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$T\overline{T}$-like Deformations of Supersymmetric Quantum Field Theories and the Design of New Tensor Algebra Software

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The work presented in this Thesis is, to the best of my knowledge and belief original, except as acknowledged in the text, and has not been submitted either in whole or in part, for a degree at this or any other university.

Jesse Woods

"The hardest things are usually the best. If it's easy, leave it to somebody else."

— Tom Platz

"My name is Jeff."

— Channing Tatum

Abstract

One of the main open problems in theoretical physics is determining the high-energy behaviour of the Standard Model and General Relativity, and how they fit together. The Standard Model - a Quantum Field Theory (QFT) at heart - is perhaps the most successful and stringently tested scientific theory of nature, describing fundamental particles and their interactions in terms of quantised excitations of fields that span spacetime. General Relativity describes gravity in a natural geometric way - however it is unknown how this theory behaves at the quantum level. A better understanding of high energy physics will allow us to see how, and to what, these two theories converge.

Recently, an operator known as $T\bar{T}$, constructed from the stress-energy tensor, has been demonstrated to create a unique flow through the space of 2D QFTs by deforming the Lagrangian of the theory [1]. This can be used to understand non-trivial interacting field theories by deforming solvable (e.g. free) models. Remarkably, this operator is solvable; it possesses properties that ensure quantities like the spectrum and $S-$matrix can be computed exactly starting from the undeformed theory. Given that all meaningful theories possess a stress-energy tensor, one can universally construct a composite (super)current-squared operator from its components, analogously to $T\bar{T}$.

One of the most powerful concepts for discovering new physics is symmetry. Imposing additional symmetries on a theory may yield enough constraints to make the theory solvable, or to demonstrate interesting new properties. There is only one possible way to impose additional, non-trivial, physical, symmetries on spacetime: supersymmetry (SUSY). This symmetry relates bosonic (integer spin) states to fermionic (half-integer spin) states by extending the Poincaré symmetry group by anti-commuting "supercharges". Supersymmetry is considered by some to be part of the resolution to the issues between the Standard Model of Particle Physics, and General Relativity (GR), and is an important component of string theory.

This naturally raises the question as to how $T\bar{T}$ behaves with supersymmetric theories. For theories possessing SUSY, the stress-energy tensor is generalised to a set of supercurrents, containing the stress-energy tensor and possibly other currents [2]. One can then create a supercurrent-squared operator, which should reduce to the $T\bar{T}$ operator when supersymmetry is truncated. This allows for $T\bar{T}$ to be extended to supersymmetric theories.

Currently, the supercurrent-squared operator has been show to be equivalent to T$\bar{\text{T}}$ for 2D $\mathcal{N} = (0,1)$, $\mathcal{N} = (1,1)$, $\mathcal{N} = (0,2)$ and $\mathcal{N} = (2,2)$ theories [3] [4] [5] [6]. We aim to extend these results to the case of 2D $\mathcal{N} = (0,4)$ and $\mathcal{N} = (4,4)$ supersymmetry and show that the extension is natural, which is a novel result. In the process of doing so, we have studied the $T\bar{T}$ and $T\bar{T}-$like terms of the aforementioned SUSY theories, and $4D$ $\mathcal{N} = 2$

due to its relationship to $2D$ $\mathcal{N} = (4,4)$ [7] supersymmetry whose supercurrent is poorly understood.

For theories with greater numbers of supercharges, the algebraic manipulations needed to determine the supercurrent multiplet structure become increasingly complex. Motivated by this, we have also created a general-purpose tensor algebra software package in Julia in order to perform algebraic computations. Given its native ability to handle covariant derivatives and graded structures, this software will serve as a powerful tool for those studying both spacetime or supersymmetric structures.

Acknowledgements

I would like to thank my supervisor Dr. Gabriele Tartaglino-Mazzucchelli not only for supervising this thesis, but also investing significant time in me for many smaller projects over the past few years. He has allowed me to learn much more throughout my degree than what is taught in just standard courses alone, and has provided me with the opportunity to pursue my curiosities. I have continued to learn more about supersymmetry, supergravity, $T\bar{T}$, and quantum field theories over the course of this thesis, and now find the possibilities associated with $T\bar{T}$ absolutely fascinating.

I'd also like to thank everyone who has been a student under Gabriele alongside me over the years; Joey, Tamera, Liam, Greg, Saurish, William, Jessica, Zejun, Christian, and Cian. It was always great to hear about what people were working on in their own projects, and inspired me to want to learn more.

The past year would have been very difficult without the support of my friends. Thanks Huddo pour les yarns et scrans, Hugo for the sound-on snaps, Violeta for always knowing what to say, Chris for things that no one would believe me if I said (except maybe Karl), Will for always checking in on me, and Jaden and Lara for literally everything.

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1 Introduction

Quantum Field Theory (QFT) is perhaps the most successful scientific theory of nature. Born out of the necessity to make Quantum Mechanics compatible with Einstein's theory of Special Relativity, it describes fundamental particles and their interactions in terms of quantised excitations of fields that span spacetime. Our current best theory of physics, the Standard Model of Particle Physics (SM), is at heart a QFT. When combined with supersymmetry (SUSY), such theories offer not only a promising avenue for the unification of the SM with Einstein's General Relativity theory (GR) of gravity - an open problem in physics, but a rich background for studying mathematics like representation theory. In general, QFTs are incredibly difficult to solve, and thus developing new mathematical tools, and learning how to exploit known tools, are at the forefront of theoretical physics research.

In quantum mechanics, a system is described by a (possibly infinite) set of vectors that span a Hilbert space. A basis may be chosen such that basis vectors correspond to a physical energy level state (eigenstate), and linear combinations (superpositions) of these vectors can be taken to describe the full state (wavefunction) of a system. So-called "ladder" operators can be used to map one vector to another, simulating the transition between two energy eigenstates induced by exciting the system with a quanta of energy. Hermitian operators then model observables (physical quantities like position, momentum, energy, spin, etc), with (real) eigenvalues corresponding to the possible values of the observable. Despite its various successes, quantum mechanics breaks in the special relativistic limit. In Special Relativity, space and time are treated on equal footing. In particular, it imposes a "universal speed limit", the speed of light, c, of which nothing can travel faster. Combined with the Lorentz group symmetry required to ensure that physics behaves the same in all inertial reference frames, i.e. the principle that physics should be independent of observers' coordinate systems, this imposes a non-Euclidean spacetime structure. This gives rise to the notion of a "lightcone", which constrains the region of spacetime one can influence without violating faster-than-light travel, implying a causal spacetime structure.

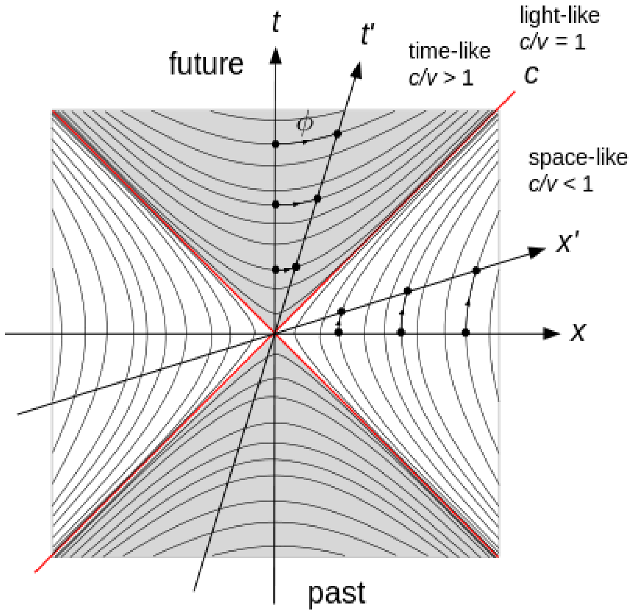


Figure 1.1: Special relativistic 2D spacetime structure. Here, the red lines represent null, or light-like, curves, which bound an observer's future and past light cones (grey). The observer cannot be effected by events outside this cone, as this would require faster than light travel. Lorentz transformations ϕ move events along the black hyperbolic trajectories. Inside the cone, events are time-like - their time ordering cannot be changed by Lorentz transformations. Outside the cone, events are space-like and cannot be causally related. Indeed, under Lorentz transformations, points on the space-like curve can have their time-ordering reversed. Source: [8].

However, one can compute the amplitude of a quantum-mechanical free particle to propagate outside the lightcone, starting from $x_0 = (t, \bar{x}_0) = (0, 0, 0, 0)$ (a position eigenstate). This calculation is nontrivial and requires contour integration over the complex plane, but yields [10]:

$$A = \langle x|e^{-i\hat{H}t}|x_0\rangle = \int d^3p \frac{1}{(2\pi)^3} e^{ip\cdot x - iE_p t} \propto e^{-m|x|}. \tag{1.1}$$

This is evidently a non-zero amplitude even for large x - a particle can potentially propagate outside of it's lightcone. This means that single-particle quantum mechanics is incompatible with special relativity, and produces a violation of causality. To remedy this, we can instead consider such a particle as an excitation of a more fundamental, underlying field object. Classically, a field is a function that assigns to each spacetime point a quantity, like a scalar or a vector. In the quantisation of a classical system, we assign to each degree of freedom (position, momentum) an operator. As a field consists of infinitely many degrees of freedom, we consider the quantised field to be operator-valued. Then one imposes that at spacelike separations, the field operators commute, meaning that two measurements of the field at these spacetime coordinates are entirely independent. This preserves the causal structure of the theory.

In order to model the physics of such a field, or a set of many fields $\{X_i\}$ and their interactions, the Lagrangian formalism is most convenient. The procedure is analogous to classical mechanics. By imposing our axiomatic constraints, like invariance under Lorentz transformations, the set of possible terms that can be included in this Lagrangian $\mathcal{L}$ is highly selective. In fact, the Lagrangian is constrained to contain (for the most part) only terms proportional to products of fields and their first derivatives. This Lagrangian can then be integrated over spacetime to obtain the action S,

$$S = \int d^4x \mathcal{L}(X_1, \partial_\mu X_1, X_2, \partial_\mu X_2, ...), \tag{1.2}$$

on which the principle of least action can be imposed, allowing for the extraction of the equations of motion of the fields via the Euler-Lagrange equation. These fields can come in many different varieties, indeed:

- Real scalar fields, usually denoted by ϕ, which are used to model chargeless spin-0 (bosonic) particles, like the Higgs Boson,
- Complex scalar fields, usually denoted by φ and $\varphi^\dagger$, which are charged particles and anti-particles,
- Vector fields, denoted A_μ, which describe force carrying bosonic particles with spin-1, like the photon,
- Spinor fields, often denoted by ψ and $\bar{\psi}$, describing charged spin-1/2 fermions, like electrons and positrons,
- Tensor fields, which describe particles of higher spin, like the graviton.

These field building blocks can then be used to construct a plethora of QFTs by including different fields and interactions. This framework is exceptionally important and has been used to describe a variety of physical phenomena like superconductors, many body quantum systems, and of course, the Standard Model [10].

However, most QFTs are incredibly difficult to solve to make physical predictions. Often sophisticated mathematical or computational machinery is required to extract results from the theory. In most scenarios, perturbation expansions of the theory must be performed in order to compute meaningful quantities. Indeed, one common quantity of interest is the S-matrix, which describes the amplitude for a quantum field in some initial state $|p\rangle$, to "scatter" to some other final state $|q\rangle$ [10],

$$A = \langle q|\hat{S}|p\rangle = \langle q|T[e^{i\int d^4x \hat{\mathcal{H}}_I(x)}]|p\rangle\,, \tag{1.3}$$

where T is the time-ordered product, and $\hat{\mathcal{H}}_I(x)$ is the interaction Hamiltonian density. One must then Dyson-expand this evolution operator in order to compute it, necessitating the calculation of infinitely many terms in order to solve exactly. As a concrete example, consider the interacting real scalar ϕ^4 theory in 4 spacetime dimensions:

$$\mathcal{L} = \frac{1}{2}[\partial_\mu\phi(x)]^2 - \frac{m^2}{2}\phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4, \tag{1.4}$$

where λ is the strength of the interaction. The free part of the theory is given by the kinetic and mass terms, $\frac{1}{2}[\partial_\mu\phi(x)]^2$ and $\frac{m^2}{2}\phi(x)^2$, that can be quantised quite straightforwardly. Using Hamilton's equations, one then has $\hat{\mathcal{H}}_I = \frac{\lambda}{4!}\phi(x)^4$. Then, expanding the expression for the S-matrix, one obtains

$$\hat{S} = T\Big[\exp\Big(-i\int d^4x\hat{\mathcal{H}}_I(x)\Big)\Big] \tag{1.5}$$

$$= T\Big[1 - i\int d^4x_0\hat{\mathcal{H}}_I(x_0) + \frac{(-i)^2}{2!}\int d^4x_1d^4x_2\hat{\mathcal{H}}_I(x_1)\hat{\mathcal{H}}_I(x_2) + \mathcal{O}(\hat{\mathcal{H}}_I^3)\Big]\,. \tag{1.6}$$

$$= T\Big[1 - \frac{i\lambda}{4!}\int d^4x_0\phi(x_0)^4 + \frac{(-i\lambda)^2}{2!\,(4!)^2}\int d^4x_1d^4x_2\phi(x_1)^4\phi(x_2)^4 + \mathcal{O}(\hat{\mathcal{H}}_I^3)\Big] \tag{1.7}$$

Then, one can compute the amplitude $A = A^{(0)} + A^{(1)} + A^{(2)} + ...$ as a sum of terms in powers of λ. Clearly, $A^{(0)} = \langle q|T[1]|p\rangle = \langle q|p\rangle$ is simply the overlap of the states $|p\rangle$ and $|q\rangle$, with no interactions occurring. For the first order term $A^{(1)}$, one has

$$A^{(1)} = \langle q|T[-\frac{i\lambda}{4!}\int d^4x_0\phi(x_0)^4]|p\rangle = -\frac{i\lambda}{4!}\int d^4x\,\langle 0|a_qT[\phi(x)\phi(x)\phi(x)\phi(x)]a_p^\dagger|0\rangle \tag{1.8}$$

In order to evaluate this vacuum expectation value, Wick's theorem is employed to turn this time-ordered expression into a sum over all possible "contractions" and "normal-orderings" of the operators inside the expectation expression, in effect commuting all the annihilation operators to the right. Such normal-ordered operators annihilate the vacuum, leaving only fully contracted terms, which are simple products of vacuum expectation values. Indeed, after applying Wick's theorem, the integrand will contain a sum of contraction terms of the form

$$\langle 0|\,a_q\phi(x)\phi(x)\phi(x)\phi(x)a_p^\dagger\,|0\rangle = \langle 0|Ta_q\phi(x)|0\rangle\,\langle 0|T\phi(x)\phi(x)|0\rangle\,\langle 0|T\phi(x)a_p^\dagger|0\rangle \tag{1.9}$$

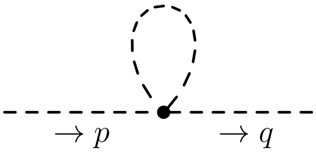
$$= \frac{e^{iq\cdot x}}{(2\pi)^{3/2}(2E_q)^{1/2}}\Delta(x-x)\frac{e^{-ip\cdot x}}{(2\pi)^{3/2}(2E_p)^{1/2}}, \tag{1.10}$$

over which the integral can be relatively easily computed. Here, $\Delta(x-y)$ is the propagator Green's function for the theory,

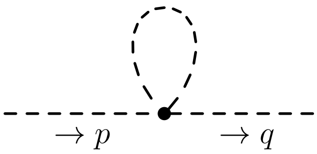
$$\Delta(x-y) = \int\frac{d^4k}{(2\pi)^4}\frac{ie^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}. \tag{1.11}$$

These vacuum expectation values are all easily computed from the field expansion of the operators. Each of these contraction terms represents a possible physical process of the field that is allowed by the theory. For example, the above amplitude describes a ϕ particle progating with momentum p, producing another "virtual" ϕ particle which interacts and annihilates (a so-called "self-energy" interaction), before continuing on its way with momentum q. This can be represented graphically by the Feynman diagram seen in (1.12)

Here, the internal and external lines and vertices represent instructions for computing the amplitude, which are dependent on the theory. It is often more convenient to work in



$$= -\frac{i\lambda}{4!}\int d^4x \frac{e^{iq\cdot x}}{(2\pi)^{3/2}(2E_q)^{1/2}} \int \frac{d^4k}{(2\pi)^4} \frac{ie^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}(2E_p)^{1/2}} \tag{1.12}$$



$$= (2\pi)^4\delta^{(4)}(q-p)\frac{(-i\lambda)}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon} \tag{1.13}$$

momentum space. Thus, after computing the Fourier transform of each vacuum expectation value, this process thus has the amplitude seen in (1.13).

Note, however, that this integral is divergent. This infinity is unphysical. Recall that these amplitudes arise from part of the perturbative expansion of the S matrix, which is a unitary operator. Thus, the exact scattering amplitude should be finite, as the particle states are themselves elements of a Hilbert space. Thus, this infinity is residual from perturbation theory. This is problematic and must be remedied in order to make physical predictions from the perturbation theory. Such divergences occur in many interacting quantum field theories and a procedure is needed in each case to rectify them. Note that the particle states that we consider here (including the vacuum $|0\rangle$) were eigenstates of the free Hamiltonian. For the interacting theory, the vacuum must instead be an eigenstate of the full interacting Hamiltonian, $(\mathcal{H}_0 + \mathcal{H}_I)\,|\Omega\rangle = 0$. Indeed, one would expect that modifying the Hamiltonian by adding an interaction term would completely change the spectrum. Thus, the field operators $\hat{\phi}(x)$, which were constructed to create single-particle states in the free theory, may no longer create a single-particle state in the interacting theory [10]. This in turn also affects the propagator. It is thus of no surprise that accounting for these changes is necessary in order to gain meaningful results from perturbation theory.

One way to remove these divergences is to impose an abritrary cutoff momentum for the theory. In reality, we expect our current theories like the Standard Model to only be an "effective" theory, suitable for low energies only. Thus, this cutoff is not entirely unreasonable. It does, however, pose issues quantum mechanically as unitarity is violated [11]. There do fortunately exist methods for removing this momentum cutoff entirely. This framework is called renormalisation of the theory, and comes in a few varieties depending on the theory. In essence, divergent terms are absorbed in the masses, field normalisations, and couplings of the theory by adding "counterterms" to the Lagrangian. These counterterms are local (depending on a single spacetime point) operators that shift the "constants" of the theory. This has the effect of making these parameters energy, or scale, dependent. The evolution of these constants can be described by the $\beta-$function - in essence, a flow equation in theory space [12].

A theory where only a finite number of counterterms are needed is called renormalisable [10]. As the presence of divergences depends on the presence of particular terms, it is possible to classify terms of the Lagrangian pertaining to their renormalisability. By dimensional analysis, one can observe the exponent of the units of momentum, or mass, in the both the operators and couplings of the Lagrangian. The value of this exponent is known as the mass dimension, denoted by $[\cdot]$. Note that each term in the Lagrangian must have total mass dimension d, where d is the spacetime dimension, in order for the action to be unitless when the Lagrangian is integrated [12]. Indeed, $[x] = -1$ as $[m] = [E]$ ($E = mc^2$ in units where $c, \hbar = 1$), and $E = \hbar\omega = h/t$ so $[t] = -1$, and thus $[x] = -1$ since $\Delta x = c\Delta t$ for a spacetime interval. Thus, $[dx^d] = -d$. For example, in our ϕ^4 theory, $[m] = 1$, obviously, so for the term $[-\frac{m^2}{2}\phi(x)^2] = d = 4$, it must be so that $[\phi(x)] = 1$. We will denote the mass dimension of the operators as Δ. For example, classically $\phi(x)^2$ has $\Delta = 2$. Then the following observations have been made:

- If the operator has $\Delta < d$ (equivalent to the coupling having positive mass dimension), then the theory is super-renormalisable: there are only a finite number of divergent diagrams, proportional to the number of types of interaction vertices in the theory. Such operators are known as *relevant* operators. As the coupling is of positive mass dimension, one can compute that this induces a momentum dependance in the propagators and vertices that render them insignificant at high energies, and significant at low energies. We thus say that the operator generates a "flow towards the infrared" (low energy) scale - the renormalised coupling constants will grow as the energy decreases (length increases). The term "relevant" corresponds to the fact that these effects will be tangible in the low energy regime in which we conduct experiments.

