(8 pages) Reg. No. : ....

Code No. : 5380 **Sub. Code : ZMAM 34**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023

Third Semester

Mathematics — Core

TOPOLOGY — I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Let $X = \{a, b, c, d\}$ which one of the following is a topology on X

(a) $\{\phi, X, \{a\}, \{c, d\}\}$

(b) $\{X, \{a\}, \{c\}, \{a, c\}\}$

(c) $\{\phi, X, \{a, b\}, \{c, d\}\}$

(d) $\{\phi, \{a\}, \{c, d\}, \{a, c, d\}\}$

2. $[0, 2)$ is

(a) open in $\mathbb{R}$ but not open in R_l

(b) open in R_l but not open in $\mathbb{R}$

(c) open in R_l and open in $\mathbb{R}$

(d) not open in R_l and not open in $\mathbb{R}$

3. If $f : R \to R$ is defined by $f(n) = 3x + 1$, then $g : R \to R$ is the inverse of f is

(a) $g(y) = 3y - 1$

(b) $g(y) = -3y + 1$

(c) $g(y) = \frac{1}{3}(y - 1)$

(d) $g(y) = 3(y - 1)$

4. Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$ and $f_2 : A \to Y$ are two functions. If $U \times V$ is a basis element for the product topology of $X \times Y$, then $f^{-1}(U \times V)$ is

(a) $f_1^{-1}(U) \cap f_2^{-1}(V)$ (b) $f_1^{-1}(U) \cup f_2^{-1}(V)$

(c) $f_1(U) \cap f_2(V)$ (d) $f_1(U) \cup f_2(V)$

5. In R, define $d(x, y) = |x - y|$. Then $B(3, 0.7)$ is

(a) (3, 3.7) (b) (2.3, 3.7)

(c) (– 0.7, 0.7) (d) (3.7, 4.4)

6. The standard bounded metric $\overline{d}$ corresponding to d is defined by

(a) $\max\{1, d(x, y)\}$

(b) $\min\{d(x, y), d(y, x)\}$

(c) $\min\{1, d(x, y)\}$

(d) $1 + d(x, y)$

7. Let $X = \{a, b, c, d\}$ and let $I = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ then

(a) $(\{a, b\}, \{c, d\})$ is a separation of X

(b) $(\{a\}, \{b, c, d\})$ is a separation of X

(c) $(\{a\}, \{b\})$ is a separation of X

(d) X has no separation

8. Which one of the following is an open covering for $\mathbb{R}$?

(a) $\{(n, n+1) / n \in z\}$ (b) $\{(n-1, n) / n \in z\}$

(c) $\{(n, n+2) / n \in z\}$ (d) $\{[n, n+1] / n \in z\}$

9. A space X is said to be limit point compact is

(a) every point is a limit point

(b) every infinite subset of X has a limit point

(c) every subset of X has a limit point

(d) every finite subset of X has a limit point

10. The one-point compactification of the real line $\mathbb{R}$ is

(a) S^1

(b) S^2

(c) the extended complex plane

(d) $\mathbb{C}$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies τ and τ' respectively on X. Prove that τ' is fines than τ if and only if for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Or

(b) Prove that every finite point set in a Hausdorff space X is closed.

12. (a) State and prove the pasting lemma.

Or

(b) Let $\{X_\alpha\}$ be an indexed family of spaces. Let $A_\alpha \subset X_\alpha$ for each α. If πX_α is given either the product or the box topology, prove that $\pi \overline{A_\alpha} = \overline{\pi A_\alpha}$.

13. (a) Let d and d' be two metrics on the set X, and let τ and τ' be the topologies they induce, respectively. Prove that τ' is fines than τ if and only if for each x in X and each $\epsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x\ \delta) \subset B_d(x, \epsilon)$.

Or

(b) State and prove the sequence lemma.

14. (a) Prove that the image of a connected space under a continuous map is connected.

Or

(b) Prove that every compact subspace of a Hausdorff space is closed.

15. (a) Give an example of a limit point compact space which is not compact.

Or

(b) Let X be a Hausdorff space. Prove that X is exactly compact if and only if given x in X and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Define a topology. Write down any three topologies on $X = \{a, b, c\}$. Define the finite complement topology on a set X and show that it is a topology on X.

Or

(b) Let X be a topological space. Prove that

(i) ϕ and X are closed

(ii) arbitrary intersections of closed sets are closed

(iii) finite unions of closed sets are closed

17. (a) Let X and Y be topological spaces, let $f : X \to Y$. Prove that the following are equivalent.

(i) For every closed set B of Y, the set $f'(B)$ is closed in X

(ii) f is continuous

(iii) for every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$.

Or

(b) If each space X_α is a Hausdorff space, prove that πX_α is a Hausdorff space in both the box and product topologies.

18. (a) Define a suitable metric D on on R^ω and show that D induces the product topology on R^ω.

Or

(b) Show that R^ω in the box topology is not metrizable.

19. (a) Prove that a finite Cartesian product of connected space is connected.

Or

(b) Prove that the product of two compact spaces is compact.

20. (a) Define a limit point compact space and a sequentially compact space. If X is metrizable, prove that if X is limit point compact then it is sequentially compact.

Or

(b) If X is a locally compact Hausdorff space that is not itself compact, then prove that X has a one-point compactification Y.