(8 pages) Reg. No. : ....

Code No. : 7381 Sub. Code : ZMAE 31

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

Third Semester

Mathematics — Elective

ALGEBRAIC NUMBER THEORY

(For those who joined in July 2021-2022 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If $ax+by=c$ is solvable then it has a solution x_0, y_0 with

(a) $0 \le x_0 < |b|$ (b) $0 \ge y_0 \ge |b|$

(c) $|a| \le x_0 \le b$ (d) $|a| > y_0 > |b|$

2. Let a,b,c be positive integers, then there is no solution of $ax+by=c$ in positive integers if

(a) $a+b<c$ (b) $a+b=c$

(c) $a+b>c$ (d) $a>b+c$

3. Consider a right angle triangle the area cannot be a perfect square then

(a) The length of the sides are integers

(b) The length of the sides are rational

(c) The length of the sides are irrational

(d) The length of the sides are in decimals

4. If x,y,z are integers such that $x^2+y^2+z^2=2xyz$ then

(a) $x \ne y \ne z$ (b) $x = y \ne z$

(c) $x \ne y = z$ (d) $x = y = z = 0$

5. If we define $r_n = \langle a_0, a_1, \ldots a_n \rangle$ for all integers $n \ge 0$ then

(a) $r_n = k_n / h_n$ (b) $r_n = \dfrac{h_n}{k_n}$

(c) $r_n = \dfrac{k_n h_n}{r}$ (d) $r_n = \dfrac{h_{n-1}}{k_n}$

6. Any finite simple continued fraction represents a/an

(a) Real number (b) Irrational number

(c) Rational number (d) Decimal

7. If α is any integer and ϵ any unit in $\mathbb{Q}(\sqrt{m})$ then

(a) ϵ / α (b) α / ϵ

(c) $\sqrt{m}/2$ (d) $\epsilon / \sqrt{m}$

8. A quadratic field $\mathbb{Q}(\sqrt{m})$ is called real if

(a) $m=1$ (b) $m<0$

(c) $m>1$ (d) $m>0$

9. Let $\mathbb{Q}(\sqrt{m})$ have a unique factorization property if $(2,m)=1$ then 2 is the associate of a square of a prime if

(a) $m \equiv 3 \pmod 4$ (b) $m \equiv 2 \pmod 4$

(c) $m \equiv 4 \pmod 3$ (d) $m \equiv 2 \pmod 5$

10. $\sqrt{3}-1$ and $\sqrt{3}+1$ are associates in

(a) $\mathbb{Q}(\sqrt{2})$ (b) $\mathbb{Q}(\sqrt{3})$

(c) $\mathbb{Q}(\sqrt{-2})$ (d) $\mathbb{Q}(\sqrt{-3})$

PART B — (5 × 5 = 25 marks)

Answer ALL the questions, choosing either (a) or (b).

11. (a) If the system of linear equations $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \ldots a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$ has a real solution and the system of congruences $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \equiv b_1 \pmod q$, $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \equiv b_2 \pmod q \ldots$ $a_{m1} + x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \equiv b_m \pmod q$ has a solution for every modules q, then prove that the equations $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n} + x_n = b_1$, $a_{2}, x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \ldots a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n = b_m$ have an integral solution?

Or

(b) Prove that the positive primitive solutions of $x^2+y^2=z^2$ with y even are $x=r^2-s^2$, $y=2rs$, $z=r^2+s^2$ where r and s are arbitrary integers and opposite parity with $r>s>0$ and $(r,s)=1$?

[P.T.O.]

12. (a) Determine whether the Diophantine equation $x^2 - 5y^2 - 91z^2 = 0$ has a non-trivial integral solution?

Or

(b) Let λ, μ, γ be positive real numbers with product $\lambda\mu\upsilon = m$ an integer then prove that any congruence $\alpha x + \beta y + \gamma z \equiv 0 (\mathrm{mod}\, m)$ has a solution x, y, z not all zero such that $|z| \le \lambda$, $|y| \le \mu$ $|z| \le \upsilon$?

13. (a) If a polynomial equation with integral co-efficients $C_n x^n + C_{n-1} x^{n-1} + \dots + C_2 x^2 + C_1 x + C_0 = 0$, $C_n \ne 0$ has a non zero rational solution a/b where the integers a and b are relatively prime then prove that a/c_0 and b/c_n?

Or

(b) Prove that two distinct infinite simple continued fractions converge to different values?

14. (a) Let Σ denote any irrational number. If there is a rational number a/b with $b \ge 1$ such that $\left|\Sigma - \frac{a}{b}\right| < \frac{1}{2b^2}$ then prove that a/b equals one of the convergents of the simple continued fraction expansion of Σ?

Or

(b) Prove that the minimal equation of an algebraic integer is monic with integral co-efficients?

15. (a) Let $\mathbb{Q}(\sqrt{m})$ have the unique factorization property then prove that any prime π in $\mathbb{Q}(\sqrt{m})$ there corresponds one and only one rational prime p such that π / p?

Or

(b) If the norm of an integer α in $\mathbb{Q}(\sqrt{m})$ is $\pm P$ where p is a rational prime then prove that α is a prime?

PART C — (5 × 8 = 40 marks)

Answer ALL the questions, choosing either (a) or (b)

16. (a) Prove that the equation $15x^2 - 7y^2 = 9$ has no solution in integers.

Or

(b) Let U be an $m \times n$ matrix with integral elements then prove the following are equivalent?

(i) U is unimodular

(ii) The inverse matrix U^{-1} exists and has integral elements

(iii) U may be expressed as a product of elementary row matrices $U = R_g R_{g-1} \dots R_2 R_1$

(iv) U may be expressed as a product of elementary column matrices $U = C_1 C_2 \dots C_{h-1} C_h$.

17. (a) Determine whether the Diophantine equation $x^2 + 3y^2 + 5z^2 + 7xy + 9yz + 11zx = 0$ has a nontrivial integral solution?

Or

(b) Let a, b and c be any arbitrary integers then prove that the congruence $ax^2 + by^2 + cz^2 \equiv 0 \,(\mathrm{mod}\, p)$ has a nontrivial solution $(\mathrm{mod}\, p)$?

18. (a) Prove that π is irrational.

Or

(b) Prove that the value of any infinite simple continued fraction $\langle a_0, a_1, a_2 \dots \rangle$ is irrational?

19. (a) Show that the set of all algebraic numbers formula field and the set of all algebraic integers forms a ring?

Or

(b) If the norm of a product equals the product of the norms then prove that $N(\alpha\beta) = N(\alpha N(\beta) N(\alpha) = 0$ if and only if $\alpha = 0$ and if the norm of an integer in $\mathbb{Q}(\sqrt{m})$ is a rational integer, then prove that if γ is an integer in $\mathbb{Q}(\sqrt{m})$ then $N(\gamma) = \pm 1$ if and only if γ is a unit?

20. (a) Let $\mathbb{Q}(\sqrt{m})$ have the unique factorization property then prove that (i) Any rational prime P is either a prime π of the field or a product π, π_2 of two primes, not necessarily distinct of $\mathbb{Q}(\sqrt{m})$. (ii) An odd rational prime P satisfying $(P, m) = 1$ is a product π, π_2 of two primes in $\mathbb{Q}(\sqrt{m})$ if and only if $\left(\frac{m}{P}\right) = 1$?

Or

(b) Prove that the fields $\mathbb{Q}(\sqrt{m})$ for $m = -1, -2, -3, -7, 2, 3$ are Euclidean and so have the Unique factorization property?

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