- If the operator has $\Delta = d$ (equivalent to the coupling being unitless), then the theory is renormalisable: there are only a finite number of divergent diagrams for each order of perturbation theory, though divergences occur at all orders of perturbation theory. However, as the diagrams have a recursive structure, they can be corrected with a finite set of counterterms. Such operators are known as *marginal* operators. Since the coupling constants have mass dimension zero, they are unaffected by the energy scale. We say that such theories are fixed points of the renormalisation flow. Note that some quantum effects can break this scale invariance and this type of operator can be further classified as exactly marginal (leading to conformal field theories); marginally relevant (quantum relevant, according to the previous classification, though classically marginal); and marginally irrelevant (classically marginal but such that they flow to zero in the infra-red, see next).

- If the operator has $\Delta > d$ (equivalent to the coupling being of negative mass dimension), then the theory is non-renormalisable: infinitely many counterterms are needed to remove divergences. These operators are known as *irrelevant* operators. Since the coupling constants have negative mass dimension, the higher order terms of the perturbation theory become more significant at higher energies. We thus say that the operator generates a "flow towards the ultraviolet" (high energy) scale, meaning the coupling constants grow at high energy (short length).

For example, the ϕ^4 theory discussed about is renormalisable as $[\phi^4] = 4$ is a marginal operator since $[\phi] = 1$. At the quantum level, it also appears to be a marginally irrelevant operator, which indeed leaves as an open question whether the ϕ^4 theory is a well-defined quantum field theory or just an effective field theory.

Marginal and relevant deformations are well-studied due to the fact that decreasing the energy scale (increasing the length scale) removes the high momentum components of the theory - essentially "blurring" out and losing the fine details of the theory. Given an initial theory, ignoring the high momentum and thus ignoring small scale detail is well-defined. However, this process is not invertible - given a theory, there are infinitely many ways to re-add high energy behaviour (to "unblur" the small-scale details, see Figure 1.2 for a practical example of this "renormalisation group" principle).

We posit, in fact, that at sufficiently high energies, all valid QFTs become (scale-invariant) Conformal Field Theories (CFTs) [11]. In other words, the theory flows to a fixed point at sufficiently high energies, and the coupling constants become fixed (rather than diverging). Adding an arbitrary irrelevant operator to a QFT doesn't guarantee that this will be the case - it is difficult to maintain analytic control of an arbitrary irrelevant deformation in general. This makes studying irrelevant operators, and high-energy theories, difficult.

These problems arise from the fact that solving for the fields of a theory themselves from the equations of motion generated by the Euler-Langrange equations can be difficult - if not impossible. The equations in an interacting theory are often non-linear. Fields also possess an infinite number of degrees of freedom, and thus solving for the fields effectively requires solving for an infinite number of variables. However, some theories possess unique properties that allow them to be solved for.

Such QFTs are said to be "solvable": exact solutions to the equations of motion can be found in a closed-form way, for all values of parameters and initial conditions. In particular, integrable Quantum Field Theories (IQFTs) are a class of solvable QFTs that are of interest. These systems contain an infinite number of conserved quantities (integrals of motion) that can be exploited via techniques like the inverse scattering method or Bethe ansatz to solve the theory [1]. For example, in $2D$, CFTs have an infinite-dimensional symmetry group, and the generators of these symmetries can be used to generate constraints that aid in solving the theory [13]. Furthermore, sometimes the integrable structure of the system provides enough constraints such that renormalisation is unnecessary. Despite these useful properties, this class of theories is quite small, and most IQFTs are in $2D$ only.

All of these compounding factors make it quite remarkable that in 2004, an irrelevant deformation for $2D$ QFTs was discovered that has well-defined analytic quantum behaviour

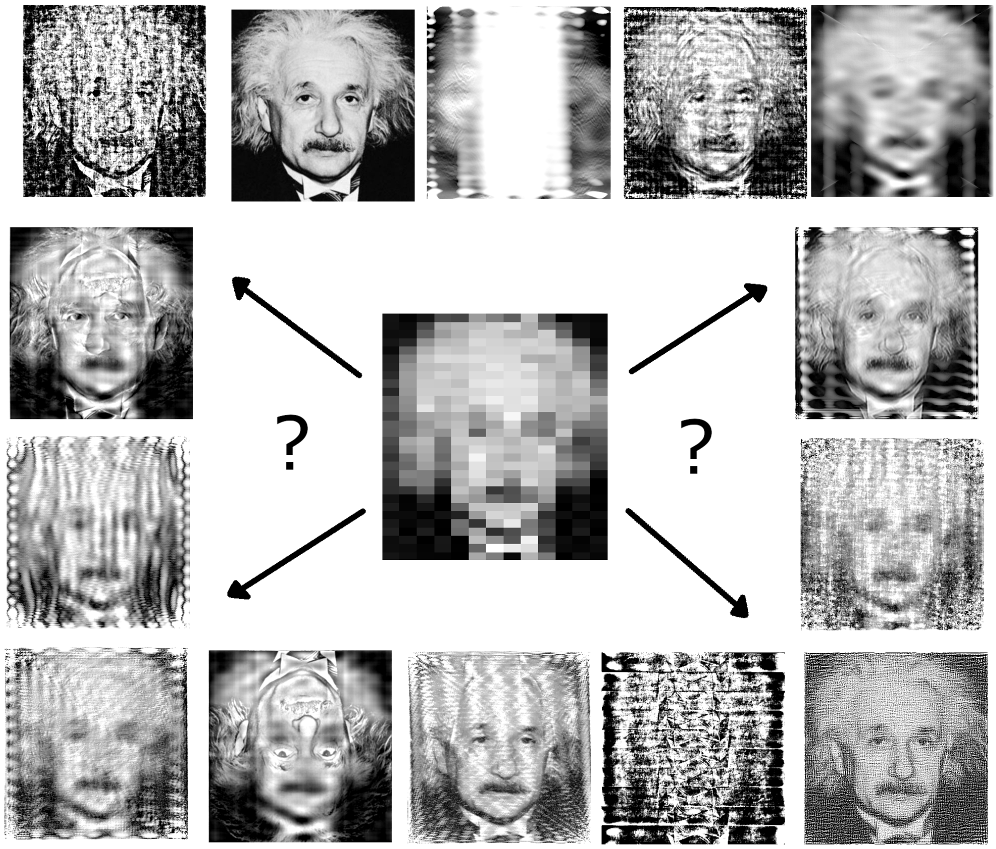


Figure 1.2: The central image has had all of its high frequency Fourier modes removed. How does one "flow to the UV" and re-add the high frequency modes in a meaningful way? There are (in QFTs, infinitely) many operators that can re-add high energy modes, but in general the result may be meaningless. Original image found at [9].

[14]. This irrelevant operator is known as the $\mathrm{T\bar{T}}$ operator, due to its relationship to the stress-energy tensor, $T_{\mu\nu}$. Because every QFT possessing translational invariance has a stress-energy tensor, the $\mathrm{T\bar{T}}$ deformation can be constructed for all $2D$ QFTs. In 2016 and 2017, it was determined that $\mathrm{T\bar{T}}$ deformed theories maintain properties like integrability if they are possessed by the undeformed theory [1]. These deformed theories exhibit analytic control also, meaning that physical quantities like the S matrix [15] and energy spectrum [14] can all be computed exactly from the undeformed theory. The $\mathrm{T\bar{T}}$ operator also has interesting links to the AdS/CFT correspondence [15], which is an exciting area of research that relates to quantum gravity. The discovery of this exciting set of properties possessed by the $\mathrm{T\bar{T}}$ deformation has made it a popular area of study in recent years.

As mentioned above, imposing additional symmetries on a theory may yield enough constraints to make the theory solvable. One such symmetry is supersymmetry. The symmetry relates bosonic (integer spin) states to fermionic (half-integer spin) states by extending the symmetry group by anti-commuting "supercharges". Supersymmetry is widely considered to be part of the resolution to the issues between the Standard Model of Particle Physics, and General Relativity (GR), and is an important component of string theory.

In 2018 onwards, the $\mathrm{T\bar{T}}$ deformation has been constructed for some classes of supersymmetric theories [3] [4] [5] [6]. The $\mathrm{T\bar{T}}$ operator itself is not manifestly supersymmetric, though it can be shown that supersymmetry is preserved by the deformation, though potentially deformed. To prove supersymmetry for the two-dimensional $\mathrm{T\bar{T}}$ case, one seeks a manifestly supersymmetric "supercurrent-squared" extension of the $\mathrm{T\bar{T}}$ operator by using so-called superspace techniques. The resulting operator proves to be equivalent to $\mathrm{T\bar{T}}$ when imposing the equations of motion - in other words, it is equivalent "on-shell". This motivates the study of $\mathrm{T\bar{T}}$ deformations of supersymmetric QFTs, where solvability could be obtained beyond simple examples like free theories.

The study of the behaviour of supersymmetry and $\mathrm{T\bar{T}}$ has led to a classification of the operator in theories with supersymmetries in two dimensions, for the $\mathcal{N} = (0,1)$, $\mathcal{N} = (0,2)$, $\mathcal{N} = (1,1)$, $\mathcal{N} = (1,2)$, $\mathcal{N} = (2,2)$ cases [3] [4] [5] [6].

In this thesis, we will aim to extend this work in the supersymmetric setting, constructing the $\mathrm{T\bar{T}}$ deformation for the cases of $\mathcal{N} = (0,4)$ and $\mathcal{N} = (4,4)$ supersymmetry, which has not been done before. The study of $\mathrm{T\bar{T}}$ deformations has also opened up an entirely new avenue of research on deformations of quantum field theories and has led to interesting results also in space-time dimensions higher than two. For instance, $\mathrm{T\bar{T}}$-like and $\sqrt{\mathrm{T\bar{T}}}$ [7] like operators have proven to be associated with important effective field theories like the (Dirac-)Born-Infeld theory of non-linear electrodynamics [16], which describes the effective behavior of open strings at low-energy in string theoretical approaches to quantum gravity, and the Modified Maxwell (ModMax) theory which has recently attracted new attention being a new class of non-linear electrodynamics that, though non-analytic in fields, preserve classical conformal invariance and electric-magnetic duality [7]. All these results have been obtained with and without a certain amount of supersymmetry. Motivated by this recent literature, we will also seek the study of $\mathrm{T\bar{T}}$-like operators in $d > 2$.

We will firstly begin in Chapter 2 by reviewing the $\mathrm{T\bar{T}}$ operator, how it is realised in the quantum setting, and its nice properties. We will then discuss the construction of the $\mathrm{T\bar{T}}$ operator in the supersymmetric setting, including the tools and formalisms needed for understanding supersymmetry. We will also discuss how $\mathrm{T\bar{T}}$-like deformations can be realised in $4D$.

In Chapter 3, we will discuss the new tensor algebra software package that we created as part of this project, and detail how it can be used to solve problems involving tensors.

In Chapter 4, we will analyse conformal supersymmetric theories in both $2D$ and $4D$, and construct the corresponding $\mathrm{T\bar{T}}$ $-$like supercurrent-squared operators.

In Chapter 5, we will consider the generalised (non-conformal) cases of $2D$ supersymmetry, and how their supercurrent-squared operators relate to $\mathrm{T\bar{T}}$.

2

Review of $\mathrm{T\bar{T}}$ and Supersymmetry

Here, we will review the $\mathrm{T\bar{T}}$ deformation for $2D$ Quantum Field Theories. We will explore the history of its discovery, including its construction and some rudimentary applications, and verify that it does indeed have the properties that it boasts. We will then review supersymmetry and the superspace formalism. This will allow us to then extend the $\mathrm{T\bar{T}}$ operator to a manifestly supersymmetric formulation in Chapter 4, and verify that, for the $2D$ cases, these are on-shell equivalent to $\mathrm{T\bar{T}}$ proving its supersymmetry. This will lay the groundwork for our original construction of the $\mathrm{T\bar{T}}$ operator for $\mathcal{N} = (0, 4)$ and $\mathcal{N} = (4, 4)$ supersymmetry in Chapters 4 and 5.

2.1 Construction of the T$\bar{\text{T}}$ Operator

The seminal T$\bar{\text{T}}$ paper was published by Zamolodchikov in 2004 [14]. In this paper, by making minimal assumptions, Zamolodchikov is able to derive a simple relation for the expectation value of the T$\bar{\text{T}}$ operator. We will derive the results of this paper here.

Recall that the stress-energy tensor $\tilde{T_{\mu\nu}}$ is the conserved Noether current of any theory that possesses translational symmetry (for example, all theories with Lorentz symmetry). It is then given by

$$\tilde{T^{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}, \tag{2.1}$$

and it can be verified that $\partial_\mu \tilde{T^{\mu\nu}} = 0$. By default, this tensor is not symmetric for any theory - however, it can be improved [10]. Note that by Lorentz invariance, the antisymmetric part of $\tilde{T^{\mu\nu}}$

$$\tilde{T^{\mu\nu}} - \tilde{T^{\nu\mu}} = -\partial_\rho X^{\rho\mu\nu}, \tag{2.2}$$

is a total derivative. Clearly, $X^{\rho\mu\nu} = -X^{\rho\nu\mu}$ by antisymmetry. Then, letting

$$A^{\rho\mu\nu} = \frac{1}{2}\Big(X^{\rho\mu\nu} + X^{\mu\nu\rho} - X^{\nu\rho\mu}\Big), \tag{2.3}$$

$A^{\rho\mu\nu}$ is clearly also antisymmetric and $\partial_\rho A^{\rho\mu\nu}$ is a total derivative (and thus vanishes under another derivative) by construction. Then, defining

$$T^{\mu\nu} = \tilde{T^{\mu\nu}} + \partial_\rho A^{\rho\mu\nu}, \tag{2.4}$$

one can then see that

$$T^{\mu\nu} - T^{\nu\mu} = \tilde{T^{\mu\nu}} + \partial_\rho A^{\rho\mu\nu} - \tilde{T^{\nu\mu}} - \partial_\rho A^{\rho\nu\mu} = -\partial_\rho X^{\rho\mu\nu} + \partial_\rho X^{\mu\nu\rho} = 0. \tag{2.5}$$

Thus, $T^{\mu\nu}$ has been improved to be symmetric and is still conserved. One assumption by Zamolodchikov is that this tensor exists. We will thus work with this symmetrised stress-energy tensor.

Given the naissance of the theory was in the context of CFTs in two dimensions (which in the Euclidean case that we consider here is intimately tied to the structure of $\mathbb{C}$), we instead express the stress energy tensor in holomorphic and anti-holomorphic coordinates of the complex plane:

$$z = x + iy; \quad \bar{z} = x - iy, \tag{2.6}$$

where (x, y) are the $2D$ Cartesian coordinates corresponding to space and time respectively. Then, this allows the components of $T_{\mu\nu}$ to be written as

$$T_{zz} = \frac{1}{4}(T_{xx} - T_{yy} - 2iT_{xy}), \quad T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{xx} - T_{yy} + 2iT_{xy}), \quad T_{z\bar{z}} = \frac{1}{4}(T_{xx} + T_{yy}). \tag{2.7}$$

Then, by the conventions of CFT, one defines

$$T(z) = -2\pi T_{zz}, \quad \bar{T}(z) = -2\pi T_{\bar{z}\bar{z}}, \quad \Theta(z) = 2\pi T_{z\bar{z}}. \tag{2.8}$$

Note that under this change of variables, the conservation equations become

$$\partial_{\bar{z}} T(z) = \partial_z \Theta(z), \quad \partial_z \bar{T}(z) = \partial_{\bar{z}} \Theta(z). \tag{2.9}$$

Then, the T$\bar{\text{T}}$ field is defined as

$$\mathrm{T\bar{T}}(z) = -\pi^2 \det T_{\mu\nu} = 4\pi^2 (T_{zz} T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2) = T(z)\bar{T}(z) - \Theta(z)^2. \tag{2.10}$$

This field then adds a term which deforms our Lagrangian,

$$\mathcal{L}^{(\lambda+\delta\lambda)} = \mathcal{L}^{(\lambda)} + \mathcal{L}^{(\delta\lambda)} = \mathcal{L}^{(\lambda)} - \frac{\delta\lambda}{\pi^2}\, \mathrm{T\bar{T}}^{(\lambda)} \tag{2.11}$$

$$\implies \frac{\mathcal{L}^{(\lambda+\delta\lambda)} - \mathcal{L}^{(\lambda)}}{\delta\lambda} = -\frac{1}{\pi^2}\, \mathrm{T\bar{T}}^{(\lambda)}. \tag{2.12}$$

Thus, as we take $\delta\lambda \to 0$, we obtain the flow equation for the T$\bar{\text{T}}$ deformation [15]:

$$\frac{\partial \mathcal{L}^{(\lambda)}}{\partial \lambda} = -\frac{1}{\pi^2}\, \mathrm{T\bar{T}}^{(\lambda)}. \tag{2.13}$$

This defines a single-parameter flow of the Lagrangian.

