Reg. No. : ....

Code No. : 7371 **Sub. Code : ZMAM 23**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Second Semester

Mathematics — Core

ADVANCED CALCULUS

(For those who joined in July 2021-2022 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. Let f and g be continuous and bounded on D. Then, for any constant C, $\iint\limits_D Cf \_ C\iint\limits_D f$.

(a) $=$ (b) $>$

(c) $\geq$ (d) $\leq$

2. The value of $\int_1^2 \sqrt{x}\,dx$ is ________

(a) $\frac{4\sqrt{2}}{3}$ (b) $\frac{4\sqrt{2}+2}{3}$

(c) $\frac{4\sqrt{2}-2}{3}$ (d) $\frac{2\sqrt{2}-2}{3}$

3. Find the rank of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ -1 & 3 & 1 \end{bmatrix}$ is ________

(a) 0 (b) 1

(c) 2 (d) 3

4. If $\begin{cases} u = x^2 + y - z \\ v = xyz^2 \\ w = 2xy - y^2 z \end{cases}$ then differential of T at $(1, 1, 1)$ is ________

(a) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$

5. Find the product of the matrices $\begin{bmatrix} \cos y & \sin y \\ -x^{-1}\sin y & x^{-1}\cos y \end{bmatrix}$. $\begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{bmatrix}$ is ________.

(a) I (b) $2I$

(c) 0 (d) $-I$

6. The value of $\begin{vmatrix} -y & 0 \\ v & u \end{vmatrix}$ is ________

(a) uv (b) vy

(c) uy (d) $-uy$

7. If $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$ then the Jacobian is ________.

(a) $\rho\sin\phi$

(b) $\sin\phi$

(c) $\rho^3\sin\phi$

(d) $\rho^2\sin\phi$

8. A curve γ is said to be smooth on interval I if γ is of class C' and the differential d_γ is always of rank ———— on I.

(a) 0 (b) −1

(c) 1 (d) 2

9. If $V = Ai + Bj + Ck$ is a vector field with components of class C'', then $div(curl(V)) =$ ————

(a) 0 (b) 1

(c) −1 (d) ∞

10. Let Σ be a suitably well-behaved orientable surface whose boundary is a curve Σ. Let ω be a 1-form of class C' defined on Σ. Then $\int_{\partial\Sigma} \omega = \iint_{\Sigma} d\omega$ is ————

(a) Green's theorem

(b) Stoke's theorem

(c) Gauss divergence theorem

(d) Chinese theorem.

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) Let f be of class C'' in a rectangle R with vertices $P_1 = (a_1, b_1)$, $Q_1 = (a_2, b_1)$, $P_2 = (a_2, b_2)$, $Q_2 = (a_1, b_2)$, where $a_1 \le a_2$ and $b_1 \le b_2$. Then prove that,

$$\iint_R f_{12} = \iint_R \frac{\partial^2 f}{\partial y \partial x} dx\, dy = f(P_1) - f(Q_1) + f(P_2) - f(Q_2).$$

Or

(b) Prove that if N' is a refinement of N, then $\underline{S}(N) \le \underline{S}(N') \le \overline{S}(N') \le \overline{S}(N)$.

12. (a) Let T be a transformation defined on a set D in n space and taking values in m space. Then, prove that T is continuous on D iff $T^{-1}(\mathcal{O})$ is open, relative to D, for every open set $\mathcal{O}$ in m space.

Or

(b) Compute the rank of the matrices $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 4 & 1 & 1 & -1 \end{bmatrix}$.

13. (a) Prove that let T be a transformation from R^n into R^n which os of class C' in an open set D, and suppose the $J(p) \neq 0$ for each $p \in D$. Then, T is locally 1-to-1 in D.

Or

(b) Compute the Jacobians transformation
$$\begin{cases} u = x^2 + 2xy + y^2 \\ v = 2x + 2y \end{cases}$$

14. (a) Prove that let γ_1 and γ_2 be smoothly equivalent smooth curves, and let p be a simple point on their trace. Then, γ_1 and γ_2 have the same direction at p.

Or

(b) With Ω and T as above, let E be a compact subset of Ω. Then prove that $\lim_{c \downarrow p} \frac{v(T(C))}{v(C)} = |J(p)|$ where C range over the family of cubes lying in Ω with center $p \in E$, and the limit is uniform for all $p \in E$.

15. (a) Prove that let Σ be a smooth surface of class C' whose domain D is a standa region, or a finite of standard region, or a finite union of standard regions, in the UV plane. Let $\omega = Adx + Bdy + Cdz$.

Or

(b) If ω is any differential form of class C'', then $dd\omega = 0$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

Each answer should not exceed 600 words.

16. (a) Let f be bounded in a closed rectangle R and continuous there except on a set E of zero area. Suppose there exists a k such that no vertical line meets E in more than k points. Then, prove that $\iint_R f = \int_a^b dx \int_c^d f(x, y)dy$.

Or

(b) Prove that let R be a closed rectangle, and let f be bounded in R and continuous at all points of R except those in a set E of zero area. Then $\iint_R f$ exists

17. (a) Let T be of class C' in an open region D, and let E be a closed bounded subset of D. Let $dT \mid p_0$ be the differential of T at a point $p_0 \in E$. Then prove that, $T(p_0 + \Delta p) = T(p_0) + dT \mid p_0(\Delta p) + R(\Delta p)$ where $\lim_{\Delta p \to 0} \frac{|R(\Delta p)|}{|\Delta p|} = 0$ uniformly for $p_0 \in E$.

Or

(b) Let the transformations S be continuous on a set A and T be continuous on a set B, and let $p_0 \in A$ and $S(p_0) = q_0 \in B$. Then, prove that the product transformation TS, defined by $TS(p) = T(S(p))$, is continuous at p_0.

18. (a) Let T be class C' on an open set D in n space, taking values in n space ; Suppose that $J(p) \neq 0$ for all $p \in D$. Then, prove that $T(D)$ is an open set ; thus, T carries every open set in D into an open set.

Or

(b) Let F be a function of three variables which is of class C' in an open set D, and let $p_0 = (x_0, y_0, z_0)$ be a point of D for which $F(p_0) = 0$. Suppose that $F_3(p_0) \neq 0$. Then, prove that there is a function ϕ of class C' in a neighbourhood N of (x_0, y_0) such that $z = \phi(x, y)$ is a solution of $F(x, y, z) = 0$ for (x, y) in N and such that $\phi(x_0, y_0) = z_0$.

19. (a) Prove that the volume of $L(D)$ is $kv(D)$, where $k = |\det(L)|$.

Or

(b) Let F be an additive set function, defined on $\mathscr{A}$ and a.c. suppose also that F is differentiable everywhere, and uniformly differentiable on compact sets, with the derivative a point function f, then, prove that f is continuous everywhere and $F(S) = \iint_S f$ holds for every rectangle S. If F is positive on an open set D (meaning that $F(S) \geq 0$ for any set $S \subset D, S \in \mathscr{A}$, then $F(S) = \iint_S f$ also holds for all sets $S \in \mathscr{A}$ contained in D.

20. (a) If ω is a differential form of class C' then prove that $T^*(d\omega) = (d\omega)^* = dT^*(\omega)$.

Or

(b) Let $a = 2i - 3j + k$, $b = i - j + 3k$, $c = i - 2j$.

Compute the vectors $(a \times b).c$, $a \times (b \times c)$,

$(a \times b) \times c$, $a(a \times b)$, $(a + b) \times (b + c)$, $(a.b)\,c - (a.c)b$.
