(6 pages)

Reg. No. : ....

Code No.: 5364 **Sub. Code: ZMAM 11**

M.Sc (CBCS) DEGREE EXAMINATION, APRIL 2023.

First Semester

Mathematics – Core

ALGEBRA – I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer:

1. If G has no nontrivial subgroups, show that G must be __________ of prime order.

   (a) Uncountable (b) Finite

   (c) Infinite (d) None of these

2. Every subgroup of an abelian group is __________.

   (a) right coset (b) last coset

   (c) normal (d) not normal

3. Let G be a group and ϕ an automorphism of G. If $a \in G$ is of order $o(a) > 0$, then $o(\phi(a)) =$ __________.

   (a) 0 (b) 1

   (c) $o(a)$ (d) ∞

4. The number of automorphisms of a cyclic group of order n is __________.

   (a) $\varphi(n)$ (b) n

   (c) n^2 (d) 1

5. Every permutation is a product of __________ cycles.

   (a) 1 (b) 2

   (c) 3 (d) 4

6. If $o(G) = p^2$ where p is a prime numbers then G is __________.

   (a) normal (b) left coset

   (c) right coset (d) abelian

7. The number of p-sylow subgroups in G, for a given prime is of the form __________.

   (a) $1 + kp$ (b) $1 - kp$

   (c) kp (d) $\frac{1+k}{p}$

8. If $p^m / o(G)$, $p^{m+1} \nmid o(G)$ then G has a subgroup of order __________.

   (a) p^2 (b) p^{m-1}

   (c) p^m (d) p^{m+1}

9. If $\phi \neq 1 \in G$ where G is an abelian group then $\sum_{g \in G} \phi(g) =$ __________.

   (a) 1 (b) 2

   (c) ∞ (d) 0

10. If $g_1 \neq g_2$ are in G, G a finite abelian group, then there is a $\phi \in G$ with $\phi(g_1)$ __________ $\phi(g_2)$.

    (a) = (b) $\neq$

    (c) > (d) <

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Show that N is a normal subgroup of G iff $gNg^{-1} = N$ for every $g \in G$.

    Or

    (b) Suppose G is a group. N a normal subgroup of G; define the mapping ϕ from G to G/N by $\phi(x) = Nx$ for all $x \in G$. Then prove that ϕ is a homomorphism of G onto G/N.

12. (a) Show that J(G) $\approx G/Z$. where J(G) is the group of inner automorphisms of G, and Z is the center of G.

    Or

    (b) If G is a finite group. and $H \neq G$ is a subgroup of G such that $o(G \nmid i(H)!$ then Prove that H must contain a nontrivial normal subgroup of G. in particular, G cannot be simple.

13. (a) If $o(G) = p^2$ where p is a prime number, then prove that G is abelian.

    Or

    (b) Show that Every permutation is the product of its cycles.

14. (a) If $p^m / o(G)$, $p^{m+1} X o(G)$ then show that G has a subgroup of order p^m.

    Or

    (b) Prove that $n(k) = 1 + p + \ldots + p^{k-1}$.

[P.T.O.]

15. (a) If G and G' are isomorphic abelian groups, then prove that for every integers, $G(s)$, and $G'(s)$ are isomorphic.

Or

(b) Suppose that G is the integral direct product of $N_1, \ldots, N_n$. Then Prove that for $i \neq j$, $N_i \cap N_j = (e)$, and if $a \in N_i$, $b \in N_j$ then $ab = ba$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Show that if ϕ is a homomorphism of G into G with kernel K. then K is a normal subgroup of G.

Or

(b) State and prove Cauchy's theorem for Abelian Groups.

17. (a) State and prove Cayley's theorem.

Or

(b) Show that if G is a group, then Prove that $\mathcal{A}(G)$ the set of automorphisms of G. is also a group.

18. (a) Prove that conjugancy is an equivalence relation on G.

Or

(b) Show that the number of conjugate classes in S_n is $p(n)$, the number of partitions of n.

19. (a) State and prove Third part of Sylow's Theorem.

Or

(b) Let G be a finite group and suppose that G is a subgroup of the finite group M. Suppose further that M has a p-sylow subgroup Q. Then Prove that G has a p-sylow subgroup P. In fact. $P = G \cap xQx^{-1}$ for some $x \in M$.

20. (a) Prove that every finite abelian group is the direct product of cyclic groups.

Or

(b) Show that the two abelian groups of order p^n are isomorphic iff they have the same invariants.
