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Reg. No. : ....

Code No.: 7756 **Sub. Code: WMAE 13**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023

First Semester

Mathematics

Elective I – ALGEBRAIC NUMBER THEORY

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — $(15 \times 1 = 15$ marks)

Answer ALL questions.

Choose the correct answer :

1. Let a and b are non zero and let $g = g.c.d.(a.b)$ in the equation $ax + by = c$. If $g \mid c$ then it has ———— solutions.

   (a) finite

   (b) no

   (c) every pair of integers is a

   (d) infinitely many

2. In the equation $ax + by = c$, if $a = b = c = 0$, then it has———— solutions.

   (a) finite

   (b) no

   (c) every pair of integers is a

   (d) infinitely many

3. One of the solution of the equation $4x + 2y = 8$ ————

   (a) (2,0) (b) (4,0)

   (c) (0,4) (d) (0,2)

4. In the equation $x^2 + y^2 = z^2$, the general value of z is ————

   (a) $r^2 + s^2$ (b) $r^2 - s^2$

   (c) $2rs$ (d) none of these

5. The equation $x^2 + y^2 = z^4$ has infinitely many solutions with $(x, y, z) =$ ————

   (a) 2 (b) 3

   (c) 0 (d) 1

6. The equation $x^4 + y^4 = z^2$ has no solution in ————

   (a) integers (b) positive integers

   (c) negative integers (d) rationals

7. The infinite simple continued fraction of $\sqrt{2}$ is ————

   (a) $\langle 2,4,4,4,....\rangle$ (b) $\langle 1,2,2,2,2,....\rangle$

   (c) $\langle 1,1,2,1,2,1,....\rangle$ (d) $\langle 0,2,2,2,2....\rangle$

8. The infinite simple continued fraction of $\sqrt{2} - 1$ is ————

   (a) $\langle 2,4,4,4,....\rangle$ (b) $\langle 1,2,2,2,2,....\rangle$

   (c) $\langle 1,1,2,1,2,1,....\rangle$ (d) $\langle 0,2,2,2,2....\rangle$

9. The value of the infinite simple continued fraction $\langle 1,1,1,1,1,...\rangle$ is————

   (a) $(2 - \sqrt{5})/2$ (b) $(1 + \sqrt{5})/2$

   (c) $(1 - \sqrt{5})/2$ (d) $(2 + \sqrt{5})/2$

10. Every good approximation to ξ is ————

    (a) convergent (b) divergent

    (c) approximation (d) fair approximation

11. Not every secondary convergent is ————

    (a) convergent (b) divergent

    (c) approximation (d) fair approximation

12. If α is any algebraic number and b is any rational integer, then $b\alpha$ is an ————

    (a) Algebraic integers

    (b) Rational integer

    (c) Gaussian integer

    (d) None of these

13. A quadratic field $Q(\sqrt{m})$ is called real if ————

    (a) $m > 0$ (b) $m < 0$

    (c) $m > 1$ (d) $m < 1$

14. The field $Q(\sqrt{-1})$ is ————

    (a) Euclidean

    (b) has the unique factorization property

    (c) both (a) and (b)

    (d) none of these

15. In $Q(\sqrt{6})$, 3, is ————

    (a) prime

    (b) not a prime

    (c) unit

    (d) not a unit

[P.T.O.]

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let a, b, c be any positive integer. Prove that there is no solution of $ax + by = c$ in positive integers, if $a + b > c$.

Or

(b) Let $ax + by = c$ has two solutions, (x_0, y_0) (x_1, y_1) with $x_1 = 1 + x_0$ and let $(a, b) = 1$. Then prove that $b = \pm 1$

17. (a) Show that the equation $x^2 + y^2 = 9z + 3$ has no integral solution.

Or

(b) Find all the solutions for the equation $x^2 + y^2 = z^2$ if $0 < z < 30$.

18. (a) Evaluate the infinite continued fraction $\langle 2, 2, 2, \ldots. \rangle$

Or

(b) Prove that two distinct infinite simple continued fractions converges to different values.

19. (a) Prove that the reciprocal of a unit is unit.

Or

(b) Show that the convergents h_n / k_n are successively closer to ξ.

20. (a) Prove that the units of $Q(\sqrt{2})$ are $\pm(1 + \sqrt{2})^n$ where n ranges over all integers.

Or

(b) Prove that $Q(\sqrt{-1})$ has the unique factorization property.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

21. (a) Find all solutions of $999x - 49y = 5000$

Or

(b) Find all solutions in integers of the Simultaneous Equations $20x + 44y + 50z = 10$, $17x + 13y + 11z = 19$.

22. (a) Prove that the equation $x^4 + y^4 = z^2$ has no solution in positive integers.

Or

(b) (i) Show that the equation $15x^2 - 7y^2 = 9$ has no solutions in integers.

(ii) Show that the equation $x^3 + 2y^3 + 4z^3 = 9w^2$ has no nontrivial solutions.

23. (a) Evaluate the infinite continued fractions $\langle 1, 2, 1, 2, \ldots\ldots \rangle$ and $\langle 1, 3, 1, 2, 1, 2 \ldots\ldots \rangle$

Or

(b) Expand the irrational numbers $\sqrt{2}/2$, $\sqrt{3}$ as infinite simple continued fractions.

24. (a) Prove that the integers of any algebraic number field form a ring.

Or

(b) Prove that the units of the rational number field Q are ± 1 and that the integers α and β are associates in this field if and only if $\alpha = \pm\beta$.

25. (a) Prove that $Q(\sqrt{-7})$ have the unique factorization property.

Or

(b) Prove that every Euclidean quadratic field has the unique factorization property.

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