(8 pages) **Reg. No. :** ....

Code No. : 5371 **Sub. Code : ZMAM 23**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Second Semester

Mathematics — Core

ADVANCED CALCULUS

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. The value of $\int_1^2 \sqrt{x}dx$ is

   (a) $\frac{4\sqrt{2}-2}{5}$ (b) $\frac{4\sqrt{2}-2}{3}$

   (c) $\frac{4\sqrt{2}-1}{3}$ (d) $\frac{\sqrt{2}-4}{3}$

2. Let D be the region between the line $y = x$ and the parabola $y = x^2$. Let $f(x, y) = xy^2$. Then $\iint_D f$ is

   (a) $\frac{1}{20}$ (b) $\frac{1}{42}$

   (c) $\frac{1}{30}$ (d) $\frac{1}{40}$

3. The image of the line $x = 0$ under the transformation $S : \begin{cases} u = x + y \\ v = x - y \\ w = x^2 \end{cases}$ is

   (a) $u = v$

   (b) the line $u + v = 0$, in the plane $w = 0$

   (c) $u = 0$, $v = 0$, $w = 0$

   (d) a circle

4. Let T be the linear transformation on R^2 into R^2 specified by the matrix $\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$. The image of the point (1, 2) is

   (a) (0, 3) (b) (3, 0)

   (c) (0, –3) (d) (–3, 0)

5. The Jacobian of the transformation $T : \begin{cases} u = x\cos y \\ v = x\sin y \end{cases}$ is

   (a) 0 (b) y

   (c) x (d) 1

6. If $T : \begin{cases} u = \cos(x + y^2) \\ v = \sin(x + y^2) \end{cases}$ then at (x, y) the Jacobian of T is

   (a) 0

   (b) $4y\sin(x + y^2)\cos(x + y^2)$

   (c) $2y\sin(x + y^2)\cos(x + y^2)$

   (d) 2

7. If F is additive on A, the $F(S_1 \cup S_2)$ is

   (a) $F(S_1) + F(S_2)$

   (b) $F(S_1) + F(S_2) + F(S_1 \cap S_2)$

   (c) $F(S_1) + F(S_2) - F(S_1 \cap S_2)$

   (d) $F(S_1) + F(S_2) - 2F(S_1 \cap S_2)$

8. The direction of the line through (1, 2, –1) towards (3, 1, 1) is

   (a) $\left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$ (b) (2, –1, 2)

   (c) $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ (d) $\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$

9. If $V = Ai + Bj + Ck$ then $curl(V)$ is

   (a) $(C_2 - B_3)i + (A_3 - C_1)j + (B_1 - A_2)k$

   (b) $(C_2 - B_3)i + (A_2 - C_1)j + (B_2 - A_1)k$

   (c) $(C_1 - B_1)i + (C_2 - B_2)j + (A_3 - B_1)k$

   (d) $(C_2 - B_3)i - (A_3 - C_1)j + (B_1 - A_2)k$

10. If α is a k form and β any differential form, then $d(\alpha\beta)$ is

    (a) $(d\alpha)\beta + (-1)^k\alpha(d\beta)$

    (b) $(d\alpha)\beta + (d\beta)\alpha$

    (c) $(d\alpha)\beta - \alpha(d\beta)$

    (d) $(-1)^k(d\alpha)\beta + (-1)^k\alpha(d\beta)$

PART B — (5 × 5 = 25 marks)

Answer ALL questions choosing either (a) or (b).

11. (a) Let f and g be continuous and bounded on D. Prove that $\iint_D |f|$ exists and $\left|\iint_D f\right| \le \iint_D |f|$.

    Or

    (b) Show that for $x > 0$
    $$\int_0^{\pi/2} \log(\sin^2\theta + x^2\cos^2\theta)d\theta = \pi\log\left(\frac{x+1}{2}\right).$$

12. (a) Let $T:\begin{cases} r = xy \\ s = 2x \\ t = -y \end{cases}$, $S:\begin{cases} u = r - s \\ v = st \end{cases}$.

Calculate the products ST and TS. Verify whether $ST = TS$ or not.

Or

(b) Define the differential of a transformation T compute the differential of

$$T:\begin{cases} u = x + 6y \\ v = 3xy \\ w = x^2 - 3y^2 \end{cases} \quad \text{at } (1, 1)$$

13. (a) Discuss the solution of the equations for u and v

$$\begin{cases} x^2 - yu = 0 \\ xy + uv = 0 \end{cases}.$$

Or

(b) Let T be of class C' in an open region D and let E be a closed bounded subset of D. Let dT/p_0 be the differential of T at a point $p_0 \in E$. Prove that

$T(p_0 + \Delta p) = T(p_0) + dT/p_0(\Delta p) + R(\Delta p)$ where $\lim_{\Delta p \to 0} \frac{|R(\Delta p)|}{|\Delta p|} = 0$ uniformly for $p_0 \in E$.

14. (a) If E is a closed bounded subset of Ω at zero volume, prove that $T(E)$ has zero volume.

Or

(b) If γ_1 and γ_2 are smoothly equivalent curves, prove that $L(\gamma_1) = L(\gamma_2)$.

15. (a) If ω is any differential form of class C'', prove that $dd\omega = 0$.

Or

(b) State and prove the divergence theorem for the case of a cube.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If f is continuous on R, prove that $\iint_R f$ exists.

Or

(b) Let R be the rectangle described by $a \le x \le b$, $c \le y \le d$ and let f be continuous on R. Prove that $\iint_R f = \int_a^b dx \int_c^d f(x, y) dy$.

17. (a) Let L be a linear transformation from R^n into R^m represented by the matrix $[a_{ij}]$. Prove that there is a constant B such that $|L(p)| \le B|p|$ for all points p. Also show that the number B is not the smallest number with this property.

Or

(b) Let T be differentiable on an open set D and let S be differentiable on an open set containing $T(D)$. Prove that ST is differentiable on D and if $p \in D$ and $q = T(p)$, then $d(ST)_p = dS/_q \, dT/_p$.

18. (a) Let T be a transformation from R^n into R^n which is of class C' in an open set D, and suppose that $J(p) \ne 0$ for each $p \in D$. Prove that T is locally 1-to-1 in D.

Or

(b) Let T be of class C' on an open set D in n space, taking values in n space. Suppose that $J(p) \ne 0$ for all $p \in D$. Prove that $T(D)$ is an open set.

19. (a) Let F be an additive set function defined on $\mathcal{G}$ and a.c. Suppose also that F is differentiable everywhere, and uniformly differentiable on compact sets, with the derivative a point function f. Prove that f is continuous everywhere and $F(S) = \iint_S f$ holds for every rectangle S.

Or

(b) Define a smooth curve. If γ is a smooth curve whose domain is the interval $[a, b]$. Prove that γ is rectifiable and $L(\gamma)$ is given by the formula $L(\gamma) = \int_a^b |r'(t)| dt$.

20. (a) Prove that $T^*(d\omega) = (d\omega)^* = d(w^*) = dT^*(\omega)$ when (i) ω is a 0-form (ii) ω is any 1-form.

Or

(b) Give a proof of Stoke's theorem by reducing it to an application of Green's Theorem.
