(7 pages) Reg. No. : ....

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M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Third Semester

Mathematics – Core

MEASURE AND INTEGRATION

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If Q is the set of rational numbers then the outer measure of Q is

(a) ∞ (b) 0

(c) 1 (d) $\frac{22}{7}$

2. A set of real numbers is said to be a G_δ set provided it is the

(a) intersection of a finite collection of open sets

(b) union of a countable collection of closed sets

(c) intersection of a countable collection of open sets

(d) union of a countable collection of open sets

3. $|f|(x)$ is defined by

(a) $\max\{f(x),0\}$ (b) $\max\{-f(x),0\}$

(c) $\max\{f(x),-f(x)\}$ (d) $f^{+}(x)-f^{-}(x)$

4. Consider the statements

(A) Pointwise limit of continuous functions is continuous

(B) Pointwise limit of Riemann integrable functions is Riemann integrable

(a) Both (A) and (B) are true

(b) Neither (A) nor (B) is not true

(c) (A) is true but (B) is not true

(d) (A) is not true but (B) is true

5. If $\phi = 2\chi_{[0,3]} + 5\chi_{[7,9]}$ then $\int_{[0,3]\cup[7,9]} \phi$ is

(a) 16 (b) 7

(c) 5 (d) 0

6. If E_0 is a subset of E of measure zero, then $\int_E f$ is

(a) $\int_{E_0} f$ (b) $\int_{E-E_0} f$

(c) $\int_{E_0-E} f$ (d) zero

7. For an extended real valued function f on E, $f+|f|$ is

(a) $f^{+}+f^{-}$ (b) $2f^{+}$

(c) $-2f^{-}$ (d) 0

8. The function $Av_h f$ is defined for all $x \in [a,b]$ and $0 < h \le 1$ by

(a) $\frac{1}{h}\int_x^{x+h} f$ (b) $\frac{f(x+h)-f(x)}{h}$

(c) $\frac{1}{h}\int_{x-h}^{x+h} f$ (d) $\frac{1}{h}\int_a^b f$

9. Which one of the following is not true?

(a) Linear combinations of absolutely continuous functions are absolutely continuous

(b) Composition of absolutely continuous functions is absolutely continuous

(c) If f is Lipschitz on [a,b] then f is absolutely continuous

(d) Absolutely continuous functions are continuous

10. If $\{A,B\}$ is a Hahn decomposition for γ, then γ^{+} is defined by

(a) $\gamma^{+}(E)=\gamma(E)$ (b) $\gamma^{+}(E)=\gamma(E\cup A)$

(c) $\gamma^{+}(E)=\gamma(E\cap A)$ (d) $\gamma^{+}(E)=-\gamma(E\cap A)$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that outer measure is countably sub additive.

Or

(b) Define a measurable set and show that the translate of a measurable set is measurable.

[P.T.O]

12. (a) Let the function f be defined on a measurable set E. Prove that f is measurable if and only if for each open set, the inverse image of the open set under f is measurable.

Or

(b) Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise *a.e.* on E to the function f. Prove that f is measurable.

13. (a) Construct a sequence $\{f_n\}$ of Riemann integrable functions whose limit function f is not Riemann integrable.

Or

(b) Let ϕ and Ψ be simple functions defined on a set of finite measure E. For any α and β, prove that $\int_E(\alpha\phi+\beta\Psi)=\alpha\int_E\phi+\beta\int_E\Psi$. Also show that if $\phi\le\Psi$ on E then $\int_E\phi\le\int_E\Psi$.

14. (a) State and prove the monotone convergence theorem.

Or

(b) Let C be a countable subset of (a,b). Prove that there is an increasing function on (a,b) that is continuous only at points in $(a,b)\sim C$.

15. (a) Define an absolutely continuous function. Give an example of a continuous function which is not absolutely continuous.

Or

(b) For an outer measure $\mu^*:2^x\to[0,\alpha]$, define a measurable set (w.r.t μ^x) and show that the union of two measurable sets is measurable.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Show that the outer measure of an interval is its length.

Or

(b) Let E be a measurable set of finite outer measure. For each $\varepsilon>0$, prove that there is a finite disjoint collection of open intervals.

$\{I_h\}_{h=1}^n$ for which it $O=\bigcup_{h=1}^n I_k$, then $m^*(E\sim O)+m^*(O\sim E)<\varepsilon$.

17. (a) State and prove Egoroff's theorem.

Or

(b) State and prove Lusin's theorem.

18. (a) State and prove the bounded convergence theorem.

Or

(b) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\}\to f$ pointwise a.e. on E, the prove that $\int_E f\le\liminf\int_E f_n$. Also show that the inequality may be strict.

19. (a) State and prove the Lebesgue dominated convergence theorem.

Or

(b) If the function f is monotone on the open interval (a,b), then prove that it is differentiable almost everywhere on (a,b).

20. (a) Let the function f be continuous on $[a,b]$. If the family of divided difference functions $\{Diff_h f\}_{0<h\le 1}$ is uniformly integrable over $[a,b]$, prove that f is absolutely continuous on $[a,b]$.

Or

(b) Let γ be a singed measure on the measurable space $(X,\mathfrak{M})$. Prove that there is a positive set A for γ and a negative set B for γ for which $X=A\cup B$ and $A\cap B=\phi$