(8 pages) Reg. No. : ....

Code No. : 7370 **Sub. Code : ZMAM 22**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Second Semester

Mathematics — Core

ANALYSIS — II

(For those who joined in July 2021-2022 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. $L(P, f, \alpha)$ ———— $L(P^*, f, \alpha)$

(a) $<$ (b) $>$

(c) $\leq$ (d) $\geq$

2. $I(x) =$ ————. If $x > 0$

(a) 0 (b) ∞

(c) -1 (d) 1

3. If x is rational then the value of $\lim_{m \to \infty} \lim_{n \to \infty} (\cos m! \pi x)^{2n}$ is ————.

(a) 0 (b) 1

(c) $-\infty$ (d) ∞

4. The value of $\Gamma(n)$ is ————

(a) n (b) $n-1$

(c) $n!$ (d) $(n-1)!$

5. The value of $\lim_{k \to \infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx =$ ————

(a) 0 (b) 1

(c) π (d) 2π

6. A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be equi-continuous on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)|$ ———— ε.

(a) $<$ (b) $\leq$

(c) $>$ (d) $\geq$

7. The value of $\lim_{x \to 0} (1+x)^{\frac{1}{x}} =$ ————

(a) 1 (b) 0

(c) e (d) e^x

8. The value of $\lim_{x \to 0} \frac{\log(1+x)}{x}$ is ————

(a) 1 (b) 0

(c) e (d) e^x

9. The value of $\int_{-\infty}^{\infty} e^{-s^2}$ is ————

(a) 0 (b) $\sqrt{\pi}$

(c) π (d) ∞

10. The value of $\Gamma(4)$ is ————

(a) 1 (b) 2

(c) 3 (d) 6

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If f is continuous on $[a, b]$ then prove that, $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

Or

(b) Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$ and α is monotonically increasing on $[a, b]$ with $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then prove that, $g \in \mathscr{R}(\beta)$ and $\int_A^B g \, d\beta = \int_a^b f \, d\alpha$.

12. (a) If f maps $[a, b]$ into R^k and if $f \in \mathscr{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then prove that $|f| \in \mathscr{R}(\alpha)$, and $\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$.

Or

(b) Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, 3, \ldots)$ then prove that $\sum f_n$ converges uniformly on E if $\sum f_n$ converges.

[P.T.O.]

13. (a) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then prove that, $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

Or

(b) If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then prove that $\{f_n\}$ is equi-continuous on K.

14. (a) Prove that for every interval $[-a, a]$ there is a sequence of real polynomial P_n such that $P_n(0) = 0$ and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a, a]$

Or

(b) Given a double sequence $\{a_{ij}\}, i = 1, 2, 3, \ldots$, $j = 1, 2, 3, \ldots$, suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ $(i = 1, 2, 3, \ldots)$ and $\sum b_i$ converges. Then prove that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

15. (a) If f is continuous and if $\varepsilon > 0$, then prove that there is a irigonometric polynomial P such that $|P(x) - f(x) < \varepsilon|$ for all real x.

Or

(b) Prove that if $x > 0$ and $y > 0$, then $\int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ this integral is the so-called beta function $\beta(x, y)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 600 words.

16. (a) Let $f \in \mathscr{R}$ on $[a, b]$, for $a \le x \le b$, put $F(x) = \int_a^x f(t)\, dt$. Then prove that F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, F is differentiable at x_0 and $F'(x) = f(x_0)$.

Or

(b) Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a, b]$, then Prove that $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

17. (a) Prove that, if γ' is continuous on $[a, b]$, then γ is rectifiable, and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

Or

(b) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t) = A_n$ $(n = 1, 2, 3, \ldots)$. Then prove that $\{A_n\}$ converges, and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. In other words, $\lim_{t\to x} \lim_{n\to\infty} f_n t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$

18. (a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

Or

(b) If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is point wise bounded and equi-continuous on K, then prove the following:

(i) $\{f_n\}$ is uniformly bounded on K

(ii) $\{f_n\}$ contains a uniformly convergent subsequence.

19. (a) Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then prove that the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

Or

(b) State and prove Taylor's theorem.

20. (a) Let $\{\phi_n\}$ be ortho-normal on $[a, b]$, let $s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$ be the nth partial sum of the Fourier series of f, and suppose $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$. Then prove that $\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx$, and equality holds iff $\gamma_m = c_m (m = 1, \ldots n)$.

Or

(b) State and prove Parseval's theorem.

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