on $[a,b]$ is absolutely continuous on $[a,b]$ if and only if it is an indefinite integral over $[a,b]$.

Or

(b) Define a positive set. Let γ be a singed measure on the measurable space $(X, \mathcal{M})$. Prove that every measurable subset of a positive set is itself positive and the union of a countable collection of positive sets is positive.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Show that every interval is measurable.

Or

(b) Prove that Lebesgue measure possesses the following continuity properties

(i) If $\{A_k\}$ is an ascending collection of measurable sets then $m\left(\bigcup_1^\infty A_k\right) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then $m\left(\bigcap_1^\infty B_k\right) = \lim_{k\to\infty} m(B_k)$.

17. (a) Let f and g be measurable functions on E that are finite a.e on E. Prove that

(i) $\alpha f + \beta g$ is measurable on E for any α and β.

(ii) fg is measurable on E.

Or

(b) State and prove Egoroff's theorem.

18. (a) State and prove the bounded covergence theorem.

Or

(b) State and prove Fatou's lemma.

19. (a) State and prove the Lebesgue dominated convergence theorem.

Or

(b) Let f be an increasing function on $[a,b]$ for each $\alpha > 0$, prove that

$$m^*\{x \in (a,b) / \overline{D}f(x) \ge \alpha\} \le \frac{1}{\alpha}[f(b) - f(a)].$$

20. (a) Let the function f be continuous on $[a,b]$. If the family of divided difference functions $\{Diff_h f\}_{0<h\le 1}$ is uniformly integrable over $[a,b]$, Prove that f is absolutely continuous on $[a,b]$.

Or

(b) State and prove the Hahn decomposition theorem.

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