(7 pages) Reg. No. : ....

Code No. : 7369 **Sub. Code : ZMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023

Second Semester

Mathematics – Core

ALGEBRA – II

(For those who joined in July 2021 – 2022 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If R is a commutative ring and $a \in R$, the $aR = \{ar \mid r \in R\}$ is a ideal of R

   (a) right ideal (b) left ideal

   (c) two-sided ideal (d) none of the above

2. The homomorphism ϕ of R into R' is ———— if and only if $I(\phi) = (0)$

   (a) one-to-one (b) onto

   (c) bijection (d) none of the above

3. Suppose m is a prime element in the Euclidean ring and $m \mid ab$ where $a, b \in R$, then m divides ————

   (a) a (b) b

   (c) (a) and (b) (d) (a) or (b)

4. The gcd of $3+4i$ and $4-3i$ in $J[i]$ is ————

   (a) $2-i$ (b) $2+i$

   (c) $1+2i$ (d) none of the above

5. Which of the following is the unique factorization domain?

   (a) Z (b) $Z(\sqrt{-5})$

   (c) (a) and (b) (d) no one of the above

6. $x^3 - 9$ is reducible over the ————

   (a) integers mod 5 (b) integers mod 7

   (c) integers mod 11 (d) none of the above

7. Let $F[[x]]$ be the ring of formal power series over a field F. Then rad $F[[x]] =$ ————

   (a) (0) (b) (1).

   (c) (x) (d) none of the above

8. Let R be any commutative regular ring. Then the J – radical of a ring R is ————

   (a) {1} (b) {0}

   (c) R (d) none of the above

9. A ring R is isomorphic to a subdirect sum of ———— if and only if R is without a prime ideal.

   (a) ideals (b) integral domain

   (c) prime ideals (d) none of the above

10. If $R^V \neq \{0\}$, then the annihilator of the set of zero divisors of R is ————

    (a) R (b) $\{0\}$

    (c) R^V (d) None of the above

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field.

    Or

    (b) (i) If U is an ideal of R and $1 \in U$, prove that $U = R$.

    (ii) If U, V are ideals of R, let $U + V = \{u + v \mid u \in U, v \in V\}$. Prove that $U + V$ is also an ideal.

12. (a) Prove that a necessary and sufficient condition that the element a in the Euclidean ring be a unit is that $d(a) = d(1)$.

    Or

    (b) Let p be a prime integer and suppose that for some integer c relatively prime to p we can find integers x and y such that $x^2 + y^2 = cp$. Then p can be written as the sum of squares of two integers, that is, there exists integers a and b such that $p = a^2 + b^2$.

13. (a) State and prove the division algorithm.

    Or

    (b) Define primitive polynomial and prove that if $f(x)$ and $g(x)$ are primitive polynomials, then $f(x)g(x)$ is a primitive.

14. (a) Prove that the ring Z of integers is semisimple.

    Or

    (b) The prime radical of the ring R coincides with the nil radical of R; that is Rad R is simply the ideal of all nilpotent elements of R.

[P.T.O.]

15. (a) An element $a \in R$ is quasi-regular if and only if $a \in I_a$.

Or

(b) For any ring $R, R/rad\ R$ is isomorphic to a subdirect sum of fields.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) (i) Let R be the ring of integers. Then the ideal $M = (n_0)$ is maximal if and only if n_0 is prime.

(ii) If R is a commutative ring with unit element and M is an ideal of R, then M is a maximal ideal of R if and only if R/M is a field.

Or

(b) Prove that every integral domain can be imbedded in a field.

17. (a) State and prove unique factorization theorem.

Or

(b) The ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R if and only if a_0 is a prime element of R.

18. (a) State and prove the Eisenstein criterion.

Or

(b) If R is a unique factorization domain and if $p(x)$ is a primitive polynomial in $R[x]$, then it can be factored in a unique way as the product of irreducible elements in $R[x]$.

19. (a) Let I be an ideal of the ring R. Further, assume that the subset $S \subseteq R$ is closed under multiplication and disjoint from I. Then prove that there exists an ideal R which is maximal in the set of ideals which contain I and do not meet S; any such ideal is necessarily prime.

Or

(b) Show that a ring R is a primary ring if and only if it has a minimal prime ideal which contain all zero divisors.

20. (a) Let $I_1, I_2, \ldots I_n$ be a finite set of ideals of the ring R. If $I_i + I_j = R$, whenever $i \neq j$, then $R/\bigcap I_i \simeq \Sigma \oplus \left(\frac{R}{I_i}\right)$.

Or

(b) If R is a ring for which $R^{\vee} \neq \{0\}$, then

(i) ann $R^{\vee}$ is a maximal ideal of R

(ii) $ann\ R^{\vee}$ consists of all zero divisors f R, plus zero

(iii) whenever R is without prime radical, R forms a field.

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