(7 pages) Reg. No. : ....

Code No.: 5022 **Sub. Code: PMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Fourth Semester

Mathematics – Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. A complete normed linear space is a ———— space.
   (a) Compact
   (b) Banach
   (c) Continuous
   (d) Hilbert

2. The set of all continuous linear transformations of a normed linear space N into R or C according as N is real or complex denoted by N^* is called ————
   (a) Banach space of N
   (b) complement of N
   (c) conjugate space of N
   (d) Hilbert space of N

3. The conjugate space of N^* is called as ———— conjugate.
   (a) second (b) dual of N
   (c) third (d) first

4. The isometric isomorphism $x \rightarrow F_x$ is called the ———— of N into N^{**}
   (a) Banach (b) Natural imbedding
   (c) Surjective (d) Injective

5. A ———— space is a complex Banach space the whose norm arises from the inner product.
   (a) Hilbert (b) Banach
   (c) Inner product (d) Banach algebra

6. Two vectors x and y in a Hilbert space H are said to be ———— if $(x, y) = 0$
   (a) orthogonal (b) inverse
   (c) complement (d) inner

7. A ———— set in a Hilbert space H is a non empty subset of H which consists of mutually orthogonal unit vectors.
   (a) Hilbert (b) Empty
   (c) Orthonormal (d) Banach

8. The value of T^{**} = ————.
   (a) T (b) T^*
   (c) T^{-1} (d) T_1

9. An operator is ———— if it commutes with its adjoint.
   (a) adjoint (b) normal
   (c) unitary (d) singular

10. The value of det(I) =
    (a) 0 (b) 1
    (c) –1 (d) 2

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that if N is a normed linear space and x_0 is a nonzero vector in N then there exists a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

Or

(b) Let M be a closed linear space of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M is defined by $\|x + M\| = \inf\{\|x + m\| : m \in M\}$ then prove that N/M is a normed linear space.

12. (a) Prove that if B and B' are Banach spaces and if T is a linear transformation of B into B' then T is continuous if and only if its graph is closed.

Or

(b) If P is a projection on a Banach space B, and if M and N are its range and null space then prove that M and N are closed linear subspaces of B such that $B = M \oplus N$.

13. (a) Prove that if x and y are any two vectors in a Hilbert space H, then $|< x, y >| \le \|x\| \, \|y\|$.

Or

(b) Prove that if M is a closed linear space of a Hilbert Space, then $H = M \oplus M^{\perp}$.

14. (a) Prove that if A_1 and A_2 are self adjoint operators on H, then their product A_1A_2 is self adjoint if and only if $A_1A_2 = A_2A_1$.

Or

(b) Prove that if T is an operator on H, for which $(Tx, x) = 0$ for all x then $T = 0$.

15. (a) Prove that if T is normal, then x is an eigen vector of T with eigen value $\lambda \Leftrightarrow x$ is an eigen vector of T^* with eigen value $\bar{\lambda}$.

Or

(b) If T is normal, then prove that the Mi's are pairwise orthongonal.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) State and prove Hahn banach theorem.

Or

(b) Let N anad N' be normed spaces and T is a linear transformation of N into N'. Then prove that the following conditions on T are all equivalent.

(i) T is continuous.

(ii) T is continuous at the origin.

(iii) There exists a real number $k \geq 0$ with the property that $\|T(x)\| \leq k\|x\|$ for every x in N.

(iv) If $S = \{x : \|x\| \leq 1\}$ is closed unit sphere in N then its image $T(S)$ is a bounded set in N'.

17. (a) State and prove open mapping theorem.

Or

(b) If N is a normed linear space then prove that the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak * topology.

18. (a) If T is an operator on a normed linear space N, then prove that its conjugate T^* defined by $[T^*(f)](x) = f(T(x))$ is an operator on N^* and the mapping $T \rightarrow T^*$ is an isometric isomorphism of $\mathfrak{B}(N)$ into $\mathfrak{B}(N^*)$ which reverses products and preserves the identity transformation.

Or

(b) If M is a proper closed linear subspace of a Hubert space H, then prove that there exists a non zero vector z_0 in H such that $z_0 \perp M$.

19. (a) Prove that if $\{e_i\}$ is an ortho normal set in a Hilbert space H, and if x is an arbitrary vector in H then $x - \sum(x, e_i)e_i \perp e_j$ for each j.

Or

(b) Prove that if H be a Hilbert space and f be an arbitrary functional in H^* then there exists a unique vector y in H such that $f(x) = (x, y)$.

20. (a) Prove that if P is a projection on H with range M and null space N then $M \perp N$ if and only if P is self adjoint and in this case $n = M^{\perp}$.

Or

(b) (i) Prove that $\|N^2\| = \|N\|^2$ if N is normal operator on H.

(ii) Also prove that if T is an operator on H, then T is normal iff its real and imaginary parts commute.

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