

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2023.

First Semester

Mathematics — Core

REAL ANALYSIS — I

(For those who joined in July 2023 onwards)

Time : Three hours

Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

- If there exists a positive number M such that $\sum_{k=1}^n |\Delta f_k| \leq M$ for all partitions of $[a, b]$, then f is said to be of _____ variation on $[a, b]$.
(a) equal (b) less than
(c) bounded (d) unbounded
- The total variation $V_f(a, b) = 0$ if and only if f is _____ on $[a, b]$.
(a) constant (b) variable
(c) equal (d) ten

- If α be continuous and $f \uparrow$ on $[a, b]$, then there exists a point x_0 in $[a, b]$ such that $\int_a^b f(x) d\alpha(x) = f(a) \int_a^{x_0} d\alpha(x) +$ _____
(a) $f(a) \int_{x_0}^b d\alpha(x)$ (b) $f(b) \int_{x_0}^b d\alpha(x)$
(c) $\int_b^{x_0} f(b) d\alpha(x)$ (d) $\int_{x_0}^b d\alpha(x)$
- For $f \in R$ and α a continuous function on $[a, b]$ whose derivative α' is Riemann integrable on $[a, b]$ then $\int_a^b f(x) d\alpha(x) =$ _____
 $\int_a^b f(x) \alpha'(x) dx$.
(a) = (b) \neq
(c) < (d) >
- The double series is said to be _____ to the sum a if $\lim_{p, q \rightarrow \infty} S(p, q) = a$.
(a) converge (b) diverge
(c) oscillate (d) all
- The radius of convergence of the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ is _____
(a) 0 (b) 1
(c) ∞ (d) 2

- A series $\sum a_n$ is called _____ convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.
(a) absolutely (b) conditionally
(c) equally (d) unequally
- The value of $\int_a^a f dx$ is _____
(a) 1 (b) ∞
(c) 0 (d) fx
- Name the condition that f satisfies on $[a, b]$ that if for every $\epsilon > 0$, there exists a partition P_ϵ such that P finer than P_ϵ implies $0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ with respect to α .
(a) Riemann's (b) Cauchy
(c) Euler (d) Newton's
- If $f \in R(\alpha)$ and $g \in R(\alpha)$, where $\alpha \uparrow$ on $[a, b]$ then the product $fg \in$ _____
(a) $R(\alpha)^2$ (b) $R(\alpha).R(\alpha)$
(c) $2R(\alpha)$ (d) $R(\alpha)$
- One of the sufficient condition for the existence of the Riemann integral $\int_a^b f(x) dx$ is _____ on $[a, b]$.
(a) f is of bounded variation
(b) f is not continuous
(c) f unbounded
(d) f is of unbounded variation

- The series $\sum (-1)^{n+1} n$ is _____
(a) (C,1) summable (b) not (C,1) summable
(c) Summable (d) not summable
- A sequence $\{f_n\}$ is said to be _____ on S if there exists a constant $M > 0$ such that $|f_n(x)| \leq M$ for all x in S and all n .
(a) Limit (b) Converge
(c) Uniformly bounded (d) Continuous
- A sequence of functions $\{f_n\}$ is said to be _____ on T if $\{f_n\}$ is pointwise convergent and uniformly bounded on T .
(a) integrable
(b) converge
(c) boundedly convergent
(d) divergent
- For the function $f_n(x) = x^n$ in $0 \leq x \leq 1$ which is continuous with discontinuous limit, the convergence is _____ on $[0, 1]$.
(a) uniform (b) not uniform
(c) both (a) and (b) (d) cannot say

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If f is monotonic on $[a, b]$, then prove that the set of all discontinuities of f is countable.

Or

- (b) Assume that f and g are each of bounded variation on $[a, b]$. Then prove that their sum and product are also of bounded variation.

17. (a) Assume that α' on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$ then prove that $f^2 \in R(\alpha)$ on $[a, b]$.

Or

- (b) If $f, g \in R(\alpha)$ on $[a, b]$ then show that $C_1f + C_2g \in R(\alpha)$ on $[a, b]$ for any two constants C_1 and C_2 and also

$$\int_a^b (C_1f + C_2g) d\alpha = C_1 \int_a^b f d\alpha + C_2 \int_a^b g d\alpha.$$

18. (a) If f is continuous on $[a, b]$ and if α is of bounded variation on $[a, b]$ then show that $f \in R(\alpha)$ on $[a, b]$.

Or

- (b) State and prove second mean value theorem for Riemann integrals.

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22. (a) Assume that α' on $[a, b]$. Then prove that the following statements are equivalent.

- (i) $f \in R(\alpha)$ on $[a, b]$
 (ii) f satisfies Riemann's condition with respect to α on $[a, b]$
 (iii) $\underline{I}(f, \alpha) = \bar{I}(f, \alpha)$.

Or

- (b) Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having end points c and d . Assume that $a = g(c)$, $b = g(d)$. Let h and β be the composite functions defined as $h(x) = f[g(x)]$, $\beta(x) = \alpha[g(x)]$ if $x \in S$. Then prove that $h \in R(\beta)$ on S and $\int_a^b f d\alpha = \int_c^d h d\beta$

23. (a) State and prove theorem on change of variable in Riemann integral.

Or

- (b) Assume that α is of bounded variation on $[a, b]$. Let $V(x)$ denote the total variation of α on $[a, x]$ if $a < x \leq b$ and $V(a) = 0$. Let f be defined and bounded on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$ then prove that $f \in R(V)$ on $[a, b]$.

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19. (a) Prove that if a series is convergent with sum S , then it is also $(C, 1)$ summable with Cesaro sum S .

Or

- (b) Let $a_n \geq 0$ then prove that the product $\prod (1 - a_n)$ converges if and only if the series $\sum a_n$ converges.

20. (a) State and prove Dirichlet's test for uniform convergence.

Or

- (b) Assume $f_n \rightarrow f$ uniformly on S . If each f_n is continuous at a point C of S then prove that the limit function f is also continuous at C .

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

21. (a) Let f be of bounded variation on $[a, b]$. If $x \in (a, b)$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Then prove that every point of continuity of f is also a point of continuity of V prove converse also.

Or

- (b) State and prove Riemann's theorem on conditionally convergent series.

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24. (a) Write Bernstein's theorem and prove it.

Or

- (b) State and prove Merten's theorem.

25. (a) State and prove three examples of sequences of real valued functions.

Or

- (b) State and prove the theorem on Cauchy condition for uniform convergence.

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