In the QFT regime, as usual, these fields become operator-valued. Then, it is important to be able to compute expectation values of these operators. In fact, a composite operator must be well-defined inside such correlation functions in order to be considered quantum-mechanically well-defined [12]. Thus, we will verify that this field is well-defined when operator-valued. As seen in the introduction, computing the expectation value of the product of two or more operators is difficult. In general, for two local operators $\mathcal{O}_i(z)$ and $\mathcal{O}_j(z')$, their product can be expressed as an operator-product expansion (OPE) [14]

$$\mathcal{O}_i(z)\mathcal{O}_j(z') = \sum_k C_{ij}^k (z - z') \mathcal{O}_k(z'). \tag{2.14}$$

Then, expectation values are given by

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(z') \rangle = \sum_k C_{ij}^k (z - z') \langle \mathcal{O}_k(z') \rangle. \tag{2.15}$$

Zamolodchikov assumes global translational symmetry: for all local fields, the expectation value is independent of the coordinates. In other words, $\langle \mathcal{O}_k(z') \rangle = \langle \mathcal{O}_k \rangle$ is constant. This implies that

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(z') \rangle = G_{ij}(z - z'), \tag{2.16}$$

some function dependent only on point separations. Finally, it is assumed that at large (infinite in at least one direction) separations, local operators are not-correlated (which implies that the spacetime background is topologically either an infinite plane, or an infinite cylinder), and that the underlying theory becomes a conformal field theory at high energies (and thus the QFT in question is the underlying CFT perturbed by a relevant operator). To verify the validity of the T$\bar{\text{T}}$ operator in the quantum regime, one must take the limit

$$\lim_{z \to z'} (T(z)\bar{T}(z') - \Theta(z)\Theta(z')) \tag{2.17}$$

and show that it leads to something well-defined. In general, products of operators at the same spacetime point lead to divergences (in fact, we saw this for the ϕ^4 theory in the introduction). This is why it is remarkable that the $\mathrm{T\bar{T}}$ operator is indeed well-defined. By taking holomorphic derivatives, applying the conservation equations and using OPEs, a non-trivial calculation leads to show that the following equation holds

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \mathcal{O}_{\mathrm{T\bar{T}}}(z') + \text{derivatives.} \tag{2.18}$$

Since the right hand side of this equation is independent of z (except for the derivatives), the limit can be taken linearly. Note that thanks to the assumed global translational symmetry, such derivative terms vanish under taking the expectation. Thus, one defines

$$\mathrm{T\bar{T}}(z) \equiv \mathcal{O}_{\mathrm{T\bar{T}}}(z), \tag{2.19}$$

which is quantum mechanically well-defined mod derivatives. Thus, the $\mathrm{T\bar{T}}$ operator is well-defined for 2D QFTs, under the stated assumptions. Furthermore, it was shown that the correlation function of the $\mathrm{T\bar{T}}$ operator can also be factorised. Following a similar procedure of taking holomorphic derivatives, it was shown that

$$\langle \mathrm{T\bar{T}} \rangle = \langle T(z)\bar{T}(z') \rangle - \langle \Theta(z)\Theta(z') \rangle \tag{2.20}$$

is not only a constant and independent of coordinates, but also

$$\langle \mathrm{T\bar{T}} \rangle = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle \langle \Theta \rangle . \tag{2.21}$$

This means that the operators are uncorrelated, regardless of separation or direction. Furthermore on cylindrical spacetime of radius R, it can be shown that this relation holds for the expectation with respect to any non-degenerate energy eigenstate, not just the vacuum. Indeed, one can show by the same process that

$$\langle n|\mathrm{T\bar{T}}|n\rangle = \langle n|T(z)\bar{T}(z')|n\rangle - \langle n|\Theta(z)\Theta(z')|n\rangle \tag{2.22}$$

is constant, and independent of the coordinates. Once also provide the same factorisation formula holds by using the spectral decomposition,

$$\langle n|T(z)\bar{T}(z')|n\rangle = \sum_{n'} \langle n|T(z)|n'\rangle \langle n'|\bar{T}(z)|n\rangle \cdot e^{(E_n - E_{n'}|y-y'| + i(P_n - P_{n'})(x-x')}, \tag{2.23}$$

and similarly for the $\Theta(z)$ term. By using the fact that the expression should be coordinate independent, terms with $n \neq n'$ vanish, and assuming that $|n\rangle$ is non-degenerate, one gets the factorisation

$$\langle n|\mathrm{T\bar{T}}|n\rangle = \langle n|T|n\rangle \langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle \langle n|\Theta|n\rangle \tag{2.24}$$

By using the definition of the components of $T_{\mu\nu}$, one can rewrite this in terms of the radius dependent spectrum,

$$\langle n|\mathrm{T\bar{T}}|n\rangle = -\frac{\pi^2}{R}\left(E_n(R,\lambda)\frac{d}{dR}E_n(R,\lambda) + \frac{1}{R}P_n^2(R)\right) \tag{2.25}$$

Then, recalling the flow equation for $\mathrm{T\bar{T}}$, one can see that

$$\langle n|\mathrm{T\bar{T}}|n\rangle = -\pi^2 \langle n|\partial_\lambda \mathcal{L}^{(\lambda)}|n\rangle = -\frac{\pi^2}{R}\partial_\lambda E_n(R,\lambda) \tag{2.26}$$

This yields an equation:

$$\partial_\lambda E_n(R,\lambda) = E_n(R,\lambda)\frac{d}{dR}E_n(R,\lambda) + \frac{1}{R}P_n^2(R). \tag{2.27}$$

Therefore, remarkably, the deformed spectrum is directly computable from the undeformed spectrum by solving this partial differential equation. In fact, the PDE is the inviscid Burgers' equation with a driving force and is well-known in the context of fluid mechanics [14]. Given a theory where the explicit dependence of energy and momentum on radius is known, for example in CFTs, the deformed energies can be solved for exactly. Note that in this derivation, we have used a Cartesian coordinate system. The ability to factorise expectation values can be proven covariantly, as seen in [15].

Another property that we claimed that the $\mathrm{T\bar{T}}$ operator boasts is that it preserves integrability, if the undeformed theory is an IQFT [1]. Recall that an IQFT is special as it contains an infinite number of commutative conserved charges, or integrals of motion, we will denote as Q_s ($\bar{Q}_s \equiv Q_{-s}$). These charges are defined as the spatial integral of some local conserved currents, which we denote as $T_{s+1}(z)$ and $\Theta_{s-1}(z)$ ($\bar{T}_{s+1}(z) \equiv \Theta_{-s-1}(z)$ and $\bar{\Theta}_{s-1}(z) \equiv T_{-s+1}(z)$), which satisfy

$$\partial_{\bar{z}}T_{s+1}(z) = \partial_z\Theta_{s-1}(z), \quad \partial_z\bar{T}_{s+1}(z) = \partial_{\bar{z}}\bar{\Theta}_{s-1}(z). \tag{2.28}$$

From these currents, it then holds that

$$Q_s = \int_C T_{s+1}(z)dz + \Theta_{s-1}(z)d\bar{z}, \quad \bar{Q}_s = \int_C \bar{T}_{s+1}(z)d\bar{z} + \bar{\Theta}_{s-1}(z)dz \tag{2.29}$$

are conserved charges. Here, s denotes the (integer) spin. For $s = 1$, these currents are precisely the components of the stress-energy tensor. Note that since

$$[Q_s, Q_{s'}] = 0 \tag{2.30}$$

by the definition of an IQFT, this implies that

$$[Q_s, T_{s'+1}] = \partial_z F_{s,s'}(z), \quad [Q_s, \Theta_{s'-1}] = \partial_{\bar{z}} F_{s,s'}(z), \tag{2.31}$$

$$[Q_s, \bar{T}_{s'+1}] = \partial_{\bar{z}} G_{s,s'}(z), \quad [Q_s, \bar{\Theta}_{s'-1}] = \partial_z G_{s,s'}(z), \tag{2.32}$$

for some local fields F and G, as each of these charges commute.

We would thus like to show that the commutator of each charge with the $\mathrm{T\bar{T}}$ operator is a total derivative, and thus the deformation preserves integrability. Using the point splitting approach from which $\mathrm{T\bar{T}}$ was defined, the commutator is given by

$$[Q_s, T(z)\bar{T}(z') - \Theta(z)\Theta(z')]$$

$$= [Q_s, T(z)]\bar{T}(z') + T(z)[Q_s, \bar{T}(z')] - [Q_s, \Theta(z)]\Theta(z') + \Theta(z)[Q_s, \Theta(z')]$$
$$= \partial_z F_{s,1}(z)\bar{T}(z') + T(z)\partial_{\bar{z}} G_{s,1}(z') - \partial_{\bar{z}} F_{s,1}(z)\Theta(z') + \Theta(z)\partial_z G_{s,1}(z') \tag{2.33}$$

Note that by continuity

$$\partial_z F_{s,1}(z)\bar{T}(z') - \partial_{\bar{z}} F_{s,1}(z)\Theta(z') = \partial_z[F_{s,1}(z)\bar{T}(z')] - \partial_{\bar{z}}[F_{s,1}(z)\Theta(z')]$$
$$= (\partial_z + \partial_{z'})[F_{s,1}(z)\bar{T}(z')] - \partial_{z'}[F_{s,1}(z)\bar{T}(z')] - \partial_{\bar{z}}[F_{s,1}(z)\Theta(z')]$$
$$= (\partial_z + \partial_{z'})[F_{s,1}(z)\bar{T}(z')] - F_{s,1}(z)\partial_{z'}\bar{T}(z') - \partial_{\bar{z}}[F_{s,1}(z)\Theta(z')]$$
$$= (\partial_z + \partial_{z'})[F_{s,1}(z)\bar{T}(z')] - F_{s,1}(z)\partial_{\bar{z}'}\Theta(z') - \partial_{\bar{z}}[F_{s,1}(z)\Theta(z')]$$
$$= (\partial_z + \partial_{z'})[F_{s,1}(z)\bar{T}(z')] - (\partial_{\bar{z}} + \partial_{\bar{z}'})[F_{s,1}(z)\Theta(z')], \tag{2.34}$$

and similarly for the terms involving $G_{s,1}$. Thus, when taking the limit $z \to z'$, one sees that the T$\bar{\text{T}}$ operator commutes with all of the conserved charges up to a combination of total derivatives, which vanishes under expectation values as reasoned previously. This implies that integrability is conserved under the deformation.

It is also useful to note that the T$\bar{\text{T}}$ operator modifies the scattering S-matrix in a predictable way. This is important, as the $S-$matrix is one of the most important obseravables for a QFT. Specifically, the deformed S-matrix can be computed from the undeformed S-matrix by multiplying by a phase factor of the Castillejo-Dalitz-Dyson type, known as a CDD factor. The factor is given by [15]

$$e^{-i\delta_{ij}^{(\lambda)}} = e^{-i\lambda\epsilon_{\mu\nu}p_i^\mu p_j^\nu}, \tag{2.35}$$

and the deformed S-matrix becomes

$$S^{(\lambda)}(\{p_i\}) = (\Pi_{i<j}e^{-i\delta_{ij}^{(\lambda)}})S(\{p_i\}), \tag{2.36}$$

where $\{p_i\}$ is the set of momenta of the incoming particles. When applied to the case of T$\bar{\text{T}}$ production, the CDD phase factor can modify the shape of the differential cross section for the process, leading to non-trivial effects that can be used to probe the structure of the underlying theory. The factor is sometimes known as a "gravitational dressing" due to its relationship with spacetime geometry. The deformed S-matrix also still satisfies unitarity, crossing symmetry, and, for integrable QFTs, the Yang-Baxter equations, implying again that the T$\bar{\text{T}}$ deformation preserves integrability [15].

In summary, it has been shown that the T$\bar{\text{T}}$ operator is not only a well-defined irrelevant operator, but that it deforms the spectrum and modifies the S-matrix of a theory in a controlled way, and also preserves the integrability of a theory.

2.2 T$\bar{\text{T}}$ Examples

The typical toy example of T$\bar{\text{T}}$ deforming a classical theory is for the free boson. The Lagrangian for such a theory is simple,

$$\mathcal{L}_{FB}^{(0)} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{2.37}$$

We would then like to find $\mathcal{L}_{FB}^{(\lambda)}$ by solving the flow equation for the classical Lagrangian.

For a general Lagrangian with an explicit dependence on a Euclidean metric, The Hilbert stress-energy tensor is known to be of the form [17]

$$T_{\mu\nu}^{(\lambda)} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\int d^2x \sqrt{-g} \mathcal{L}^{(\lambda)} \right)}{\delta g^{\mu\nu}} = g_{\mu\nu} \mathcal{L}^{(\lambda)} - 2 \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\mu\nu}}. \tag{2.38}$$

One can compute the T$\bar{\text{T}}$ deformation from its definition using this stress-energy tensor,

$$-\pi^2 \, \mathrm{T}\bar{\mathrm{T}} = \frac{1}{2} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} T_{\mu\rho} T_{\nu\sigma} = (\mathcal{L}^{(\lambda)})^2 - 2\mathcal{L}^{(\lambda)} g^{\mu\nu} \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\mu\nu}} + 2\epsilon^{\mu\nu} \epsilon^{\rho\sigma} \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\mu\rho}} \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\nu\sigma}}, \tag{2.39}$$

and thus the flow equation reads

$$\frac{\partial \mathcal{L}^{(\lambda)}}{\partial \lambda} = (\mathcal{L}^{(\lambda)})^2 - 2\mathcal{L}^{(\lambda)} g^{\mu\nu} \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\mu\nu}} + 2\epsilon^{\mu\nu} \epsilon^{\rho\sigma} \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\mu\rho}} \frac{\partial \mathcal{L}^{(\lambda)}}{\partial g^{\nu\sigma}}. \tag{2.40}$$

For the free boson, we can observe that the tensor

$$X_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi \tag{2.41}$$

is symmetric with trace $X = g^{\mu\nu} X_{\mu\nu}$. Then, our Lagrangian becomes

$$\mathcal{L}_{FB}^{(0)} = \frac{1}{2} X. \tag{2.42}$$

Note that this action is diffeomorphism invariant. In order to preserve this symmetry, the deformed Lagrangian can be only a function of scalar variables, X and λ, $\mathcal{L}_{FB}^{(\lambda)} = \mathcal{L}(X, \lambda)$. This greatly simplifies the expression for the flow equation [17]:

$$\frac{\partial \mathcal{L}_{FB}^{(\lambda)}}{\partial \lambda} = (\mathcal{L}_{FB}^{(\lambda)})^2 - 2\mathcal{L}_{FB}^{(\lambda)} g^{\mu\nu} \frac{\partial \mathcal{L}_{FB}^{(\lambda)}}{\partial g^{\mu\nu}} = (\mathcal{L}_{FB}^{(\lambda)})^2 - 2\mathcal{L}_{FB}^{(\lambda)} g^{\mu\nu} \frac{\partial \mathcal{L}_{FB}^{(\lambda)}}{\partial X} \frac{\partial X}{\partial g^{\mu\nu}} = (\mathcal{L}_{FB}^{(\lambda)})^2 - 2\mathcal{L}_{FB}^{(\lambda)} X \frac{\partial \mathcal{L}_{FB}^{(\lambda)}}{\partial X} \tag{2.43}$$

This can be written in the form

$$\partial_\lambda \mathcal{L}_{FB}^{(\lambda)} + (X\partial_X - 1) \mathcal{L}_{FB}^{(\lambda)\,2} = 0, \tag{2.44}$$

which is the Burgers' equation. Solving this PDE thus yields the deformed Lagrangian

$$\mathcal{L}_{FB}^{(\lambda)} = -\frac{1}{2\lambda} + \frac{1}{2\lambda} \sqrt{1 + 2\lambda X}. \tag{2.45}$$

However, this deformed Lagrangian is actually the Lagrangian that describes a free bosonic string of tension $1/\lambda$ living in a 3D space, the Nambu-Goto Lagrangian, in the static gauge. This demonstrates the UV flow behaviour of the $\mathrm{T\bar{T}}$ operator: the theory flows towards the high-energy limit string theory. Clearly, when $\lambda = 0$, the tension becomes infinite, and the Lagrangian reduces to that of a point particle. For finite λ, the Lagrangian describes a non-local extended object. This is just one instance of the connection between the $\mathrm{T\bar{T}}$ deformation and string theory, which suggests that the operator may be the key to interpolating between local quantum field theories and string theory. For more details, see [15].

2.3 Supersymmetry and Superspace

2.3.1 Supersymmetry and the Superpoincaré Algebra

Any field theory defined on Minkowski spacetime, which regards space and time as opposite metric components, must possess certain symmetries in order to comply with Special Relativity. The Poincare group, which includes rotations, translations, and boosts, is the sensible choice of group to describe the symmetry of these theories. If a field theory is invariant under the Poincare group, it means that the physical laws are unchanged under the group's transformations. Indeed, this aligns with the axiom that the laws of physics should be the same regardless of the choice of coordinates system, or reference frame. The Poincare algebra, described by Lie brackets

$$[P^\mu, P^\nu] = 0, \tag{2.46}$$

$$[M^{\mu\nu}, P^\sigma] = i(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma}), \tag{2.47}$$

and

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\mu\sigma} g^{\nu\rho} + M^{\nu\rho} g^{\mu\sigma} - M^{\mu\rho} g^{\nu\sigma} - M^{\nu\sigma} g^{\mu\rho}\right), \tag{2.48}$$

where P is the generator of translations, g is the Minkowski metric, and M is the generator of rotations and Lorentz boosts, generates the Poincare group. The study of the symmetries of a theory can reveal important information about its physics. In particular, continuous symmetries that belong to Lie groups can give rise to conserved quantities, as described by Noether's theorem. To discover new physics, one approach is to examine the theory and look for additional symmetries. Another way to explore new physics is to extend the underlying symmetry group and see if a corresponding physical theory can be established. Naturally, the question arises, "how can one extend a symmetry group or algebra while maintaining a physically realisable theory?"

According to the Coleman Mandula Theorem [18], the Poincare algebra is the largest Lie algebra that corresponds to a physically realisable theory, up to the addition of arbitrarily many generators that commute with the entire algebra. For example, in the Standard Model, this "internal" symmetry group is $SU(3) \otimes SU(2) \otimes U(1)$, which allow for enough physics to unify 3 out of 4 fundamental forces. There is no non-trivial way of extending the Poincare Lie algebra. Instead, however, one can generalise the Lie algebra to a graded Lie algebra, specifically a ($\mathbb{Z}_2$-graded) "super" Lie algebra. This bypasses the Coleman Mandula Theorem, as established by the Haag, Lopuszanski-Sohnius Theorem [19]. In this case, one can extend the superalgebra non-trivially by adding anticommuting "fermionic" generating elements to the algebra, known as the supercharges, Q. These elements are spinorial in nature, and are often denoted Q^i_α and $\overline{Q}^{\dot{\alpha}}_i$, where α and $\dot{\alpha}$ are spinor indices in the complex Weyl spinor and complex conjugate Weyl spinor representation respectively, and i is the internal isospin flavour index.

In the simplest case, there is one such pair of generators, known as $\mathcal{N} = 1$ supersymmetry. In this project, we will work with classes of supersymmetry with more generators. The Lie brackets between the generators of this superalgebra are given by (here we present the $4D$ $\mathcal{N}$-extended case):

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}) \\
[P_\mu, Q^i_\alpha] &= 0 \\
[M_{\mu\nu}, Q^i_\alpha] &= \frac{i}{2}(\sigma_{\mu\nu})_\alpha^\beta Q^i_\beta \\
[P_\mu, \overline{Q}^{\dot{\alpha}}_i] &= 0 \\
[M_{\mu\nu}, \overline{Q}^{\dot{\alpha}}_i] &= -\frac{i}{2}(\overline{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}\overline{Q}^{\dot{\beta}}_i \\
\{Q^i_\alpha, \overline{Q}^{\dot{\beta}}_j\} &= 2\delta^i_j P^{\dot{\beta}}_\alpha \\
\{Q^i_\alpha, Q^j_\beta\} &= \epsilon_{\alpha\beta}Z^{ij} \\
\{\overline{Q}^{\dot{\alpha}}_i, \overline{Q}^{\dot{\beta}}_j\} &= \epsilon_{\dot{\alpha}\dot{\beta}}(Z_{ij})^*,
\end{aligned} \tag{2.49}$$

where $\sigma^{\mu\nu}$ are the spin generators of the Lorentz group, $\eta_{\mu\nu}$ is the Minkowski metric and Z^{ij} are some central charges that commute with the rest of the algebra [20].

When the Q-generators act on bosonic states, they are transformed into fermionic states by modifying the spin by $\frac{1}{2}$. Note that each generator can be applied only once to a state due to their anticommuting nature. This means that applying each of the supercharges in the supersymmetry theory facilitates the construction of a finite-sized "multiplet" of bosonic and fermionic states. This in essence unifies particles of matter (fermions) and mediators of forces (bosons).

One can construct such a particle multiplet by considering "on-shell" (where the particle satisfies its equations of motion and has its physical mass) representations of this algebra. Firstly, one can observe that the operator P^2 commutes with all elements of the algebra, making it a Casimir of supersymmetry. The consequence of this is that all fields in a multiplet must have the same mass. The construction of the multiplet is quite straightforward, by treating the supercharges as ladder operators that increment spin rather than energy.

For field representations, one must consider "off-shell' representations, where the field may not satisfy the energy-momentum relation. These representations in essence describe the full physics of the theory. This is important for QFTs; for example, virtual particles are considered off-shell [12]. Such representations are more complex than on-shell ones. As they do not satisfy the energy-momentum relation, these representations have less constraints -

they have more degrees of freedom than physical degrees of freedom. For example, a massive spin 1 particle in 4D spacetime is described by a vector field. Although this vector field has four components, there are only three physical polarisations for this particle. This makes the theory more complex to study, and often one must impose additional constraints (such as gauge fixing) which can lead to different physical interpretations.

