(8 pages)

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Code No.: 7377 **Sub. Code: ZMAM 31**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023

Third Semester

Mathematics – Core

ADVANCED ALGEBRA – I

(For those who joined in July 2021–2022)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Let U, W be subspaces of V such that $U \subset W$. Let $A(U)$ be the annihilator of U. Then
   (a) $A(U) \supset A(W)$
   (b) $A(U) \subset A(W)$
   (c) $A(U)$ and $A(W)$ are not comparable
   (d) $A(U) = A(W)$

2. If $\dim V = 5$, $\dim W = 6$ then $\dim Hom(V, F) + \dim Hom(W, F)$ is
   (a) 60 (b) 30
   (c) 61 (d) 11

3. The number of operations is an algebra is
   (a) 1 (b) 2
   (c) 3 (d) 4

4. Let V be two-dimensional over the filed F or real number with a basis v_1, v_2. If T is defined by $v_1T = v_1 + v_2$, $v_2T = v_1 - v_2$, which one of the following is a characteristic root of T
   (a) $\sqrt{2}$ (b) 3
   (c) 2 (d) 1

5. If M, of dimension 16, is cyclic w.r.t. T, then the dimension of MT^4 is
   (a) 64 (b) 4
   (c) 12 (d) 20

6. Which of the following is a Jordan block

   (a) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

7. The companion matrix of $-5 + 6x + 7x^2 + x^3 + x^4$ is

   (a) $\begin{pmatrix} -5 & 6 & 7 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & -6 & -7 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 6 & 7 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 6 & 7 & 0 \end{pmatrix}$

8. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ then
   (a) $\det C = \det A + \det B$
   (b) $\det C = \det A \bullet \det B$
   (c) $\det C = \det A - \det B$
   (d) $\det C = 2\det A + \det B$

9. If N is normal and if for $\lambda \in F$, $v(N - \lambda)^k = 0$ then vN is
   (a) $\lambda^k v$ (b) λv
   (c) $(-\lambda)^k v$ (d) 0

10. If 2,3,–4,–5,–7,0,0 are the characteristic roots of a real symmetric matrix A then the signature of A is
    (a) 5 (b) 1
    (c) –1 (d) 7

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) If V is finite dimensional, prove that $\psi : V \to \hat{\hat{V}}$ defined by $v\psi = T_v$ for every $v \in V$, where $T_v(f) = f(v)$ for any $f \in \hat{V}$, is an isomorphism of V on $\hat{\hat{V}}$.

Or

[P.T.O.]

(b) If V is a finite dimensional inner product space and if W is a subspace of V, prove that V is the direct such of W and $W^{\perp}$.

12. (a) If V is finite dimensional over F, prove that $T \in A(\upsilon)$ is invertible if and only if the constant term of the minimal polynomial for T is not zero

Or

(b) Prove that $\lambda \in F$ is a characteristic root of $T \in A(\upsilon)$ if an only if for some $\upsilon \neq 0$ is V, $\upsilon T = \lambda \upsilon$.

13. (a) If V is n-dimensional over F and if $T \in A(\upsilon)$ has all its characteristic roots in F, prove that V satisfies a polynomial of degree n over F.

Or

(b) If $T \in A(\upsilon)$ is nilpotent, prove that $\alpha_0 + \alpha_1 T + \ldots\ldots + \alpha_m T^m$ where the $\alpha_i \in F$ is invertible if $\alpha_0 \neq 0$.

14. (a) If V is cyclic relative to T and if the minimal polynomial of T in $F(x)$ is $p(x)$, then prove that for some basis of V the matrix of T is $C(p(x))$.

Or

(b) State and prove Jacobson lemma.

15. (a) If $(\upsilon T, \upsilon T) = (\upsilon, \upsilon)$ for all $\upsilon \in \upsilon$, prove that T is unitary.

Or

(b) If N is normal and $AN = NA$, prove that $AN^* = N^* A$.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b)
Each answer should not exceed 600 words.

16. (a) Define an inner product space V with an example and show that if $u, \upsilon \in V$ then $|(u,\upsilon)| \leq \|u\| . \|\upsilon\|$

Or

(b) Let R be Euclidean ring. Prove that any finitely generated R-module is direct sum of a finite number of cyclic submodules.

17. (a) If V is finite dimensional over F then for S, $T \in A(\upsilon)$ prove that $r(ST) \leq n(T)$, $r(TS) \leq r(T)$ and $r(ST) = r(TS) = r(T)$ for S regular in $A(\upsilon)$

Or

(b) Suppose V is the vector space of all polynomials over F of degree 30 less and let D be the differentiation operator : Compute the matrix of D in the basis

(i) $1, x, x^2, x^3$

(ii) $1, 1+x, 1+x^2, 1+x^3$

(iii) If the matrix in part (1) is A and that in part (2) is B, find a matrix C so that $B = cAc^{-1}$.

18. (a) If $T \in A(\upsilon)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) Let $T \in A(\upsilon)$ and let $p(x) = q_1(x)^{l_1}\, q_2(x)^{l_2} \ldots\ldots q_k(x)^{l_k}$ be the minimal polynomial of T over F.

Let $V_i = \{\upsilon \in V | \upsilon q_i(T)^{l_i} = 0\}$. For each i = 1,2, k, prove that $V_i \neq (0)$ and $V = V_1 \oplus V_2 + \ldots\ldots \oplus V_k$.

19. (a) If F is a field of characteristic 0 and if $T \in A_F(\upsilon)$ is such that $trT^i = 0$ for all $i \geq 1$, prove that T is nilpotent.

Or

(b) Prove that (i) $\det A = \det(A')$ (ii) A is invertible if and only if $\det A \neq 0$.

20. (a) If $T \in A(\upsilon)$, prove that (i) $T^* \in A(\upsilon)$ (ii) $(T^*)^* = T$ (iii) $(S+T)^* = S^* + T^*$.

Or

(b) State and prove Sylvester's law.