(7 pages) Reg. No. : ....

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M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

First Semester

Mathematics — Core

ALGEBRA — I

(For those who joined in July 2021–2022 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If G is a group and H is a subgroup of index 2 in G then
   (a) H is a normal subgroup of G
   (b) H is a abelian in G
   (c) $aHa^{-1} \in H, a \in G$
   (d) None of these

2. Let N be a subgroup of G and every left coset of N in G is the right coset of N in G then which one of the following is false?
   (a) N is a normal subgroup of G
   (b) $N_a N_b = N_{ab}$ for all $a, b \in G$
   (c) G/N is a group
   (d) None of these

3. Let G be a group of order 36 and let H be a sub group of order 9, then H contains a normed sub group of order
   (a) 3 or 4 (b) 5 or 7
   (c) 3 or 6 (d) 3 or 9

4. A group G is said to be solvable if there exist subgroups $G = N_0 \supset N_2 \supset \ldots \supset N_i = (e)$ such that
   (a) each N_i is normal in N_{i-1}
   (b) N_{i-1}/N_1 is abelian
   (c) Both (a) and (b) are true
   (d) None of these

5. Let A_n be the set of all even permutations in S_n Then $(s, A_n) =$
   (a) $n!$ (b) $\frac{n!}{2}$
   (c) n (d) $\frac{n}{2}$

6. S_p^n has a P-sylow subgroup of order
   (a) $p^{N(s)}$ (b) p^n
   (c) p (d) None

7. Let G be a group of order 72 Then
   (a) G must have a non trivial normal subgroups
   (b) G is simple
   (c) G is not having any normal subgroup
   (d) None

8. Product of an even permutation and an odd permutation is
   (a) odd (b) even
   (c) both (d) none

9. Every finite abelian group is the direct product of ________
   (a) Subgroups (b) Abelian groups
   (c) Cyclic groups (d) None

10. The number of non isomorphic abelian groups of order p^n, p prime, equals
    (a) $p(n)$ (b) p^2
    (c) p (d) None

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that HK is a subgroup of G iff HK = KH.

    Or

    (b) A subgroup N of a group G is a normal subgroup of $G \Leftrightarrow$ every left cosets of N in G is a right cosets of N in G. prove this.

12. (a) Prove that $I(G) \cong G/Z$, where $I(G)$ is the group of inner automorphisms of G and Z is the centre of G.

    Or

    (b) Prove that an automorphism group of an infinite cyclic group is a group of order 2.

13. (a) Prove that S_n has A_n as normal subgroup of index 2.

    Or

    (b) If $O(G) = p^2$ where p is a prime number, then show that G is abelian.

14. (a) Prove that any two p-sylow subgroups of a group G are conjugate to each other.

Or

(b) Find the number of 11 sylow subgroups and 13 sylow subgroups of a group of order $11^2 \times 13^2$ and show that the group is abelian.

15. (a) Suppose G is the internal direct product of $N_1, N_2, \ldots N_n$. Then for $i \neq j$, show that $Ni \cap Nj = \{e\}$ and if $a \in Ni, b \in Nj$ then $ab = ba$.

Or

(b) Let G and G' are isomorphic abelian groups then show that for every integer s, $G(s)$ and $G'(s)$ are isomorphic.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) (i) If H and K are finite subgroups of orders $O(H)$ and $O(K)$ then Show that $O(HK) = O(H)O(K)/O(H \cap K)$.

(ii) Suppose H and K are the subgroup of a group G and $O(H) > \sqrt{O(G)}$, $O(K) > \sqrt{O(G)}$. Then show that $H \cap K \neq \{e\}$.

Or

(b) State and prove Cauchy's theorem for abelian groups.

17. (a) If G is a group, H is a subgroup of G, and S is the set of all right cosets of H in G, then show that there is a homomorphism θ of G into $A(S)$, and the Kernal of θ is the largest normal subgroup of G, which is contained in H.

Or

(b) (i) Let G be a group. Prove that $\mathcal{J}(G)$, then set of all inner automorphisms of G, is subgroup of $A(G)$.

(ii) Also prove that $A(G) \cong G/Z$ is the centre of the group G.

18. (a) State and prove Cauchy's theorem for general group.

Or

(b) Prove that the number of conjugate class on S_n is $p(n)$, the number of partions of n. Also Prove that $a \in Z$ if and only if $N(a) = G$.

19. (a) State and prove Sylow's theorem for general groups.

Or

(b) State and prove third part of sylow's theorem.

20. (a) Let G be a group and suppose that G is the internal direct product of $N_1, N_2 \ldots N_n$. Let $T = N_1 \times N_2 \times \ldots N_n$. Then Prove that G and T are isomorphic.

Or

(b) Prove that two abelian groups of order p^n are isomorphic iff they have the same invariants.

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