2.3.2 Superspace

Fortunately, in the case of supersymmetric theories, in the 1970s, the superspace formalism was introduced as a tool for describing off-shell supersymmetric theories [20]. In superspace, spacetime is extended by additional fermionic coordinates, which allow for supersymmetric fields to be described in a natural way.

In superspace, one has both commuting (that we are familiar with) and anticommuting coordinates,

$$[x, x] = 0, \quad [x, \theta] = 0, \quad \{\theta, \theta\} = 0. \tag{2.50}$$

where x are commuting coordinates and θ (and $\bar{\theta}$, depending on the number of SUSY generators) is used for the anticommuting coordinates. These anticommuting coordinates are Grassmann numbers and obey a particular set of rules with respect to integration and differentiation [20].

In order to understand the utility of this extension, first one must understand what is meant by spacetime. Recall that the Poincaré group is the group that preserves the structure of spacetime: translations, rotations, boosts, etc. This contains the Lorentz group as a subgroup, which is the group that preserves the structure of spacetime, and the origin. Because both of these groups are Lie groups, one can think of them as smooth manifolds. One can then think of a point in spacetime as a coset of the quotient group (Poincaré)/(Lorentz)- in other words, physical spacetime is the manifold in which points cannot be related by a Lorentz transformation. Effectively, one is left with a manifold corresponding to the translational subgroup of the Poincaré group, as expected. One is left with a manifold parametrised by

$$s(x) = e^{ix^\mu \hat{P}_\mu}, \tag{2.51}$$

where $\hat{P}_\mu$ are the generators of the quotient group.

Similarly, superspace is the corresponding coset space one obtains from taking the quotient (super-Poincaré)/(Lorentz) [20]. For the case of 4D $\mathcal{N} = 1$ supersymmetry, for example, one is left with a manifold parametrised by

$$S(x, \theta, \bar{\theta}) = e^{i(x^{\alpha\dot{\beta}} \hat{P}_{\alpha\dot{\beta}} + \theta^\alpha \hat{Q}_\alpha + \bar{\theta}^{\dot{\alpha}} \bar{\hat{Q}}_{\dot{\alpha}})}. \tag{2.52}$$

Again, $\hat{O}$ are the generators of the quotient group, and thus x, θ and $\bar{\theta}$ have the natural interpretation of coordinates. Note that the number of Grassmann coordinates in superspace is dependent on the dimension and number of SUSY generators. It is worth noting that these θ's and x's do not mix with each other under Lorentz transformations. This parameterisation also gives the action of the supercharge the interpretation of translation in superspace.

Then, to obtain supersymmetric representations of fields, we consider superfields $\Psi(x, \theta, \bar{\theta})$ as a function over superspace. The convienience of this is that one can expand the superfield as a Taylor series in θ (and $\bar{\theta}$) that will terminate at finite order due to the anticommuting nature of the Grassmann variables. The coefficient functions in these expansions can be thoought of the physical fields in the supersymmetry multiplet. For example, for the case of 3D $\mathcal{N} = 1$ SUSY, a scalar superfield $\Phi(x, \theta)$ can be expanded in terms of two real Grassmann coordinates as [20]

$$\Phi(x, \theta) = A(x) + \theta^\alpha \psi_\alpha(x) - \theta^2 F(x). \tag{2.53}$$

This Taylor series terminates at this order thanks to the anticommuting nature of θ and the fact that in 3D the spinor indix α takes values $\alpha = \pm$, which implies $\theta^\alpha\theta^\beta\theta^\gamma = 0$. This allows for one to express the field content of the theory in a single field over superspace. Rather than using this θ-expansion directly, one can rather extract the field content by taking covariant derivatives:

$$\begin{aligned} A(x) &= \Phi(x, \theta)|, \\ \psi_\alpha(x) &= D_\alpha \Phi(x, \theta)|, \\ F(x) &= D^2 \Phi(x, \theta)|, \end{aligned} \tag{2.54}$$

where the bar $|$ denotes projection by setting $\theta = 0$ after evaluating the derivatives, where the covariant derivative over superspace is (see Appendix A for notation)

$$D_\alpha = \partial_\alpha + \theta^\beta i \partial_{\alpha\beta}. \tag{2.55}$$

One can then construct a Lagrangian and an invariant action

$$\mathcal{L} = \int d^2\theta f(\Phi, D_\alpha \Phi, ...), \tag{2.56}$$

$$S = \int d^3x d^2\theta f(\Phi, D_\alpha \Phi, ...), \tag{2.57}$$

where f is known as the superspace Lagrangian. Using the properties of Grassmann integration, one can alternatively (and more usefully) write the action as

$$S = \int d^3x D^2 f(\Phi, D_\alpha \Phi, ...)|. \tag{2.58}$$

In our $d = 3$ example of the scalar superfield, by dimensional analysis, one obtains the action

$$S = -\frac{1}{2} \int d^3x d^2\theta (D_\alpha \Phi)^2 = \frac{1}{2} \int d^3x d^2\theta \Phi D^2 \Phi. \tag{2.59}$$

One can then use projections to find the spacetime Lagrangian density,

$$\Longrightarrow = \frac{1}{2} \int d^3x D^2 [\Phi D^2 \Phi]| = \frac{1}{2} \int d^3x (F^2 + \psi^\alpha i \partial_\alpha^\beta \psi_\beta + A \Box A). \tag{2.60}$$

From the Euler-Lagrange equations, one can see that $F = 0$ as an equation of motion. This auxiliary field can thus be neglected from the action - however, this means that the supersymmetry transformations will only close on-shell for A and ψ_α. Thus it is clear that the superspace formalism is a powerful tool for constructing off-shell representations of supersymmetry.

2.3.3 Extended Supersymmetry, Dimensional Dependancy of Representations, and Superspace Parametrisation

In general, a supersymmetric theory may possess more than one pair of supercharges. How representations of this algebra are realised, and thus the way in which we parameterise superspace, is highly dependent on the dimension under consideration. Indeed, the transformation properties of spinors are dimension-dependent. Spinors transform as representations of the Lorentz group $SO(d-1,d)$.

In $4D$, spinors transform under $so(3,1) \cong sl(2,\mathbb{C})$, and minimal representations can either be described in terms of two-component chiral Weyl spinors related by complex conjugation, or four-component real Majorana spinors. Thus, there are four independent real degrees of freedom associated with a spinor. So for the fermionic supercharges, there are four real associated Grassmann coordinates. This means for $n-$extended supersymmetry, there are $4n$ associated Grassmann coordinates. Since a reality condition cannot be imposed on the Weyl spinors, $4D$ extended SUSY theories are labelled by a single integer, $\mathcal{N}=n$.

In $2D$, spinors transform under $so(1,1) \cong \mathbb{R}$. The minimal representations can thus be described by two, one-component, chiral Weyl spinors. Due to the above isomorphism, it is possible to impose reality conditions on the Weyl spinors, as they stay real under the action of the Lorentz group. Thus the two Weyl spinors are independent and real. These two spinors are labeled by their chirality $\pm$, which determines the representation (fundamental/anti-fundamental) under which they transform. The spinors are thus dubbed as left-moving or right-moving, which will be further discussed below. Due to these reasons, in simple supersymmetry, the spinorial supercharges can be real, and labelled by their chirality eigenvalues, and thus have one Grassmann coordinate each. In extended supersymmetry, there may be p left-moving supercharges, and q right-moving supercharges, which are independent. Thus, in $2D$, extended supersymmetry theories must be labelled by two integers, $\mathcal{N}=(p,q)$, and superspace is parameterised by p real left-moving Grassmann coordinates and q real right-moving Grassmann coordinates.

2.4 Lightcone Coordinates

Before discussing T$\bar{\text{T}}$ and SUSY, we will first introduce a more convenient coordinate system for the 2D Lorentzian theories which are the subject of our studies. These are the lightcone coordinates, given by

$$x^{\pm\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \tag{2.61}$$

where (x^0, x^1) are the usual space and time coordinates in 2D. These coordinates describe the position of a particle on the lightcone, and thus are natural for describing massless particles in a 2D theory. They are the real analogue of holomorphic and antiholomorphic complex coordinate of the Euclidian plane. These coordinates are termed either left-moving (x^{--}), or right-moving (x^{++}), as a massless particle moving left $(-x^1$ direction) will be described entirely by the x^{--} coordinate with $x^{++}=0$ along its trajectory, and likewise for a right-moving massless particle. Note that these coordinates are invariant under Lorentz transformations, and obey

$$\partial_{\pm\pm} x^{\mp\mp} = 0. \tag{2.62}$$

Usefully, the number of $\pm$ indices on objects in this coordinate system describe how the object transforms under Lorentz transformations. For example, spinorial objects will have only a single $\pm$ index, while vectors will have two $\pm$ indices, and so-on for higher spin objects. A description of the index rules of such objects in this coordinate system can be found in Appendix A.1. Note also that in contrast to the usual Einstein summation notation, repeated lightcone indices do not denote summation.

2.5 T$\bar{\text{T}}$ and Supersymmetry

The usual T$\bar{\text{T}}$ deformation is constructed from the stress-energy tensor, the conserved current generated by spacetime translations. In order to obtain a supersymmetric analogue to the deformation, it would be desirable to work with the conserved currents generated by translations in superspace. To demonstrate how this works, consider $\mathcal{N} = (1,1)$ supersymmetry, in 2D [6] (which is a direct analogue of 3D $\mathcal{N} = 1$ described by two real supercharges and two real Grassmann spinors θ^+ and θ^-). The Lagrangian for such a theory, in its most general form, is given by

$$\mathcal{L} = \int d^2\theta f(\Phi, D_+\Phi, D_-\Phi, \partial_{++}\Phi, \partial_{--}\Phi, D_+D_-\Phi). \tag{2.63}$$

This Lagrangian has the associated equation of motion

$$\frac{\delta f}{\delta \Phi} = D_+\left(\frac{\delta f}{\delta D_+\Phi}\right) + D_-\left(\frac{\delta f}{\delta D_-\Phi}\right) + \partial_{++}\left(\frac{\delta f}{\delta \partial_{++}\Phi}\right) + \partial_{--}\left(\frac{\delta f}{\delta \partial_{--}\Phi}\right) - D_+D_-\left(\frac{\delta f}{\delta D_+D_-\Phi}\right). \tag{2.64}$$

Once can then express the variation of f, δf, in terms of $\delta\Phi$ by using the product rule,

$$\begin{aligned}
\delta f = \delta\Phi &\left[D_+\left(\frac{\delta f}{\delta D_+\Phi}\right) + D_-\left(\frac{\delta f}{\delta D_-\Phi}\right) + \partial_{++}\left(\frac{\delta f}{\delta \partial_{++}\Phi}\right) + \partial_{--}\left(\frac{\delta f}{\delta \partial_{--}\Phi}\right) - D_+D_-\left(\frac{\delta f}{\delta D_+D_-\Phi}\right)\right] \\
&= D_+\left(\delta\Phi\frac{\delta f}{\delta D_+\Phi}\right) + D_-\left(\delta\Phi\frac{\delta f}{\delta D_-\Phi}\right) + \partial_{++}\left(\delta\Phi\frac{\delta f}{\delta \partial_{++}\Phi}\right) + \partial_{--}\left(\delta\Phi\frac{\delta f}{\delta \partial_{--}\Phi}\right) \\
&\quad + \frac{1}{2}\left(D_+\left(\frac{\delta f}{\delta D_+D_-\Phi}D_\delta\Phi\right) + D_-\left(\delta\Phi D_+\frac{\delta f}{\delta D_+D_-\Phi}\right)\right) \\
&\quad - \frac{1}{2}\left(D_-\left(\frac{\delta f}{\delta D_+D_-\Phi}D + \delta\Phi\right) + D_+\left(\delta\Phi D_-\frac{\delta f}{\delta D_+D_-\Phi}\right)\right) \\
-\delta\Phi&\left(-\frac{\delta f}{\delta\Phi} + D_+\left(\frac{\delta f}{\delta D_+\Phi}\right) + D_-\left(\frac{\delta f}{\delta D_-\Phi}\right) + \partial_{++}\left(\frac{\delta f}{\delta \partial_{++}\Phi}\right) + \partial_{--}\left(\frac{\delta f}{\delta \partial_{--}\Phi}\right) - D_+D_-\left(\frac{\delta f}{\delta D_+D_-\Phi}\right)\right).
\end{aligned} \tag{2.65}$$

Note that the last line is equal to zero when the variations are on-shell, and thus vanishes. Now, consider some small spacetime translation, $\delta x^{\pm\pm} = \epsilon^{\pm\pm}$. Then, one has the variations $\delta f = \epsilon^{++}\partial_{++}f + \epsilon^{--}\partial_{--}f$, and $\delta\Phi = \epsilon^{++}\partial_{++}\Phi + \epsilon^{--}\partial_{--}\Phi$. Given that the former is a total derivative, it vanishes under integration when computing the action. Then, in order for the action to remain invariant, the righthand side of (2.65) must be equal to 0 under the variation of Φ. Applying this variation yields the constraint

$$\begin{aligned}0 = \epsilon^{++}D_+\Big[&\partial_{++}\Phi\frac{\delta f}{\delta D_+\Phi}+D_+\Big(\partial_{++}\Phi\frac{\delta f}{\delta\partial_{++}\Phi})+\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_-(\partial_{++}\Phi)-\frac{1}{2}\partial_{++}\Phi D_-(\frac{\delta f}{\delta D_+D_-\phi}\Big)-D_+f\Big]\\
+\epsilon^{++}D_-\Big[&\partial_{++}\Phi\frac{\delta f}{\delta D_-\Phi}+D_-\Big(\partial_{++}\Phi\frac{\delta f}{\delta\partial_{--}\Phi})-\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_+(\partial_{++}\Phi)+\frac{1}{2}\partial_{++}\Phi D_+(\frac{\delta f}{\delta D_+D_-\phi}\Big)\Big]\\
+\epsilon^{--}D_+\Big[&\partial_{--}\Phi\frac{\delta f}{\delta D_+\Phi}+D_+\Big(\partial_{--}\Phi\frac{\delta f}{\delta\partial_{++}\Phi})+\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_-(\partial_{--}\Phi)-\frac{1}{2}\partial_{--}\Phi D_-(\frac{\delta f}{\delta D_+D_-\phi}\Big)\Big]\\
+\epsilon^{--}D_-\Big[&\partial_{--}\Phi\frac{\delta f}{\delta D_-\Phi}+D_-\Big(\partial_{--}\Phi\frac{\delta f}{\delta\partial_{--}\Phi})-\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_+(\partial_{--}\Phi)+\frac{1}{2}\partial_{--}\Phi D_+(\frac{\delta f}{\delta D_+D_-\phi}\Big)-D_-f\Big].\end{aligned} \tag{2.66}$$

Making the definitions

$$\mathcal{T}_{++-} = \partial_{++}\Phi\frac{\delta f}{\delta D_+\Phi}+D_+\Big(\partial_{++}\Phi\frac{\delta f}{\delta\partial_{++}\Phi})+\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_-(\partial_{++}\Phi)-\frac{1}{2}\partial_{++}\Phi D_-(\frac{\delta f}{\delta D_+D_-\phi}\Big)-D_+f, \tag{2.67}$$

$$\mathcal{T}_{+++} = \partial_{--}\Phi\frac{\delta f}{\delta D_+\Phi}+D_+\Big(\partial_{--}\Phi\frac{\delta f}{\delta\partial_{++}\Phi})+\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_-(\partial_{--}\Phi)-\frac{1}{2}\partial_{--}\Phi D_-(\frac{\delta f}{\delta D_+D_-\phi}\Big), \tag{2.68}$$

$$\mathcal{T}_{---} = \partial_{--}\Phi\frac{\delta f}{\delta D_+\Phi}+D_+\Big(\partial_{--}\Phi\frac{\delta f}{\delta\partial_{++}\Phi})+\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_-(\partial_{--}\Phi)-\frac{1}{2}\partial_{--}\Phi D_-(\frac{\delta f}{\delta D_+D_-\phi}\Big), \tag{2.69}$$

$$\mathcal{T}_{--+} = \partial_{--}\Phi\frac{\delta f}{\delta D_-\Phi}+D_-\Big(\partial_{--}\Phi\frac{\delta f}{\delta\partial_{--}\Phi})-\frac{1}{2}\frac{\delta f}{\delta D_+D_-\Phi}D_+(\partial_{--}\Phi)+\frac{1}{2}\partial_{--}\Phi D_+(\frac{\delta f}{\delta D_+D_-\phi}\Big)-D_-f, \tag{2.70}$$

one can plainly see that these are (divergenceless) conserved quantities for each lightcone coordinate satisfying the conservation equations

$$D_+\mathcal{T}_{++-} + D_-\mathcal{T}_{+++} = 0, \tag{2.71}$$

$$D_+\mathcal{T}_{---} + D_-\mathcal{T}_{--+} = 0. \tag{2.72}$$

Now, consider the flow equation

$$\frac{\partial}{\partial\lambda}f^{(\lambda)} = \mathcal{T}^{(\lambda)}_{+++}\mathcal{T}^{(\lambda)}_{---} - \mathcal{T}^{(\lambda)}_{--+}\mathcal{T}^{(\lambda)}_{++-}. \tag{2.73}$$

We claim that this supercurrent-square deformation is the $\mathcal{N} = (1,1)$ manifestly supersymmetric analogue to the T$\bar{\text{T}}$ deformation [6]. To explicitly see this, consider the free scalar superspace Lagrangian (this time in lightcone coordinates),

$$f = D_+\Phi D_-\Phi, \tag{2.74}$$

with superfield expansion

$$\Phi(x,\theta^+,\theta^-) = A(x) + i\theta^+\psi_+(x) + i\theta^-\psi_-(x) + \theta^+\theta^- F(x). \tag{2.75}$$

Then, the supercurrent takes the form

$$\mathcal{T}_{++-} = \partial_{++}\Phi D_-\Phi - D_+(D_+\Phi D_-\Phi), \tag{2.76}$$

$$\mathcal{T}_{+++} = -\partial_{++}\Phi D_+\Phi, \tag{2.77}$$

$$\mathcal{T}_{---} = \partial_{--}\Phi D_-\Phi, \tag{2.78}$$

$$\mathcal{T}_{--+} = -\partial_{--}\Phi D_+\Phi - D_-(D_+\Phi D_-\Phi). \tag{2.79}$$

Integrating out the Grassmann coordinates, one gets the term

$$\int d^2\theta(\mathcal{T}_{+++}\mathcal{T}_{---} - \mathcal{T}_{--+}\mathcal{T}_{++-}) = -\Big((\partial_{++}\phi)^2 + \psi_+\partial_{++}\psi_+\Big)\Big((\partial_{--}\phi)^2 + \psi_-\partial_{--}\psi_-\Big)$$
$$-2\partial_{++}\phi\partial_{--}\phi(\psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-) - \psi_-\partial_{++}\psi_-\psi_+\partial_{--}\psi_+. \tag{2.80}$$

The first term is precisely the usual T$\bar{\text{T}}$ field for the scalar theory in question. The remaining two terms are proportional to the fermion equations of motion, $\partial_{\pm\pm}\psi_\mp = 0$. These terms thus vanish on-shell, and thus the supercurrent-squared deformation is equivalent to T$\bar{\text{T}}$. It is also possible to prove that this deformation indeed has the same solvability properties as the T$\bar{\text{T}}$ deformation [6].

One can construct similar supercurrent-squared deformations for SUSY theories for different numbers of generators. This has been done for $\mathcal{N} = (1,1)$ supersymmetry, as seen above, which can easily be reduced to $\mathcal{N} = (0,1)$ SUSY [3]. A similar, but more involved, approach can be followed to obtain the deformation for $\mathcal{N} = (2,2)$ SUSY and $\mathcal{N} = (0,2)$ SUSY [4] [5] [6]. In this thesis, we aim to extend this construction to the $\mathcal{N} = (0,4)$ and $\mathcal{N} = (4,4)$ cases.

