(8 pages) Reg. No. : ....

Code No. : 7758 **Sub. Code : WMAE 15**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

First Semester

Mathematics – Elective – II

ANALYTIC NUMBER THEORY

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. $d/n \Rightarrow$ ____________.

   (a) n/d (b) $d+1/n$

   (c) ad/an (d) $d/n+1$

2. $(a, b) =$ ____________.

   (a) (b, a) (b) $(a+1, b)$

   (c) $(a, b+1)$ (d) $(a+1, b+1)$

3. $(8, 20) =$ ____________.

   (a) 2 (b) 8

   (c) 20 (d) 4

4. $\mu(4) =$ ____________.

   (a) 1 (b) 0

   (c) 8 (d) -1

5. $I(7) =$ ____________.

   (a) 0 (b) 1

   (c) 4 (d) -1

6. $\phi(6) =$ ____________.

   (a) 1 (b) -3

   (c) 2 (d) 4

7. If f is multiplicative, then $f(1) =$ ____________.

   (a) 0 (b) -1

   (c) 1 (d) 6

8. $\lambda(1) =$ ____________.

   (a) 6 (b) 3

   (c) -1 (d) 1

9. $d(n)$ is the ____________.

   (a) number of divisiors of n

   (b) n

   (c) n^2

   (d) sum of the divisiors of n

10. Euler's constant $c =$ ____________.

    (a) $\lim_{n\to\infty} \log n$

    (b) $\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)$

    (c) $\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$

    (d) n^2

11. If $f(x)=0(g(x))$, $g(x)>0$, then ____________.

    (a) $\frac{f(x)}{g(x)}$ is bounded for all $x \geq a$

    (b) $\frac{g(x)}{f(x)}$ is bounded for all $x \geq a$

    (c) $\frac{f(x)+1}{g(x)+1}$ is bounded for all $x \geq a$

    (d) $f(x)g(x)$ is bounded for all $x \geq a$

12. If $f(x) \sim g(x)$ as $x \to \infty$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ ____________.

    (a) 0 (b) 7

    (c) 10 (d) 1

13. If $a(n)$ is the characteristic function of prime, then $a(7) =$ ____________.

    (a) 1 (b) 0

    (c) -1 (d) 2

14. $U\left(\frac{1}{2}\right) =$ ____________.

    (a) -1 (b) -2

    (c) 0 (d) 1

15. For $x>0$, the chebyshev's χ-function $\chi(x) =$ ____________.

    (a) $\wedge(n)$ (b) $\sum_{n\leq x} \wedge(n)$

    (c) $\sum_{n=0}^{x} \wedge(n)$ (d) $\prod_{n\leq x} \wedge(n)$

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that every integer $n > 1$ is either a prime number or a product of prime numbers.

Or

(b) If $(a, b) = 1$, then prove that $(a + b, a^2 - ab + b^2)$ is either 1 or 3.

17. (a) If $n \geq 1$, prove that

$$\sum_{d/n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & if \quad n = 1 \\ 0 & if \quad n > 1 \end{cases}$$

Or

(b) If $n \geq 1$ prove that $\phi(n) = \sum_{d/n} \mu(d) \frac{n}{d}$.

18. (a) If f and g are multiplicative, then prove that their Dirichlet product $f * g$ is also multiplicative.

Or

(b) Let $f(n) = [\sqrt{n}] - [\sqrt{n-1}]$. Then prove that f is multiplicative but not completely multiplicative.

19. (a) For all $x \geq 1$, prove that $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2}$ and $(2) x^2 + 0(x \log x)$.

Or

(b) If $\beta > 0$, let $\delta = Min\{0, 1 - \beta\}$. If $x > 1$, prove that $\sum_{n \leq x} \sigma_{-\beta}(n) = \xi(\beta + 1) + 0(x^{\delta})$ if $\beta \neq 1$ = $\xi(2) x + 0(\log x)$ if $\beta = 1$.

20. (a) For $x \geq 1$, prove that $\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1$ and $\sum_{n \leq x} \wedge(n) \left[\frac{x}{n}\right] = \log[x]!$

Or

(b) If $x \geq 2$, prove that $\log[x]! = x \log x - x + 0(\log x)$ and hence $\sum_{n \leq x} \wedge(n) \left[\frac{x}{n}\right] = x \log x - x + 0(\log x)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

21. (a) State and prove Fundamental theorem of arithmetic.

Or

(b) State and prove the Euclidean algorithm.

22. (a) For $n \geq 1$, prove that $\phi(n) = n \prod_{p/n} \left(1 - \frac{1}{p}\right)$.

Or

(b) If f is an arithmetical function with $f(1) \neq 0$, then prove that there is a unique arithmetical function f^{-1} such that $f * f^{-1} = f^{-1} * f = I$. Moreover f^{-1} is given by $f^{-1}(1) = \frac{1}{f(1)}$ and $f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d/n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d)$, for $n > 1$.

23. (a) Let f be multiplicative. Then prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$.

Or

(b) Prove that $\sigma_1(n) = \sum_{d/n} \phi(d) \sigma\left(\frac{n}{d}\right)$ and derive a generalization involving $\sigma_\alpha(n)$.

24. (a) State and prove Euler's summation formula.

Or

(b) For all $x \geq 1$, prove that $\sum_{n \leq x} d(n) = x \log x + (2c - 1)x + o(\sqrt{x})$.

25. (a) For $x > 0$, prove that $0 \leq \frac{\chi(x)}{x} - \frac{\gamma(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x} \log 2}$.

Or

(b) Prove that the following are logically equivalent

(i) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$

(ii) $\lim_{x \to \infty} \frac{\gamma(x)}{x} = 1$

(iii) $\lim_{x \to \infty} \frac{\chi(x)}{x} = 1$.

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