(7 pages) **Reg. No. :** ....

Code No. : 5377 **Sub. Code : ZMAM 31**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Third Semester

Mathematics – Core

ADVANCED ALGEBRA – I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If V is a vector space of dimension 5 then dim $\dim Hom(Hom(V,F),F)$ is

   (a) 125

   (b) 10

   (c) 25

   (d) 5

2. If $U \subseteq W$ are subspaces of a vector space V and if $A(W)$ is the annihilator of W then

   (a) $A(U) \subseteq A(W)$

   (b) $A(U) \supseteq A(W)$

   (c) $A(U) = A(W)$

   (d) $A(U)$ and $A(W)$ are not comparable

3. If V is finite-dimensional over F, then $T \in A(V)$ is singular if and only if there exists a

   (a) $v \neq 0$ in V such that $vT = 0$

   (b) $v \neq 0$ in V such that $vT \neq 0$

   (c) $v = 0$ in V such that $vT = 0$

   (d) $v = 0$ in V such that $vT \neq 0$

4. If $m(s) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $m(T) = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ then $m(ST)$ is

   (a) $\begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 6 \\ 5 & 12 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 6 \\ 5 & 12 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ 5 & 7 \end{pmatrix}$

5. The matrix M_4 is

   (a) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

6. If M, of dimension 8, is cyclic w.r.t T, then the dimension of MT^3 is

   (a) 11 (b) 5

   (c) 24 (d) 8^3

7. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 7 & -6 & 3 & 5 \end{pmatrix}$ is the companion matrix of the polynomial

   (a) $7 - 6x + 3x^2 + 5x^3 + x^4$

   (b) $-7 + 6x - 3x^2 - 5x^3 - x^4$

   (c) $-7 + 6x - 3x^2 - 5x^3 + x^4$

   (d) $7 - 6x + 3x^2 + 5x^3$

8. If $A = \begin{pmatrix} 1 & 15 & 4 \\ 3 & -1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$ then $tr(A)$ is

   (a) 6 (b) 9

   (c) 5 (d) 7

9. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ then $(-1)^{\sigma}$ is

   (a) 1 (b) −1

   (c) σ (d) $-\sigma$

10. $T \in A(V)$ is normal if and only if

    (a) $TT^* = TT^*$ (b) $(T^*)^* = T$

    (c) $TT^* - T^*T = 0$ (d) $TT^* = 1$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If S, $T \in Hom(V,W)$ and $\lambda \in F$, define $S+T$ and λS. Also show that $S+T \in Hom(V,W)$.

    Or

    (b) If u, $v \in V$, Prove that $|u,v| \leq \|u\| \cdot \|v\|$.

12. (a) Let A be an algebra with unit element over F and suppose that A is of dimension m over F. Prove that every element in A satisfies some nontrivial polynomial in $F[x]$ of degree at most m.

Or

(b) If V is finite dimensional over F and S, $T \in A(V)$ prove that $r(ST) = r(TS) = r(T)$ for S regular in $A(V)$.

13. (a) Let V_1 be the subspace of V spanned by v, vT, $vT^2, \ldots, vT^{n_1-1}$, where $T \in A(V)$. If $u \in V_1$ is such that $uT^{n_1-k} = 0$ where $0 < k \le n_1$, prove that $u = u_0 T^k$ for some $u_0 \in V_1$.

Or

(b) If $W \subseteq V$ is invariant under T, prove that T induces a linear transformation $\overline{T}$ on V/W defined by $(v+W)\overline{T} = vT + W$.

14. (a) State and prove Jacobson lemma.

Or

(b) Prove that the determinant of a triangular matrix is the product of its entries on the main diagonal.

15. (a) If $T \in A(V)$ is Hermitian, prove that all its characteristic roots are real.

Or

(b) Show that congruence is an equivalence relation.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If V is finite-dimensional, prove that there is an isomorphism ψ of V onto $\hat{\hat{V}}$, the second dual of V.

Or

(b) If V is a finite dimensional inner product space, prove that V has an orthonormal set as a basis.

17. (a) If A is an algebra with unit element over F, prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F.

Or

(b) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ in F are distinct characteristic roots of $T \in A(U)$ and if $v_1, v_2, \ldots v_k$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2 \ldots, \lambda_k$ respectively, then prove that $v_1, v_2, \ldots v_k$ are linearly independent over F.

18. (a) If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) "Two nilpotent linear transformation are similar if and only if they have the same invariants" – prove.

19. (a) If the elements S and T in $A(V)$ are similar in $A(V)$, prove that they have the same elementary divisors.

Or

(b) If F is a field of characteristic 0 and if $T \in A(V)$ is such that $trT^i = 0$ for all $i \ge 1$, prove that T is nilpotent.

20. (a) If $T \in A(V)$ is such that $(vT, v) = 0$, for all $v \in V$, prove that $T = 0$. If V is an inner product space over the real field, prove that the above result may be false.

Or

(b) State and prove Sylvester's law.

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