2.6 4D T$\bar{\text{T}}$-like Deformations

$4D$ supersymmetry has interesting links to $2D$ supersymmetry, and thus can be a potentially helpful tool in studying $\mathcal{N} = (4,4)$ and $\mathcal{N} = (0,4)$ supercurrent-squared deformations - they should be related to analogous 4D deformation via dimensional reductions and truncation. Thus, we shall also review these $4D$ deformations.

In two dimensions, the T$\bar{\text{T}}$ operator is given by

$$\text{T}\bar{\text{T}} = T_{\mu\nu}T^{\mu\nu} - (T^\mu_\mu)^2 = T^{\mu\nu}T_{\mu\nu} - \Theta^2 \propto \det T_{\mu\nu}. \tag{2.81}$$

One can attempt to generalise this operator to higher dimensions by considering the operator

$$O^{[r]}_{T^2} = T^{\mu\nu}T_{\mu\nu} - r\Theta^2, \quad r \in \mathbb{R}. \tag{2.82}$$

In $2D$, choosing $r = 1$ uniquely yields a well-defined operator at the quantum level [16]. Unfortunately, in higher dimensions, there is no such known r that yields as well-defined operator. However, certain values of r make interesting choices when deforming a classical theory. For example, consider a constant shift in the vacuum energy of the theory (for example, generated by a cosmological constant),

$$\mathcal{L} \to \mathcal{L} + c, \quad T^{\mu\nu} \to T^{\mu\nu} - c\eta^{\mu\nu}. \tag{2.83}$$

This transforms the operator (2.82) as [16]

$$O_{T^2}^{[r]} \to O_{T^2}^{[r]} + 2c(2r-1)\Theta + 4c^2(1-r). \tag{2.84}$$

This Θ trace term causes the operator to transform in a non-trivial way, and vanishes only in the case $r = \frac{1}{2}$. Note that any other choice is peculiar, as the dynamics of the undeformed theory are modified by a shift in energy. For this choice of r, the dynamics of the theory are unaffected. Thus, we will consider henceforth

$$O_{T^2} = T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}\Theta^2. \tag{2.85}$$

One can then consider the generalisation of this operator to $4D$ $\mathcal{N} = 1$ SUSY, as seen in [16]. To do so, we will assume there exists a Ferrara-Zumino multiplet of currents, described by a vector superfield $\mathcal{J}_\mu$ and complex scalar superfield $\mathcal{X}$ satisfying the constraints

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha \mathcal{X}, \qquad \bar{D}^{\dot{\alpha}}\mathcal{X} = 0. \tag{2.86}$$

Solving these constraints gives a supercurrent of the form

$$\begin{aligned}
\mathcal{J}_\mu(x) =& j_\mu(x) + \theta\left(S_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar{\chi}\right) + \bar{\theta}\left(\bar{S}_\mu + \frac{1}{\sqrt{2}}\bar{\sigma}_\mu\chi\right) + \frac{i}{2}\theta^2\partial_\mu\bar{\mathrm{x}} - \frac{i}{2}\bar{\theta}^2\partial_\mu\mathrm{x} \\
&+ \theta\sigma^\nu\bar{\theta}\left(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}\Theta - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) \\
&- \frac{i}{2}\theta^2\bar{\theta}\left(\bar{\partial\!\!\!/}S_\mu + \frac{1}{\sqrt{2}}\bar{\sigma}_\mu\ \partial\!\!\!/\bar{\chi}\right) - \frac{i}{2}\bar{\theta}^2\theta\left(\partial\!\!\!/\bar{S}_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar{\partial\!\!\!/}\chi\right) \\
&+ \frac{1}{2}\theta^2\bar{\theta}^2\left(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\right),
\end{aligned} \tag{2.87}$$

and scalar field of the form

$$\begin{aligned}
\mathcal{X}(y) &= \mathrm{x}(y) + \sqrt{2}\theta\chi(y) + \theta^2\mathrm{F}(y), \\
\chi_\alpha &= \frac{\sqrt{2}}{3}\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\bar{S}_\mu^{\dot{\alpha}}, \quad \mathrm{F} = \frac{2}{3}\Theta + i\partial_\mu j^\mu,
\end{aligned} \tag{2.88}$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and $\partial\!\!\!/ = \sigma^\mu\partial_\mu$, $\bar{\partial\!\!\!/} = \bar{\sigma}^\mu\partial_\mu$. Then, considering the supercurrent-squared operator over supersapce given by

$$\mathcal{O}_{T^2} = -\frac{1}{2}(\eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu + \frac{5}{4}\chi\bar{\chi}), \tag{2.89}$$

one can integrate over the Grassmann coordinates to obtain the supersymmetric operator

$$\mathrm{O}_{T^2} = \int d^\theta \mathcal{O}_{T^2} = T^2 - \frac{1}{2}\Theta^2 + \frac{3}{8} j_\mu \partial^2 j^\mu + \frac{3}{8}\partial_\mu \mathrm{x} \partial^\mu \bar{\mathrm{x}} - \frac{i}{2}\Big(S_\mu \ \partial\!\!\!/ \bar{S}^\mu - \frac{9}{4}\bar{\chi}\ \bar{\partial\!\!\!/}\chi\Big) + \text{EOM} + \text{tot. derivs.}, \tag{2.90}$$

which is clearly the natural extension of the operator O_{T^2} in the supersymmetric setting. It is worth noting that for $2D$, $\mathrm{O}_{T^2} = O_{T^2}$ up to terms that vanish on-shell, but clearly for $4D$ O_{T^2} has extra current contributions. It seems as if it is not possible to construct an O_{T^2} such that $\mathrm{O}_{T^2} = O_{T^2}$ from the combinations of the supercurrent superfield squared in $4D$ $\mathcal{N} = 1$ SUSY. This suggests that the two operators will lead to two different flows, in contrast to the $2D$ case.

Despite the fact that the flow generated by this operator O_{T^2} is not well-defined at the quantum level, it can still generate a classical flow. For example, it has been shown that the $4D$ $\mathcal{N} = 1$ Supersymmetric Born-Infeld Lagrangian can be obtained as a supercurrent-squared flow of the $\mathcal{N} = 1$ free Maxwell Lagrangian [16]. What's interesting about these $4D$ flows is their relationship to $2D$ flows. In particular, the non-linearly realized supersymmetry in two-dimensional $\mathcal{N} = (2, 2)$ theories can be described by a supercurrent-squared term, which is similar to the one in the four-dimensional $\mathcal{N} = 1$ supersymmetric Born-Infeld Lagrangian. Moreover, to some extent, 4D supersymmetry is better studied than extended supersymmetry in 2D. The 4D $\mathcal{N} = 1$ case is related by a dimensional reduction to 2D $\mathcal{N} = (2, 2)$. The case of 2D $\mathcal{N} = (4, 4)$ is directly connected to 4D $\mathcal{N} = 2$ though the former is much less understood. For instance, the general structure of the supercurrent multiplet for 2D $\mathcal{N} = (4, 4)$ is not known yet. This is a problem since the recent classification of manifestly supersymmetric forms of the $T\bar{T}$ operators is based on supercurrent squared operators. A large part of our work is then to construct supercurrents for extended supersymmetry and then explore $T\bar{T}$ operators. Since the constraints for the 4D $\mathcal{N} = 2$ supercurrent are better understood, analysis of this theory is a good starting point for understanding the dimensionally reduced case in 2D.

3 Design and Usage of the Tensor Algebra Software

Working in the superspace formalism, one can see that as the number of supersymmetry generators of the theory increases, the number of Grassmann coordinates that are needed to describe superspace increases in step. Since the superfield expansion involves all possible products of these Grassmann coordinates, the number of terms in the superfield expansion increases exponentially with the number of generators. Indeed, for $\mathcal{N} = (p, q)$ supersymmetry, there are roughly $O(2^{p+q})$ possible terms. Additionally, in a general study of spacetime and curved space (outside the scope of our project) if one would like to take (potentially many applications of) covariant derivatives on such an expression, then the number of terms in the final expression will be roughly $O((1 + k)^n \cdot 2^{p+q})$, where k is the number of connection terms, and n is the number of applications of the derivative. Clearly, in a theory with even a modest number of connection terms, these expressions grow quickly and become exceptionally unwieldy.

Additionally, when performing algebra in the supersymmetric setting, one must keep in mind the grading (whether the objects are bosonic or fermionic) of objects, including operators, as anticommuting objects acquire a minus sign when swapped. This makes by-hand computations potentially error-prone when commuting variables in large expressions.

This motivates the use of computer algebra software to perform such computations.

Although there are a few pieces of software that can do similar manipulations already [21] [22], they each have their limitations; either they are unwieldy to use, require licensed software, don't handle graded objects natively (or act unpredictable when performing operations like the graded Leibniz rule), or don't allow for the use of multi-index covariant derivatives. For these reasons we opted to create our own software package that supports all of the operations needed to perform algebraic manipulations in the supersymmetric setting required for our project. This includes many features that would be useful for any manipulation of algebraic expressions involving tensors.

This package is written in Julia. Julia is a high-level computer language, that runs with speeds comparable to low-level languages [23] [24]. It was designed with scientific computing in mind, and thus includes many convenient features for those in the sciences. This language was thus chosen as it possesses the advantages of being fast, modular, easy to program in, and has the ability to use Unicode characters when writing code. Speed is important when analysing long and complex expressions like those that arise in our use case, and the use of Unicode characters is convenient in a mathematics/physics context as the user can name variables with Greek letters, for example.

In this chapter, we will discuss how to use the package, as well as some of the design philosophy and some runtime analysis of some of the main algorithms. Note that this package was written entirely from scratch; everything presented in this chapter is original work.

3.1 Basics

3.1.1 Setup

To start using the package, firstly run the code

```
include("alakazam.jl")
```

This will automatically import all functions and initialise all internal variables appropriately. For example, the package automatically creates variables `upper` and `lower`, which are used to denote index positions when creating a tensor, and `zero` representing an instance of the empty expression.

Note that the package has dependancies `Symbolics`, `OrderedCollections`, `StatsBase`, `SymbolicUtils`, and `DataStructures`. These should be installed first via the Julia package manager in order for this package to run.

3.1.2 `IndexSet`

Next, unless your theory contains only scalars, you should define indices and index sets. Firstly, one must create an `IndexSet`. Index sets are necessary, as they tell the package which indices can be drawn upon as dummy indices, or whether a replacement pattern (discussed later) is equivalent to an expression modulo index names. The constructor is given by

```
IndexSet(name::String, range::Int=1, default_position::IndexPosition=upper)
```

Here, `name` is the name of the index set, `range` is the number of values the index ranges over in summation (the dimension of the space the indices represent), and `default_position` is the natural position an index should be found in on an object used when deciding how to expand an expression using metrics into a canonical form. To create this object, one could use the below code for example

```
flavour_indices = IndexSet("flavour", 2,lower)
spacetime_indices = IndexSet("spacetime", 4,lower)
```

3.1.3 `Index`

Now, one must create the indices that can be attached to tensors. The user should ensure that they create enough indices in order to perform all algorithms (for example, many extra indices to be used as dummy indices may be needed depending on the algorithm) required for their problem. The constructor is given by

```
Index(name::String,index_set::IndexSet=IndexSet(""), grading::Int=0)
```

Here, `name` will be used to identify the index and should be a single unique character. There are no restrictions on the name of this index, except that it should not use special characters `* ? _ { } [ ] ^` which are all reserved for pattern matching logic. This means that unicode characters are also valid, including dotted and barred characters. Upon construction, the new index will be automatically assigned to the `IndexSet`. In this software, information on whether tensors commute or anti-commute is encoded in the indices. The `grading` parameter can be set to 0 if the index is bosonic, and 1 if the index is fermionic. By default, indices are bosonic. For example,

```
i = Index("i", flavour_indices, 0)
j = Index("j", flavour_indices)
α = Index("α", spinor_indices, 1)
β = Index("β", spinor_indices, 1)
αdot = Index("˙α", spinor_conj_indices, 1)
βdot = Index("˙β", spinor_conj_indices, 1)
```

3.1.4 `Coordinate`

If the user wishes to study tensor fields, rather than constant tensors, coordinates can be defined. Tensors can then be specified to be a function of these coordinates. This information is used later in computing derivatives, for example. The coordinate constructor is given by

```
Coordinate(name::String, index_set::IndexSet)
Coordinate(name::String, index_set::Vector{IndexSet})
```

Here, `name` is the name of the coordinate, which can be used when printing an expression. `index_set` can be either a single `IndexSet` or a vector of `IndexSet`s, which allow for coordinates to have multiple associated indices. For example, one may wish to represent the vector of spacetime coordinates in terms of spinor indices (bispinor convention), and would thus use two different associated spinor indices. To see the usage in practice,

```
    x = Coordinate("x", spacetime_indices)
    θ = Coordinate("θ", [flavour_indices, spinor_indices])
```

3.1.5 `Tensor`

The package contains a hierarchy of objects that allow tensor algebra to be performed. At the lowest level, there are `Tensor` objects, representing a single tensor. These single tensors can then be multiplied together to create a `TensorTerm`. These terms can then be added together to get a `TensorExpression`. Most functions are defined on a `TensorExpression`, but the package automatically will parse a `Tensor` or `TensorTerm` as a `TensorExpression` if needed for a particular function.

To create a `Tensor`, one can use the constructor

```
Tensor(name::String,
indices::Vector{Pair{I, IndexPosition}}
    =Vector{Pair{IndexSuperType, IndexPosition}}(),
function_of::Coordinate=Vector{Coordinate}(),
weights::Dict{String,Number}=Dict{String, Number}())
```

Here, `name` is the string that represents the tensor when doing pattern matching or display. `indices` is a vector containing pairs of indices and their positions. `function_of` is used to specify that the `Tensor` is a tensor field, dependent on the specified coordinates. `weights` can be used to keep track of other arbitrary user-defined properties on the tensor, for example, polynomial powers. For a basic usage example, consider

```
    A = Tensor("A", [i => upper, j => lower], x)
```

This will produce the tensor field

$$A^{i}{}_{j}(x). \tag{3.1}$$

It is important to stress that index order is important. There are no implicit assumptions about symmetry in the indices, which means

$$A^{i}{}_{j}(x) \neq A_{j}{}^{i}(x). \tag{3.2}$$

One should keep this in mind when using the package, as it will treat a tensor with a lower then upper index, differently from a tensor with an upper then lower index (for example, when doing algebra or pattern matching).

For totally symmetric and totally antisymmetric tensors, there exist tensor objects in the package which are treated with these symmetries in mind when performing calculations and simplifications. These have similar constructors to the regular `Tensor` type,

```
SymmetricTensor(name::String,
indices::Vector{Pair{I, IndexPosition}}
    =Vector{Pair{IndexSuperType, IndexPosition}}(),
```

```
function_of::Coordinate=Vector{Coordinate}(),
weights::Dict{String,Number}=Dict{String, Number}())
```

```
AntisymmetricTensor(name::String,
indices::Vector{Pair{I, IndexPosition}}
    =Vector{Pair{IndexSuperType, IndexPosition}}(),
function_of::Coordinate=Vector{Coordinate}(),
weights::Dict{String,Number}=Dict{String, Number}())
```

In the simplification algorithms in the software, terms involving `SymmetricTensor` or `AntisymmetricTensor`s will be merged appropriately if the terms are equivalent up to permutation of indices. `AntisymmetricTensor`s with repeated indices on the same level will also be annihilated.

3.1.6 `TensorTerm`

Products of `Tensor`s are stored as a `TensorTerm`. To create a `TensorTerm`, one can use either of the constructors

```
    TensorTerm(terms::Vector{T} where {T<:TensorSuperType})
    TensorTerm(term::TensorSuperType)
```

Note that upon construction, the field `indices` in `TensorTerm` will be initialised to contain the full set of indices of each of the `Tensor`s it contains in `terms`.

However, it is more natural to simply create a `TensorTerm` through multiplication of two `Tensor`s, for example

```
        A = Tensor("A", [i => upper, j => lower], x)
        B = Tensor("B", [j => upper, k => lower], x)
        AB = A*B
```

Here, AB will be a `TensorTerm`, AB= ${A^i}_j(x){B^j}_k(x)$.

3.1.7 `TensorExpression`

In order to represent a full equation involving tensors, one must also be able to add tensors and products of tensors. Such equation objects are stored as a `TensorExpression`. The `TensorExpression` object has many possible constructors for all possible ways of creating such an equation. However, it is more intuitive to create a `TensorExpression` using algebraic operations on `Tensor`s.

```
      A = Tensor("A", [i => upper, j => lower], x)
      B = Tensor("B", [j => upper, k => lower], x)
```

```
    AB = A*B #This will be a TensorTerm
    C = Tensor("C", [i => upper, k => lower], x)

    exp=AB+2*C #This will be a TensorExpression
```

The above example encodes the expression

$$A^i{}_j(x)B^j{}_k(x) + 2C^i{}_k(x). \tag{3.3}$$

Any type of number, or symbol from the Symbolics package, may be used as coefficients for terms in an expression. Note that when creating a `TensorExpression` in this way, the package will throw an error when trying to add together terms that have different free indices:

```
    AB = Tensor("A", [i => upper, j => lower], x)
        *Tensor("B", [j => upper, k => lower], x)

    CD =  Tensor("C", [i => upper, l => lower], x)
        *Tensor("D", [l => upper, k => lower], x)
    exp1 = AB+CD #This is ok, terms have the same free indices,
                # even though they use different dummy indices

    EF =  Tensor("E", [i => upper, j => lower], x)
        *Tensor("F", [j => upper, l => lower], x)
    exp2 = AB+EF #This is will error, free index mismatch

    GH =  Tensor("G", [i => lower, j => lower], x)
        *Tensor("H", [j => upper, k => upper], x)
    exp3 = AB+GH #This is will error, index position mismatch

```

Note also that the tensors need not be dependent on the same coordinates.

3.1.8 Operators

Often, for a variety of applications, operators need to be used. The simplest example would be a derivative which has a Leibniz rule action, but the package should also be able to handle operators that act with a "homomorphism-like" action, for example $O(AB) = O(A)O(B)$. All operators descend from the `OperatorSuperType` type, which allow them to share a common set of functions. Users can extend `OperatorSuperType` to create their own custom operators with other unique behaviour. All operators extending `OperatorSuperType` are however assumed to possess a set of common parameters,

```
    Operator(name::String, operand::{TensorExpression, Tensor, TensorTerm},
        operator_indices::Vector{Pair{Index, IndexPosition}}
            =Vector{Pair{Index, IndexPosition}}())
```

Here, `name` will be the name of the operator used in pattern matching and display. `operand` can be any type of `TensorExpression`, `Tensor`, `TensorTerm`, or even another `Operator`, noting however that regardless of the type, operand will be internally stored (and thus must be accessed as) a `TensorExpression`. An operator is free to have its own `operator_indices`, though it should be understood that an `Operator` object also contains a parameter `operand_indices` automatically extracted from the `operand`, and a parameter `indices` which is the union of the operand and operator indices. Note that `OperatorSuperType` is a subtype of `TensorSuperType`, and thus an `Operator` is treated as single tensor under functions. Some functions (for example simplify algorithms) may check if the tensor in question is an operator, and recursively call the function on its operand.

3.1.9 Typing Hierarchy

In Julia, objects can only extend abstract types. Thus, abstract "super"-types are used that possess the necessary fields and methods that all subtypes can extend and use. This means that if the user would like to define a new type of tensor with custom behaviour, they cannot extend the `Tensor` type, but must extend `TensorSuperType` in order to use any default methods defined on tensors. The current typing hierarchy in the package can be seen in Figure 3.1.

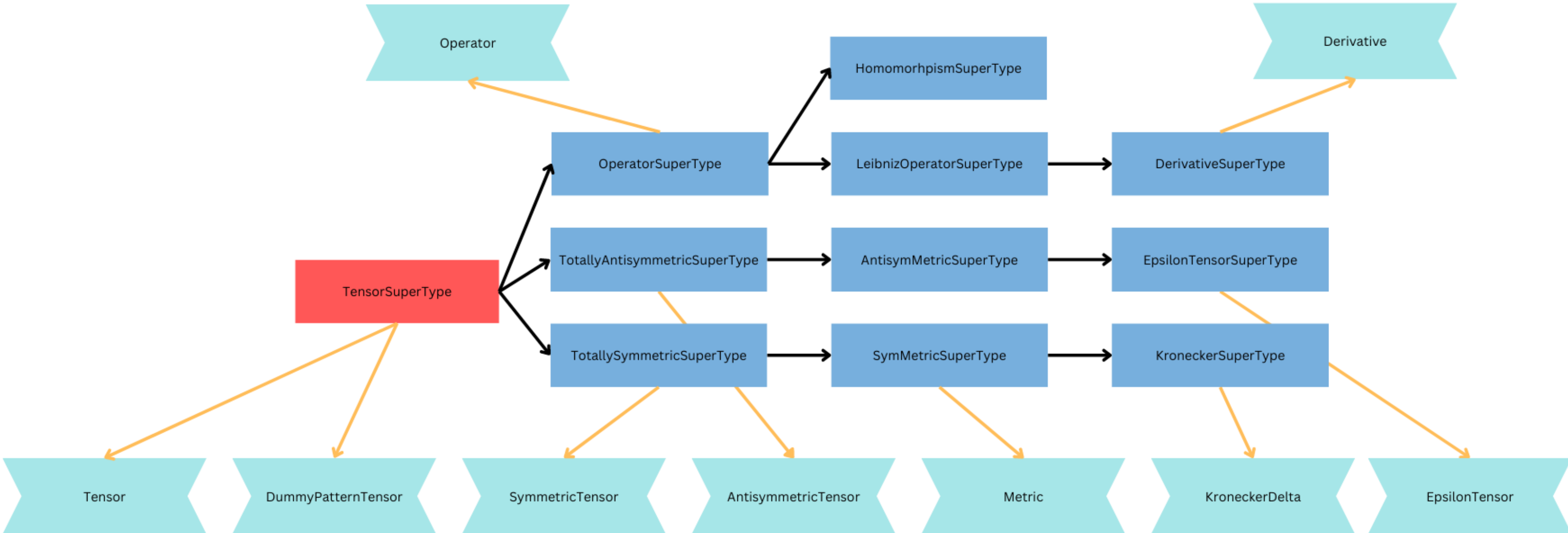


Figure 3.1: Type Hierarchy. Arrows denote inheritance. Cyan types cannot be extended. Blue types are abstract types and can be extended. All types extend the red `TensorSuperType`.

3.2 Useful Functions and Objects

Mathematical operators, like addition, subtraction and multiplication of tensors, and addition, subtraction, multiplication, and division by numbers (or Symbolics) all function in the natural way. There are in addition numerous other functions implemented, specific to tensors, in this package.

3.2.1 Tensor Appension (`×`)

Suppose one has tensors,

```
A = Tensor("A", [i => upper, j => lower], x)
B = Tensor("B", [i => upper], x)
C = Tensor("C", [i => lower, j => lower], x)
```

Then, if the user would like to multiply these tensors, this would produce an error as the resulting expression would have an invalid index structure. If the user would like to multiply such tensors, without performing any contractions over the indices, instead of creating new tensors such that the indices differ, the tensor appension binary operator `×` can be used:

```
exp=A×B×C
```

which produces output `A^{i*}_{*j}B^{k}C_{lm}`. Indices are automatically labelled by indices from the correct `IndexSet` and dummy indices are preserved. For example,

```
E = Tensor("E", [i => upper, j => lower, α => upper], x)
F = Tensor("F", [i => upper, α => lower, β => upper], x)
G = Tensor("G", [i => lower, i => upper, j => lower], x)

exp=E×F×G
```

produces output `E^{i*α}_{*j*}F^{k*β}_{*γ*}G^{*l*}_{l*m}`.

3.2.2 `Metric`

The ability to raise and lower tensor indices is a requirement for any useful sort of manipulation of tensor expressions. To create a `Metric`, one can use the constructor

```
Metric(name::String="η",
    indices::Vector{Pair{Index, IndexPosition}}
        =Vector{Pair{Index, IndexPosition}}())
```

Note that this `Metric` is a subtype of `SymMetricSuperType`. If the user needs an antisymmetric metric, for example, they can use an `EpsilonTensor` (see below), or extend `AntisymMetricSuperType`. This `Metric` tensor can then be multiplied with expressions. To perform a contraction, one can use the `contract_metrics()` function,

```
contract_metrics(T::TensorExpression, typ::Type=Metric)
```

Note that the parameter `typ` can be used to selectively contract only certain kinds of metrics depending on the application. To contract all sorts of metrics, one can use `SymMetricSuperType` and `AntisymMetricSuperType` as `typ`, as all metrics are a subtype of either one of these.

As an example of the usage, consider

```
A = Tensor("A", [i => upper, j => lower], x)
ηjk = Metric([j => upper, k => upper])

exp = ηjk*A

out = contract_metrics(exp)
```

This will have the output `A^{ik}`.

3.2.3 `KroneckerDelta`

The `KroneckerDelta` type in this package is a tensor that is a subtype of `TotallySymmetricSuperType`. To create a `KroneckerDelta`, one can use the constructor

```
KroneckerDelta(indices::Vector{Pair{Index, IndexPosition}})
```

Note that an error will be thrown if the two indices contained in `indices` are not on opposite levels, or if they are from different `IndexSet`s. Expressions containing contractions between `KroneckerDelta`s and tensors can be simplified using the `eliminate_kronecker()` function, a single argument function taking a `TensorExpression`. Note that when `eliminate_kronecker()` is executed, `KroneckerDelta`s with repeated indices will be replaced by a constant determined by the `range` parameter of the `IndexSet` associated with the indices of the `KroneckerDelta` (ie, summation will be performed over the indices).

3.2.4 `EpsilonTensor`

In this package, objects of the type `EpsilonTensor` are a subtype of the `AntisymMetricSuperType`, and thus can be used as an antisymmetric metric if needed. It can be created via the constructor

```
EpsilonTensor(indices::Vector{Pair{Index, IndexPosition}})
```

Expressions containing contractions between `EpsilonTensor`s and an arbitrary tensor can then be simplified using the `eliminate_epsilon()` function, a single argument function taking a `TensorExpression`. This will raise and lower indices, with antisymmetry in mind. Note that for `EpsilonTensor`s contacted in either or both indices, the `eliminate_epsilon()`

function will replace such contractions with either a constant or a `KroneckerDelta` where appropriate.

3.2.5 `rename_dummies()`

In Einstein summation notion, repeated indices in an expression denote summation and are thus "dummy" indices: the name of their label is irrelevant. Often in the process of performing algebraic manipulations, being able to recognise terms that are equivalent up to dummy index labels is important. Thus, the built-in function `rename_dummies()` is able to relabel dummy indices in an expression to some canonical ordering, which allows for like terms to be combined. Indices are guaranteed to be taken from the correct `IndexSet`, and will be replaced with the last indices from the set that are not already used in the expression to aid the user to quickly determine which indices are dummies in a convoluted expression.

3.2.6 `pop_metric()`

The `pop_metric()` function can be used to restore all indices in an expression to their canonical position as defined by the `IndexSet`. This has utility in helping to recognise like-terms in an expression, as well as simplifying expressions with explicit symmetry or antisymmetry in indices, as this information will be contained in the metric. This function takes a single `TensorExpression` argument.

3.2.7 Weights

All `Tensor`s allow for a arbitrary weight properties to be defined. For example, by default, a `Tensor` will be assigned a weight of `1` to a weight labelled by each `Coordinate` that the `Tensor` is a `function_of`. Weights are additive, thus a weight of a `TensorTerm` will be the sum of the weights of its component `Tensors`. Thus the aforementioned example will determine the polynomial weight of a term for each coordinate.

The `get_weighted_terms()` function allows for a particular weight to be extracted from an expression. This property can be useful for example in getting the nth degree of a polynomial. For example,

```
x = Coordinate("x", [spacetime_indices])
X=Tensor("X",x)
exp= 2*X*X+8*X+9*X*X*X
get_weighted_terms(exp,"x",3)
```

This will have output `9*XXX`. This is a simple example, however the weights allow for the encoding of any sort of extra information, like grading (which is already kept track of) or another numerical property associated with the tensor.

Additionally, there is also the `drop_weight_above()` function, which could be useful for example to extract terms up to a certain order below (excluding) n in a Taylor series.

3.2.8 `sort()`

The `sort()` function sorts a `TensorExpression` into some canonical form. By default, this function will sort a `TensorTerm` such that the `Tensor`s are in alphabetical order. However, the user my also define a custom sort order via `custom_sort_order()`, which takes a `Vector` of names which `sort()` will sort into that specified order. Any name not found in this user-specified order will be sorted alphabetically after the user-specified names. The `sort()` function will also take grading into account. To do this, the function uses a modified bubble sort, running in $O(n^2)$ time, for a `TensorTerm` with n `Tensor`s, and thus $O(kn^2)$ for a k term `TensorExpression`. The fact that grading is important when performing swaps restricts the usage of faster sorting algorithms.

3.2.9 `sort_symmetric()`

Some `Tensor`s have implicit symmetries in their indices, which allow for expressions to be simplified. Currently, for totally symmetric and anti-symmetric `Tensor`s, the `sort_symmetric()` function sorts the indices into alphabetical order, allowing for like-terms to be recognised by other algorithms. If this function is run on any other sort of `Tensor`, the input will be returned without changes.

3.2.10 `simplify()`

The `simplify()` function performs a series of other functions in a particular order to simplify a `TensorExpression`.

In particular, this function:

- Contracts all metrics in the expression
- Pops all metrics in the expression
- Sorts the expression
- Eliminates `EpsilonTensor`s
- Renames dummy indices
- Sorts symmetric and antisymmetric tensor indices into alphabetical order
- Eliminates `KroneckerDelta`s
- Renames dummy indices again
- Contracts all remaining metrics

This has worst case performance $O\Big(kn(n+m+i^2)\Big)$, where k is the number of `TensorTerms` in the `TensorExpression`, m is the maximal number of `Metric`s in a `TensorTerm` in the initial expression, n is the maximal number of non-metric `Tensor`s in a term in the initial expression, and i is the maximal number of indices attached a `Tensor` in the `TensorExpression`. In practice, the algorithm will run much faster: this asymptotic runtime is for extreme pathological inputs.

This particular routine was chosen to be able to successfully simplify a variety of expressions under different use cases. The user is, of course, free to define their own simplify function with a subset of these operations, or with additional operations, more specific to their use case.

3.3 Performing Derivatives

The motivation for the creation of this package was to be able to apply an arbitrary number of covariant derivatives to a tensor expression in a quick and simple way.

Derivatives are handled without the need for replacement patterns in expressions. The main idea is that the action for a particular type of derivative is defined on a set of basis objects, and then at runtime, (if needed) the derivative is applied linearly to the expression and the (graded) Leibniz rule is performed, and the derivative is peformed on objects on which it is defined. To create a working derivative requires a few steps.

3.3.1 `Derivative`

A `Derivative` is a subtype of the `LeibnizOperatorSuperType` type. To create a `Derivative`, one can use the constructor

```
    Derivative(name::String, operand::TensorTerm, coords::Vector{Coordinate},
        operator_indices::Vector{Pair{Index, IndexPosition}}
            =Vector{Pair{Index, IndexPosition}}())
```

These parameters are the same as that of the basic `Operator` type, with the addition of `coords`. This parameter is the set of coordinates that the derivative is with respect to. Any operand that is not a function of at least one of these coordinates will be annihilated by the `Derivative` automatically. Note that the `operator_indices` can be any number and combination of indices, however, it should be noted that having n indices from the same `IndexSet` will not imply that this is the nth derivative with respect to the associated coordinates, contrary to the behaviour of other software.

3.3.2 Derivative Action

Before applying the derivative to an actual expression, the action of the derivative should be defined. To do this, one should define a `Derivative` with the appropriate indices, and with respect to the appropriate coordinates, with an expression input that will be ignored (the expression here is not used to define the action). For example, to create a spatial derivative,

```
spacetime_indices = IndexSet("spacetime", 4)
a = Index("a", spacetime_indices, 0)
x = Coordinate("x", spacetime_indices)

spatial_deriv=Derivative("∂",zero, [x], [a => lower])
```

Note that each different type of derivative should be given a different name. Now, for example, suppose one has a tensor

```
b = Index("b", spacetime_indices, 0)
X = Tensor("X", [b => upper], x)
```

Then, to define the action of our derivative on this tensor, one can use the `add_derivative_action()` function

```
add_derivative_action(derivative_operator::DerivativeSuperType,
    operand::TensorSuperType,
    result<:Union{TensorSuperType,TensorExpression,TensorTerm})
```

Here, `derivative_operator` is the `Derivative` we'd like to define the action of, `operand` is the basis `Tensor` we'd like to define the action on, and `result` is the outcome of applying the derivative to this object. Note that care should be taken to ensure that the inputs to this function have the correct indices. The index names used are not important - this function simply encodes what behaviour to perform when tensors of a particular index structure are encountered. In other words, registering the action of ${D_i}^{\alpha}(A^i)$ will allow expressions like ${D_j}^{\alpha}(A^j)$ and ${D_i}^{\beta}(A^i)$ to be computed, but not ${D_i}^{\alpha}(A^j)$, nor ${D^{\alpha}}_i(A^i)$, $D^{i\alpha}(A_i)$ or ${D^i}_{\alpha}(A_i)$. Thus, it is recommended to register the derivatives in some sort of canonical form, and then use metrics to move indices to the canonical positions before applying the action.

Returning to our example, we could thus define the derivative action

```
add_derivative_action(spatial_deriv,X,KroneckerDelta([a=>lower,b=>upper]))
```

This defines the action of the derivative on `X`, for any pattern of indices of `X` and `spatial_deriv`.

3.3.3 `product_rule()`

An important part of taking derivatives is the Leibniz product rule. For operators of the `LeibnizOperatorSuperType`, as `TensorExpresion` will be created by the `product_rule()` that respects the graded Leibniz rule. This is particularly useful when working with fermionic objects.

3.3.4 Derivative Application

Once the action of a derivative on the desired tensors has been specified, applying the derivative is simple. Continuing from the above example, suppose we have the expression

```
eq=(SymmetricTensor("A", [a => lower, b => lower])
  *Tensor("X", [a => upper], x)*Tensor("X", [b => upper], x)
  +Tensor("B", [a => lower])*Tensor("X", [a => upper], x)
  +Tensor("C"))
```

This represents a quadratic equation in N-D. Then, to compute the derivative of this expression as defined before, we can execute the code

```
first_deriv=Derivative("∂", eq, [x], [c => lower])
result=apply_derivative(first_deriv)
result=simplify(result)
```

This has the output

```
2*A^{a*}_{*c}X_{a}+B_{c}
```

as expected: a linear equation in each direction x_c.

3.3.5 Covariant Derivatives

Often times when working with field theories, covariant derivatives must be used to account of the curvature of the underlying spacetime, and fields. For theories with many connection terms, this expression for the derivative may be lengthy. To improve performance and readability, this package allows the user to define a `Derivative` operator that can later be expanded and replaced by the full covariant derivative at a later step in computation. This makes things like the Leibniz rule faster to run, as well as makes the code more readable. To see this in action, consider the following code. Suppose we have defined two different derivative operators, corresponding to two terms of the covariant derivative for some kind of gauge theory (loosely speaking),

```
x = Coordinate("x", spacetime_indices)
θ = Coordinate("θ", internal_space_indices)
X = Tensor("X", [b => upper], x)
Θ = Tensor("Θ", [j => upper], θ)
spatial_deriv=Derivative("∇",zero, [x], [a => lower])
connection_deriv=Derivative("∂",zero, [θ], [i => lower])
add_derivative_action(spatial_deriv,X,KroneckerDelta([a=>lower,b=>upper]))
add_derivative_action(connection_deriv,Θ,KroneckerDelta([i=>lower,j=>upper]))

covar_expansion=spatial_deriv
    -2*im*Tensor("A",[a => lower, i => upper])*connection_deriv

```

Rather than using this full derivative in computations, one can instead apply just a single derivative operator,

```
        exp=Derivative("D",X*Θ, [x,θ], [c => lower])
```

The user can then perform whatever manipulations that are required. Once the user wants to compute the derivative, they can define a derivative representing which derivative they'd like to expand, then use the `expand_derivative()` function,

```
        covar=Derivative("D",zero, [x,θ], [a => lower])

        out=expand_derivative(exp,covar,covar_expansion)
```

Note that the free indices used in the patterns `covar` and `covar_expansion` must match with each other in order to have a valid pattern, but they do not need to use the same free indices as the actual expression - the function `expand_derivative()` finds derivatives in `exp` that match the derivative name and index structure of `covar` modulo index names, and then expands the derivative using `covar_expansion` as a blueprint.

3.4 Pattern Matching and Replacements

The most powerful part of the software is the ability to make algebraic substitutions in expressions. The software employs its own syntax to do this, and will be explained below.

3.4.1 Usage

To make a replacement, one must first search an expression to find the tensor(s) that should be replaced. Patterns in this software allow the user to create expressions with arbitrary wildcards, that allow the user to find very specific (or very general) matches. This is most simply explained through examples. For any type of wildcard object, whether it be a `Tensor` or `Index`, the object's name must start with a ?, to denote the object is a wildcard. Subsequent characters after the ? can be used to denote different wildcards. For example, a pattern like $?H?H$ would search specifically for tensors in an expression that are squared (two tensors of the same name), whereas $?G?H$ will match any pair of tensors (with same or different name).

Let us now consider some basic usage. To create a basic pattern one can create a `DummyPatternTensor` object,

```
        H=DummyPatternTensor("?H",[n => upper, m => upper, m => lower, α => lower])
```

Note that even though `H` is a wildcard, the indices on `H` are not, and thus this pattern will only match tensors with these exact index names (modulo summation dummy index names).

To instead specify these indices as wildcard characters, one can use `DummyPatternIndex` objects,

```
G = DummyPatternTensor("?G", [i => upper, k => upper, j => lower,
    DummyPatternIndex("?γ") => lower], x)
```

Here, the last index in the pattern is a wildcard index and will match with any index in the 4th position, in the lower slot of a tensor.

To see how to use this, let us consider a `TensorTerm` representing our original expression, a `TensorTerm` representing the pattern to look for, and a `TensorTerm` for what we should replace the found pattern for. Note that the pattern and replacement must be of the type `TensorTerm`.

```
J = Tensor("F", [i => upper, k => upper, j => lower, β => lower], x)
Z = Tensor("Z", [i => upper, k => upper, j => lower, β => lower], x)
K = Tensor("K", [n => upper, m => upper, m => lower, α => lower], θ)
L = Tensor("L", [q => upper, p => upper], θ)

exp1=K*J*Z*L

G = DummyPatternTensor("?G", [i => upper, k => upper, j => lower,
    DummyPatternIndex("γ") => lower], x)

pat = J*G

rep= TensorTerm(G)

out=substitute_terms(exp1,pat1,rep,true)

```

This will have output `-Z^{ik**}_{**jβ}K^{nm**}_{**mα}L^{qp}`. Indeed, the pattern specified here looks for occurances of the `Tensor` J, and any tensor $?G$ with the specified indices. Note that expressions are assumed to be (anti)commutative, and thus J and $?G$ need not be next to eachother in an expression in order for a match to be found. Indeed, $?G$ matches with Z (and $?\gamma$ with β), and thus the `substitute_terms()` function makes the replacement $JZ \to Z$. This Z is then (anti)commuted to the front of the expression, and the new resulting expression is returned. Note that the last input argument of the `substitute_terms()` is a boolean which dictates if the replacement should be repeatedly applied until no matches are found.

3.4.2 Explanation of the Algorithm

In general, pattern-matching problems such as this are computationally difficult. Indeed, the fact the expressions are (anti)commutative means that the usual improvements in runtime gained in text matching algorithms from techniques like preprocessing cannot be performed, as the expression has no definite order. Preprocessing usually reduces the time complexity of pattern matching in text significantly - for algorithms like KMP the runtime reduces to $O(n+m)$ for an expression of n characters and a pattern of m characters. It should be noted that the naive brute-force approach to pattern matching for a(n) (anti)commutative expression and (anti)commutative pattern is computationally intensive, running in exponential time. In fact, this problem is equivalent to a set covering problem, which is known to be NP-complete. Thus, the naive approach is not adequate for our purposes. This means that algorithms must be implemented in a clever fashion lest they run in exponential time. Thus, particular care was taken in designing this pattern-matching algorithm in order to reduce the computational complexity.

This algorithm is the most complex feature of the package. Our algorithm runs in a few main steps, following a dynamic programming approach to solving the problem. For a tensor term of n tensors, and a pattern of m tensors,

1. **Construct Histograms**: Firstly, the expression and the pattern are converted to histograms, by firstly counting the number of occurrences of each tensor with a given name, then inverting the histogram so that the result has keys denoting frequencies, and values that are a set of tensor names (as well as their associated positions in the original expression). This allows for quick comparison between the pattern and the expression to find possible candidates for matches based on frequency. This step runs in $O(n + m)$ (amortised) time.

2. **Find a Match**: The algorithm then recursively checks for matches, by checking combinations of tensors with the correct frequency. If there is a single frequency in the pattern, then two cases are considered

   - *One type of tensor in the single pattern frequency*: If there is only one type of tensor (name) in the pattern, we loop through all frequencies in the expression histogram where the frequency of the pattern is less than the expression frequency (a requirement for a match). For each of these valid expression frequencies, the algorithm loops over the associated tensors. If the names match (or if the tensor in the pattern is a dummy), the indices are checked (baring in mind any wildcards from `DummyPatternIndices`. If the pattern does not match, the next tensor is checked. Otherwise, a match is registered, and the function returns true, along with information about which wildcards match to which index/tensor.
   - *Many types of tensors in the single pattern frequency*: The algorithm runs similarly, by first calling itself on the expression, but on only the first type of tensor in the pattern. This runs the algorithm and returns information about wildcards. If a match was found, it is removed from the pattern, and the rest of the pattern is now checked against the expression, now with the appropriate wildcards fixed. If no match is found at any step, then we return up a level in the algorithm and consider the next possible match of the pattern tensors with the expression (as perhaps there is a match that exists with a different set of wildcard matches). In this sense, the algorithm is doing a greedy search for matches.

   Otherwise, if there are multiple frequencies in the pattern histogram, we loop over the frequencies and recursively call the algorithm on the single frequency case, and record information about the wildcards to use at each step. Note that if the highest frequency in the pattern has no matches with the expression, the search can terminate in the negative. In the worst case, this step runs in $O(nm^2)$ time - but in reality, it may be faster as this asymptotic bound assumes that in the worst case all tensors in the pattern have the same frequency.

3. **Make the replacement**: If a match is found, the replacement must be made. Firstly, the grading due to rearranging terms to make the replacement is computed. Then the replacement must be constructed, by creating new tensors given information about the wildcards. In total, this runs in roughly $O(nm)$ time.

To summarise, this algorithm runs in $O(nm^2)$ total, and thus at absolute worst $O(n^3)$ time (the pattern length cannot exceed the expression length lest the algorithm terminates). In reality, it is likely that m is much less than n in standard operation. Thus, we have effectively written an algorithm that is linear in the expression length, and quadratic in the pattern length - a vast improvement on the exponential runtime of the naive approach. Bare in mind that these are asymptotic bounds, and that the actual performance will be much faster, an advantage gained by both the use of the histograms to rule out many unnecessary comparisons. It may be possible to improve the runtime of the algorithm slightly by using mimoisation techniques, however, it is unlikely as the algorithm should by construction avoid repeated computations.

3.5 Lightcone Coordinates

In usual operation, repeated indices denote summation. When dealing with lightcone coordinates in 2D, the indices $\pm$ can be attached an arbitrary number of times to an object, and do not denote summation. This means special treatment is needed for lightcone coordinates.

For this reason, the software comes with a predefined set of lightcone coordinates, which can be accessed by running `include("lightcone.jl")`,

```
lightcone_indices = IndexSet("lightcone", 2,lower)
± = Index("±", lightcone_indices, 1)
∓ = Index("∓", lightcone_indices, 1)
p = Index("+", lightcone_indices, 1)
m = Index("-", lightcone_indices, 1)
```

These indices are treated specially. These indices should be used in conjunction with another `IndexSet`; this 2nd "working set" should be used for algebraic manipulations, and then replaced with the lightcone coordinates at the end. To see this in action, consider a simple expression

```
a = Index("a", spacetime_indices, 1)
b = Index("b", spacetime_indices,1)
c = Index("c", spacetime_indices,1)

x = Coordinate("x", [spacetime_indices, spacetime_indices])
θ = Coordinate("θ", [spacetime_indices])
```

```
    field=Tensor("θ",[c=>upper],θ)
        *Tensor("S",[c => lower, a =>lower, b => lower], x)
    Dfield=Derivative("∇",field, [x], [i => lower, j => lower])
    out=apply_derivative(Dfield)
```

The result here will be in terms of `spacetime_indices`; indeed, it has output `θ^{c}∇_{ij}[[S_{cab}]]`. To write the result in terms of `lightcone_indices`,

```
    res=to_lc_expression(out,Dict(a=>m,b=>m,i=>p))
```

This defines a mapping of the free indices of the expression to `lightcone_indices`, and automatically expands sums over dummy indices. The result of this code would thus be `θ^{+}∇_{++}[[S_{+--}]]+θ^{-}∇_{++}[[S_{---}]]`.

For chiral theories, we further have the function `to_chiral_part()` which takes an expression and a lightcone index `p` or `m`, and returns the expression with only `p` or `m` indices respectively.

3.6 Superspace

The ability to perform integration over superspace is important for the construction of many types of expressions. Fortunately, by virtue of the properties of Grassmann numbers, integration behaves the same as differentiation over superspace. We can thus perform Grassmann integration over our expressions without much extra effort. We have defined convenient functions and variables relevent to superspace in its own file. To use these, one can firstly run `include("superspace.jl")`.

The main part of this extension is the `project_component()` function. This takes many inputs,

```
    project_component(exp::TensorExpression, thet::Coordinate,
    derivinds::Vector{Pair{Index,IndexPosition}}, theta_weight::Integer,
    theta_bar_weight::Integer, thet2::Coordinate=nothing,
    deriv2inds::Vector{Pair{Index,IndexPosition}}=nothing,
    theta1lev::Integer=1, theta2lev::Integer=1)
```

Here, `exp` is the expression to integrate (differentiate), `thet` is the coordinate to integrate with respect to (for example, if the user has parametrised superspace in terms of complex Grassmann numbers), `derivinds` encode the index structure of the relevant superspace derivative, `theta_weight` is the number of times to integration before setting $\theta = 0$, and likewise for `theta_bar_weight`. There are also optional arguments for integrating over a 2nd types of Grassmann variable in the case where superspace is parametrised by complex Grassmann numbers. Finally, since there are two independent θ^2 structures in most of the theories we consider, we have `theta1lev` and `theta2lev` to dictate which θ^2 to integrate with respect to.

4
Construction of T$\bar{\text{T}}$ and T$\bar{\text{T}}$-like Terms for Conformal Theories

4.1 General Approach and Procedure

For a theory possessing the symmetries of a particular (super-)Lie group, the generators of the group are conserved charges by Noether's theorem. In an ordinary theory, momentum is a conserved charge, and can be expressed as the spatial integral of a conserved current,

$$P^\mu = \int d^3x T^{0\mu}. \tag{4.1}$$

Analogously, the supercharge can be expressed as the integral

$$Q^i_\alpha = \int d^3x S^{0i}_\alpha, \tag{4.2}$$

of the so-called supersymmetry current $S^{\mu i}_\alpha$. A general theory may possess additional conserved charges. Indeed, it is possible to non-trivially extend the super-Poincaré algebra with (for example, in $4D$ $\mathcal{N}=1$) [2]

$$\begin{aligned}\{Q_\alpha, \overline{Q}_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}(P_\mu + Z_\mu)\\ \{Q_\alpha, Q_\beta\} &= \sigma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu},\end{aligned} \tag{4.3}$$

where Z_μ and $Z_{\mu\nu}$ are known as brane charges, and are nonzero for strings and domain walls respectively. Note that these are not central charges - they do not commute with the Lorentz generators. These charges are usually infinite, and thus it is more meaningful to examine the local version,

$$\{\overline{Q}_{\dot\beta}, S_{\alpha\mu}\} = 2\sigma^\nu_{\alpha\dot\beta}(T_{\nu\mu} + C_{\nu\mu}) + ...,$$

$$\{Q_\beta S_{\alpha\mu}\} = \sigma^{\nu\rho}_{\alpha\beta} C_{\nu\rho\mu} + ..., \tag{4.4}$$

where $C_{\mu\nu}$ and $C_{\mu\nu\rho}$ are known as brane currents. The algebra implies that these currents live inside a supercurrent multiplet, along with $S_{\alpha\mu}$ and $T_{\mu\nu}$.

In this thesis, we aim to describe this supercurrent multiplet. In principle, in the same way the stress-energy tensor can be determined from coupling to gravity, the supercurrent can be determined from coupling to the supergravity superfield [20]. However, for the theories that we would like to consider, the supergravity formulation is not well-understood.

Thus we take an alternate approach, using superspace techniques.

We assume, firstly, that the super-Poincaré algebra exists and has associated local currents. We in particular assume that the stress-energy tensor and the supersymmetry current exist, and are conserved. We know from the fact that the currents transform non-trivially under supersymmetry, that they must close to form a supercurrent multiplet. We also assume for physical reasons that the stress-energy tensor is the highest spin (2) current in the multiplet. Since the supercurrents permit a finite order superfield expansion in superspace, we can thus determine relationships between the component currents by imposing consistency conditions. We thus aim to construct a, in a sense minimal, indecomposable supercurrent multiplet that obeys these requirements.

In $2D$, it is still unknown what the most general supercurrent multiplet obeying these requirements are for $\mathcal{N} = (0, 4)$ and $\mathcal{N} = (4, 4)$, as well as $4D$ $\mathcal{N} = 2$.

In order to work towards a classification of the supercurent for these theories, we will work first in the conformal case. The conformal theory is much simplified, and possesses a traceless stress-energy tensor. Many of the currents in addition will be constrained to be zero thanks to strong constraints on the algebra.

Using these conformal constraints, we will use the software that we developed to determine a set of constraint equations on the component fields of the supercurrent multiple after using superspace to perform a superfield expansion. We will then solve these constraints, determining the form of the multiplet.

Once the multiplet is solved, we can then construct the supercurrent-squared term, and integrate over superspace to obtain the operator defined on spacetime. We can then compare this to T$\bar{\text{T}}$. We expect that this operator should be equivalent T$\bar{\text{T}}$, on-shell and up to a total derivative in the $2D$ case, and possibly different in the $4D$, inline with the findings for other theories in [3] [4] [5] [6] [7]. For the conformal theory, this should consist of only one term, $T^{\mu\nu}T_{\mu\nu}$, and no $T^\mu_\mu = \Theta^2$ term, as the stress-energy tensor is traceless for conformal theories.

Note that in this chapter, we present results for $2D$ $\mathcal{N} = (0, 2)$ and $\mathcal{N} = (2, 2)$ which are already known, along with results for $2D$ $\mathcal{N} = (0, 4)$ and $\mathcal{N} = (4, 4)$, $4D$ $\mathcal{N} = 2$. We include these known results as a verification of the correct performance of our software and our general procedure.

4.2 2D $\mathcal{N} = (0, 2)$ Supersymmetry

As stated earlier, in two dimensions, it is convenient to work in lightcone coordinates. We will do so for all $2D$ theories for the remainder of this thesis.

For $\mathcal{N} = (0, 2)$ supersymmetry, we need only consider a superfield expansion in one of either Grassman numbers with + or − indices. Without loss of generality, we will consider the + chiral theory. Thus, we consider $2D$ $\mathcal{N} = (0, 2)$ superspace parameterised by $x^M = (x^{\pm\pm}, \theta^+, \overline{\theta}^+)$, where θ^+ is a complex Grassmann coordinate and $\bar{\theta}^+$ is its complex conjugate. The spinor covariant derivatives are given by

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - \frac{\mathrm{i}}{2}\bar{\theta}^+\partial_{++}, \quad \overline{\mathcal{D}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + \frac{\mathrm{i}}{2}\theta^+\partial_{++}, \tag{4.5}$$

and obey the anti-commutation relations

$$\left\{\mathcal{D}_+, \overline{\mathcal{D}}_+\right\} = \mathrm{i}\partial_{++}. \tag{4.6}$$

It is known that the supercurrent is described in terms of a vector supercurrent $\mathcal{S}_{++}$ and a spin-2 supercurrent $\mathcal{T}_{----}$ [4]. In general we can expand these as a supercurrent expansion

$$\mathcal{S}_{++}(x, \theta^+, \overline{\theta}^+) = j_{++}(x) + \theta^+ S_{+++}(x) - \overline{\theta}^+ \overline{S}_{+++}(x) + \theta^+\overline{\theta}^+ T_{++++}(x) \tag{4.7}$$

and

$$\mathcal{T}_{----}(x, \theta^+, \overline{\theta}^+) = T_{----}(x) + \theta^+ B_{+----}(x) - \overline{\theta}^+ \overline{B}_{+----}(x) + \theta^+\overline{\theta}^+ C_{++----}(x) \tag{4.8}$$

Then, the superconformal current constraint is [4]

$$\partial_{--}\mathcal{S}_{++} = 0, \tag{4.9}$$

and

$$D_+\mathcal{T}_{----} = \overline{D}_+\mathcal{T}_{----} = 0, \tag{4.10}$$

Which immediately yields for $\mathcal{S}_{++}$

$$\partial_{--}T_{++++} = 0, \tag{4.11}$$

$$\partial_{--}S_{+++} = 0, \tag{4.12}$$

$$\partial_{--}\overline{S}_{+++} = 0, \tag{4.13}$$

and

$$\partial_{--}j_{++} = 0. \tag{4.14}$$

And for $\mathcal{T}_{----}$,

$$C_{++----} = 0, \tag{4.15}$$

$$B_{+----} = 0, \tag{4.16}$$

$$\partial_{++}T_{----} = 0. \tag{4.17}$$

Then, the supercurrent-squared term is given by [4]

$$\mathcal{O}_{T^2} = \mathcal{T}_{----}\mathcal{S}_{++}, \tag{4.18}$$

and then computing the projection

$$\mathrm{O}_{T^2} = \int d\theta d\bar{\theta} \mathcal{O}_{T^2}\Big|_{\theta=\bar{\theta}=0}, \tag{4.19}$$

yields, after imposing the equations of motion,

$$\mathrm{O}_{T^2} = T_{++++}T_{----}. \tag{4.20}$$

Thus, in the conformal case, the supersymmetric supercurrent squared operator precisely corresponds to $T\bar{T}$ for a traceless stress-energy tensor as would be expected for a conformal theory. This matches the known result, and thus our approach successfully reproduces the correct result.

4.3 2D $\mathcal{N} = (2,2)$ Supersymmetry

In flat $2D$ $\mathcal{N} = (2,2)$, we parametrise superspace by $x^M = (x^{\pm\pm}, \theta^{\pm}, \overline{\theta}^{\pm})$, where $\theta^{\pm}$ is a complex Grassmann coordinate and $\bar{\theta}^{\pm}$ is its complex conjugate. Note that the covariant derivatives are given by [5]

$$D_{\pm} = \partial_{\pm} - \frac{i}{2}\overline{\theta}^{\pm}\partial_{\pm\pm} \qquad \overline{D}_{\pm} = -\overline{\partial}_{\pm} + \frac{i}{2}\theta^{\pm}\partial_{\pm\pm}, \tag{4.21}$$

and satisfy

$$\left\{D_{\pm}, \bar{D}_{\pm}\right\} = i\partial_{\pm\pm}, \tag{4.22}$$

with all other (anti-)commutators vanishing.

It is known that for this theory, there exists two supercurrents, $\mathcal{S}_{++}$ and $\mathcal{S}_{--}$, which we collectively denote $\mathcal{S}_{\pm\pm}$, that contain the stress-energy tensor and supersymmetry current [5]. The most general form of these supercurrents are given by a real vector superfield with superfield expansion

$$\begin{aligned}\mathcal{S}_{\pm\pm}(x, \theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-) = {} & j_{\pm\pm}(x) + \theta^+ S_{+\pm\pm}(x) + \theta^- S_{-\pm\pm}(x) - \overline{\theta}^+ \overline{S}_{+\pm\pm}(x) - \overline{\theta}^- \overline{S}_{-\pm\pm}(x) \\ & + \theta^+\overline{\theta}^+ T_{++\pm\pm} + \theta^+\overline{\theta}^- T_{+-\pm\pm}(x) + \theta^-\overline{\theta}^+ T_{-+\pm\pm}(x) \\ & + \theta^-\overline{\theta}^- T_{--\pm\pm}(x)(x) + \theta^+\theta^- K_{+-\pm\pm}(x) - \overline{\theta}^+\overline{\theta}^- \overline{K}_{+-\pm\pm}(x) \\ & + \theta^+\theta^-\overline{\theta}^+ H_{+-+\pm\pm}(x) + \theta^+\theta^-\overline{\theta}^- H_{+--\pm\pm}(x) - \overline{\theta}^+\overline{\theta}^-\theta^+ \overline{H}_{+-+\pm\pm}(x) \\ & - \overline{\theta}^+\overline{\theta}^-\theta^- \overline{H}_{+--\pm\pm}(x) + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^- F_{+-+-\pm\pm}(x)\end{aligned} \tag{4.23}$$

We will then impose the conformal constraint on the supercurrent,

$$\overline{D}_{\pm}\mathcal{S}_{\mp\mp} = 0, \quad D_{\pm}\mathcal{S}_{\mp\mp} = 0. \tag{4.24}$$

It should be noted here that there is no implicit Einstein summation over the lightcone indices, contrary to usual notation.

Applying these constraints yields a set of constraints on each component of the supercurrent expansion. By using the software we developed, we can easily compute the constraints for the theory, and then solve them. This in fact yields (with analogous equations for the complex conjugates),

$$S_{\pm\mp\mp} = T_{\pm\pm\mp\mp} = T_{\pm\mp\pm\pm} = K_{+-\pm\pm} = H_{+-+\pm\pm} = H_{+--\pm\pm} = F_{+-+-\pm\pm} = 0$$
$$\partial_{\pm\pm} j_{\mp\mp} = \partial_{\pm} S_{\mp\mp\mp} = \partial_{\pm\pm} T_{\mp\mp\mp\mp} = 0 \tag{4.25}$$

This reduces the supercurrent to consist of entirely all left-moving (annihilated by ∂_{++}) or all right-moving (annihilated by ∂_{--}) currents, of the form

$$\mathcal{S}_{\pm\pm}(x, \theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-) = j_{\pm\pm}(x) + \theta^{\pm} S_{\pm\pm\pm}(x) - \overline{\theta}^{\pm} \overline{S}_{\pm\pm\pm}(x) + \theta^{\pm}\overline{\theta}^{\pm} T_{\pm\pm\pm\pm}(x), \tag{4.26}$$

where the indices are chosen to be either all $+$, or all $-$.

This quite easily allows for the construction of the supercurrent squared operator by firstly taking

$$\mathcal{O}_{T^2} = \mathcal{S}_{++}\mathcal{S}_{--}, \tag{4.27}$$

and then computing the projection

$$\mathrm{O}_{T^2} = \int d^2\theta d^2\bar{\theta} \mathcal{O}_{T^2}\Big|_{\theta=\bar{\theta}=0}. \tag{4.28}$$

This yields

$$\mathrm{O}_{T^2} = T_{++++}T_{----}. \tag{4.29}$$

Thus, in the conformal case, the supersymmetric supercurrent squared operator precisely corresponds to $T\bar{T}$ for a traceless stress-energy tensor as would be expected for a conformal theory. This again matches the known result [5].

4.4 4D $\mathcal{N} = 2$ Supersymmetry

The structure of the $\mathcal{N} = (4, 4)$ supersymmetry is not well-understood in 2D. However, the structure of 4D $\mathcal{N} = 2$ supersymmetry is better understood. There is also a strong relationship between these two categories of theories, as explored in [16]. We will thus begin by understanding the supercurrent in 4D $\mathcal{N} = 2$ supersymmetry, and thus create the $\mathrm{T\bar{T}}$-like supercurrent-squared term that exists in this theory, which we can later, in future works, relate to the $2D$ theory via dimensional reduction and truncation.

In 4D $\mathcal{N} = 2$ supersymmetry, the algebra contains generators Q^i_α and $\bar{Q}^{\dot\beta}_j$, where $i, j \in \{1, 2\}$ are isospin ($SU(2)$) flavour indices, and $\alpha \in \{1, 2\}$ and $\dot\beta \in \{\dot1, \dot2\}$ are spinor indices in for the fundamental and antifundamental representations respectively. Working in the superspace formalism, these generators have associated complex Grassmann superspace coordinates θ^α_i, $\bar{\theta}^j_{\dot\beta}$, and thus superspace coordinates $z^M = (x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot\alpha})$, $a \in \{0, 1, 2, 3\}$. This has associated superspace covariant derivatives

$$D^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i} - \mathrm{i}\left(\sigma^b\right)_\alpha^{\ \dot{\beta}} \bar{\theta}^i_{\dot{\beta}} \partial_b, \quad \bar{D}^{\dot{\alpha}}_i = \frac{\partial}{\partial \bar{\theta}^i_{\dot{\alpha}}} - \mathrm{i}\left(\sigma^b\right)^{\dot{\alpha}}_{\ \beta} \theta^\beta_i \partial_b, \quad \partial_a = \frac{\partial}{\partial x^a}. \tag{4.30}$$

The relevant algebra for this theory can be found in Appendix A, as well as some useful rules and definitions regarding Grassmann variables and their contractions and complex/hermitian conjugation.

Then, the most generic form of the supercurrent is given by

$$\begin{aligned}\mathcal{J} = {} & j(x) + \theta^\alpha_i \psi^i_\alpha(x) + \theta_{ij} F^{ij}(x) + \theta^\alpha_j \theta^{ij} G_{i\alpha}(x) + \theta^4 H(x) + \theta^\alpha_j \overline{\theta}^k_{\dot{\alpha}} K^{j\dot{\alpha}}_{k\alpha}(x) + \theta^\alpha_i \overline{\theta}^{jk} L^i_{jk\alpha}(x) \\ & + \theta^\alpha_i \overline{\theta}^k_{\dot{\beta}} \overline{\theta}_{jk} M^{\dot{\beta}ij}_\alpha(x) + \theta^\alpha_i \overline{\theta}^4 N^i_\alpha(x) + \theta_{ij} \overline{\theta}^{kl} P^{ij}_{kl}(x) + \theta_{ij} \overline{\theta}^l_{\dot{\alpha}} \overline{\theta}_{kl} Q^{ijk\dot{\alpha}}(x) + \theta_{ij} \overline{\theta}^4 R^{ij}(x) \\ & + \theta^\alpha_j \theta^{ij} \overline{\theta}^l_{\dot{\alpha}} \overline{\theta}_{lk} S^{k\dot{\alpha}}_{i\alpha}(x) + \theta^\alpha_j \theta^{ij} \overline{\theta}^4 E_{i\alpha}(x) + \theta^4 \overline{\theta}^4 V(x) + \overline{\theta}^i_{\dot{\alpha}} \overline{\psi}^{\dot{\alpha}}_i(x) + \overline{\theta}^{jk} \overline{F}_{jk}(x) \\ & + \overline{\theta}^j_{\dot{\alpha}} \overline{\theta}_{jk} \overline{G}^{k\dot{\alpha}}(x) + \overline{\theta}^4 \overline{H}(x) + \overline{\theta}^i_{\dot{\alpha}} \theta_{jk} \overline{L}^{jk\dot{\alpha}}_i(x) + \overline{\theta}^i_{\dot{\alpha}} \theta^\beta_k \theta^{jk} \overline{M}^{\dot{\alpha}}_{\beta ij}(x) + \overline{\theta}^i_{\dot{\alpha}} \theta^4 \overline{N}^{\dot{\alpha}}_i(x) \\ & + \overline{\theta}^{ij} \theta^\alpha_l \theta^{kl} \overline{Q}_{ijk\alpha}(x) + \overline{\theta}^{ij} \theta^4 \overline{R}_{ij}(x) + \overline{\theta}^j_{\dot{\alpha}} \overline{\theta}_{ij} \theta^4 \overline{E}^{i\dot{\alpha}}(x) + \theta^{\alpha\beta} \Omega_{\alpha\beta}(x) + \overline{\theta}_{\dot{\alpha}\dot{\beta}} \overline{\Omega}^{\dot{\alpha}\dot{\beta}}(x) \\ & + \theta^{\alpha\beta} \overline{\theta}^i_{\dot{\alpha}} A^{\dot{\alpha}}_{i\alpha\beta}(x) + \theta^{\alpha\beta} \overline{\theta}^{jk} B_{jk\alpha\beta}(x) + \theta^{\alpha\beta} \overline{\theta}^j_{\dot{\alpha}} \overline{\theta}_{jk} U^{\dot{\alpha}k}_{\alpha\beta}(x) + \theta^{\alpha\beta} \overline{\theta}^4 C_{\alpha\beta}(x) \\ & + \theta^{\alpha\beta} \overline{\theta}_{\dot{\alpha}\dot{\beta}} T^{\dot{\alpha}\dot{\beta}}_{\alpha\beta}(x) + \overline{\theta}_{\dot{\alpha}\dot{\beta}} \theta^\alpha_i \overline{A}^{\dot{\alpha}\dot{\beta}i}_\alpha(x) + \overline{\theta}_{\dot{\alpha}\dot{\beta}} \theta_{ij} \overline{B}^{\dot{\alpha}\dot{\beta}ij}(x) + \overline{\theta}_{\dot{\alpha}\dot{\beta}} \theta^\alpha_j \theta^{ij} \overline{U}^{\dot{\alpha}\dot{\beta}}_{i\alpha}(x) + \overline{\theta}_{\dot{\alpha}\dot{\beta}} \theta^4 \overline{C}^{\dot{\alpha}\dot{\beta}}(x).\end{aligned} \tag{4.31}$$

We seek constraints on this supercurrent to simplify the expression (4.31). To do so, we impose the conformal constraints[25]

$$D^{\alpha(i} D^{j)}_\alpha \mathcal{J} = 0, \quad \bar{D}^{\dot{\alpha}}_{(i} \bar{D}_{j)\dot{\alpha}} \mathcal{J} = 0. \tag{4.32}$$

We can thus compute these derivatives of our supercurrent to generate a set of constraints on the field content. Indeed, one can compute these derivatives to obtain another superspace expansion, and then equate each coefficient function (which will be some combination of our fields and their derivatives) of the expansion to zero.

In this $4D$ $\mathcal{N} = 2$ case, the resulting superfield equation is many, many, pages long - too long to detail here. However, after performing these computations, analysing the resulting equation term by term in powers of θ yields a set of constraints.

After an arduous calculation to solve each of these constraint equations, notably, many fields are constrained to be zero. Indeed,

$$\begin{aligned} F^{ij}(x) = G_{i\alpha}(x) = R^{ij}(x) = H(x) = N^i_\alpha(x) = B_{jk\alpha\beta}(x) = C_{\alpha\beta}(x) \\ = {M_\alpha}^{\dot{\beta}ij}(x) = {L^i}_{jk\alpha}(x) = {U_{\alpha\beta}}^{\dot{\alpha}k}(x) = Q_{ijk\alpha}(x) = E_{i\alpha}(x) = {S_{i\alpha}}^{k\dot{\alpha}}(x) = V(x) = 0. \end{aligned} \tag{4.33}$$

The remaining currents obey the relations

$$P_{klsr}(x) = \frac{1}{8} \epsilon_{k(r|} \epsilon_{l|s)} \Box j(x), \tag{4.34}$$

$$\partial_{\dot{\alpha}}{}^{\alpha} \Box j(x) = 0, \tag{4.35}$$

$$\partial^\beta_{\dot{\alpha}} {\overline{\psi}_s}^{\dot{\alpha}}(x) = 0, \tag{4.36}$$

$$\partial_{\dot{\alpha}}{}^{\alpha} T^{\dot{\alpha}\dot{\beta}}{}_{\alpha\beta}(x) = 0, \tag{4.37}$$

$$\partial_{\dot{\alpha}}{}^{\beta} \overline{\Omega}^{\dot{\alpha}\dot{\beta}}(x) = 0, \tag{4.38}$$

$$\partial_{\dot{\alpha}}{}^{\alpha} K_{\alpha sr}{}^{\dot{\alpha}}(x) = 0, \tag{4.39}$$

$$\partial_{\dot{\alpha}}{}^{\alpha} \overline{A}_{r\alpha}{}^{\dot{\alpha}\dot{\beta}}(x) = 0, \tag{4.40}$$

where we define $\partial^{\mu}_{\dot{\mu}} \equiv \sigma^{a\mu}{}_{\dot{\mu}}\partial_a$, and $\Box \equiv \partial^{\mu}_{\dot{\mu}}\partial^{\dot{\mu}}_{\mu}$. Analogous equations hold for the complex conjugate fields.

The supercurrent then reduces to

$$\begin{aligned} \mathcal{J} = j(x) + \theta^{\alpha}_i \psi^i_{\alpha}(x) + \theta^{\alpha}_j \overline{\theta}^k_{\dot{\alpha}} K^{j\dot{\alpha}}_{k\alpha}(x) + \frac{1}{8}\theta_{ij}\overline{\theta}^{ij}\Box j(x) + \overline{\theta}^i_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}}_i(x) + \theta^{\alpha\beta}\Omega_{\alpha\beta}(x) \\ + \overline{\theta}_{\dot{\alpha}\dot{\beta}}\overline{\Omega}^{\dot{\alpha}\dot{\beta}}(x) + \theta^{\alpha\beta}\overline{\theta}^i_{\dot{\alpha}} A^{\dot{\alpha}}_{i\alpha\beta}(x) + \theta^{\alpha\beta}\overline{\theta}_{\dot{\alpha}\dot{\beta}} T^{\dot{\alpha}\dot{\beta}}_{\alpha\beta}(x) + \overline{\theta}_{\dot{\alpha}\dot{\beta}}\theta^{\alpha}_i \overline{A}^{\dot{\alpha}\dot{\beta}i}_{\alpha}(x). \end{aligned} \tag{4.41}$$

Now that we have an expression for the supercurrent, we can construct the supercurrent squared operator by firstly taking

$$\mathcal{O}_{T^2} = \mathcal{J}\mathcal{J}, \tag{4.42}$$

and then computing the projection

$$\mathrm{O}_{T^2} = \int d^4\theta d^4\bar{\theta}\mathcal{O}_{T^2}\Big|_{\theta=\bar{\theta}=0}. \tag{4.43}$$

This yields (after applying the equations of motion)

$$\begin{aligned} \mathrm{O}_{T^2} &= T^{\dot{\alpha}\dot{\beta}}{}_{\alpha\beta} T_{\dot{\alpha}\dot{\beta}}{}^{\alpha\beta} + P^{ijlk}P_{ijlk} \\ &= T^{\dot{\alpha}\dot{\beta}}{}_{\alpha\beta} T_{\dot{\alpha}\dot{\beta}}{}^{\alpha\beta} + \frac{3}{64}(\Box j)^2 \\ &= T^{\dot{\alpha}\dot{\beta}}{}_{\alpha\beta} T_{\dot{\alpha}\dot{\beta}}{}^{\alpha\beta} + \frac{3}{64}\partial^{\dot{\alpha}}_{\alpha}(\Box j \partial^{\alpha}_{\dot{\alpha}} j). \end{aligned} \tag{4.44}$$

Restoring this expression to spacetime indices, we have the result

$$\mathrm{O}_{T^2}(x) = T^{ab}(x)T_{ab}(x) \quad + \text{total derivatives}. \tag{4.45}$$

This is a novel result, and shows that at least in the conformal case, the 4D $\mathcal{N} = 2$ supercurrent squared operator reduces to the $\mathrm{T}\bar{\mathrm{T}}$ operator up to a total derivative and equations of motion. However, this result is not expected to extend to the non-conformal case, as discussed earlier in Section 2.6.

4.5 2D $\mathcal{N} = (0, 4)$ Supersymmetry

For 2D $\mathcal{N} = (0, 4)$ supersymmetry, superspace can be parametrised by four real grassmann coordinates, which we will denote θ^+_{ia}, with $SU(2)$ indices $i = \{1, 2\}$, $a = \{1, 2\}$.

These have associated covariant derivatives

$$D^{ia}_+ = \frac{\partial}{\partial\theta^+_{ia}} + \frac{i}{2}\theta^{+ia}\partial_{++}, \tag{4.46}$$

satisfying

$$\{D^{ia}_+, D^{jb}_+\} = i\epsilon^{ij}\epsilon^{ab}\partial_{++}\ , \quad [\partial_{\pm\pm}, D^{ia}_+] = 0. \tag{4.47}$$

Then, we posit that the supercurrent multiplet for 2D $\mathcal{N} = (0, 4)$ theories can be described by a scalar supercurent $\mathcal{S}$ and spin-2 supercurrent $\mathcal{T}_{----}$. The most general form such superfields can take are

$$\begin{aligned}\mathcal{S}(x, \theta^+_{ia}) = j(x) + \theta^+_{ia}\psi^{ia}_+(x) + \theta^+_{ia}\theta^{+i}_b F^{ab}_{++}(x) + \theta^+_{ia}\theta^{+a}_j K^{ij}_{++}(x)\\ + \theta^{+a}_j\theta^+_{ia}\theta^{+i}_b S^{jb}_{+++}(x) + \theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}T_{++++}(x),\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}\mathcal{T}_{----}(x, \theta^+_{ia}) = T_{----}(x) + \theta^+_{ia}A^{ia}_{+----}(x) + \theta^+_{ia}\theta^{+i}_b B^{ab}_{++----}(x) + \theta^+_{ia}\theta^{+a}_j C^{ij}_{++----}(x)\\ + \theta^{+a}_j\theta^+_{ia}\theta^{+i}_b G^{jb}_{+++----}(x) + \theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}H_{++++----}(x).\end{aligned} \tag{4.49}$$

For the conformal theory, these supercurrents obey

$$\partial_{--}\mathcal{S} = 0, \tag{4.50}$$

and

$$D^{ia}_+\mathcal{T}_{----} = 0. \tag{4.51}$$

Note that thanks to the algebra of covariant derivatives,

$$D^{ia}_+\mathcal{T}_{----} = 0 \implies \{D^{ia}_+, D^{jb}_+\}\mathcal{T}_{----} = i\epsilon^{ij}\epsilon^{ab}\partial_{++}\mathcal{T}_{----} = 0 \implies \partial_{++}\mathcal{T}_{----} = 0. \tag{4.52}$$

Applying the above constraints to the superfields yields the trivial conservation constraints

$$\partial_{--}j(x) = 0, \tag{4.53}$$

$$\partial_{--}\psi^{ia}_+(x) = 0, \tag{4.54}$$

$$\partial_{--}F^{ab}_{++}(x) = 0, \tag{4.55}$$

$$\partial_{--}K^{ij}_{++}(x) = 0, \tag{4.56}$$

$$\partial_{--}S^{jb}_{+++}(x) = 0, \tag{4.57}$$

$$\partial_{--}T_{++++}(x) = 0, \tag{4.58}$$

as well as the equations

$$\partial_{++}T_{----}(x) = 0, \tag{4.59}$$

$$C^{ij}_{++----}(x) = 0, \tag{4.60}$$

$$B^{ab}_{++----}(x) = 0, \tag{4.61}$$

$$A^{ia}_{+----}(x) = 0, \tag{4.62}$$

$$G^{ia}_{+++----}(x) = 0, \tag{4.63}$$

and

$$H_{++++----}(x) = 0. \tag{4.64}$$

Now that we have an expression for the supercurrent, we can construct the supercurrent squared operator by firstly taking

$$\mathcal{O}_{T^2} = \mathcal{S}\mathcal{T}_{----}, \tag{4.65}$$

and then computing the projection

$$\mathrm{O}_{T^2} = \int d^4\theta \mathcal{O}_{T^2}\Big|_{\theta=\bar{\theta}=0} \tag{4.66}$$

This yields, after applying the equations of motion,

$$\mathrm{O}_{T^2} = T_{++++}T_{----}. \tag{4.67}$$

This novel result shows that indeed, at least in the conformal case, the supercurrent-squared operator reduces to the T$\bar{\mathrm{T}}$ operator for the $2D$ $\mathcal{N} = (0,4)$ theory.

4.6 2D N=(4,4) Supersymmetry

For $\mathcal{N} = (4,4)$, consider superspace parameterised by 8 Grassmann coordinates, θ^+_I and θ^-_A, where $I = \{1,2,3,4\}$, $A = \{1,2,3,4\}$ are $SO(4)$ indices.

These have associated covariant derivatives

$$D^I_+ = \frac{\partial}{\partial\theta^+_I} + \frac{i}{2}\theta^{+I}\partial_{++}, \tag{4.68}$$

and

$$D^A_- = \frac{\partial}{\partial\theta^-_A} + \frac{i}{2}\theta^{-A}\partial_{--}. \tag{4.69}$$

One can see that these obey

$$\{D^I_+, D^J_+\} = i\delta^{IJ}\partial_{++}, \tag{4.70}$$

and

$$\{D^A_-, D^B_-\} = i\delta^{AB}\partial_{--}, \tag{4.71}$$

while commuting with $\partial_{\pm\pm}$.

Then, consider the supercurrents

$$\begin{aligned}\mathcal{R}(x,\theta^+_I,\theta^-_A) = r(x) + \theta^+_I\psi^I_+(x) + \theta^+_I\theta^+_J\epsilon^{IJ}F_{++}(x) + \theta^+_I\theta^+_J\theta^+_K\epsilon^{IJK}S_{+++}(x)\\ + \theta^+_I\theta^+_J\theta^+_K\theta^+_L\epsilon^{IJKL}T_{++++}(x) + \ \theta^-_A \text{ terms and cross terms}\end{aligned} \tag{4.72}$$

$$\begin{aligned}\mathcal{L}(x,\theta^+_I,\theta^-_A) = l(x) + \theta^-_A\psi^A_-(x) + \theta^-_A\theta^-_B\epsilon^{AB}F_{--}(x) + \theta^-_A\theta^-_B\theta^-_C\epsilon^{ABC}S^{ABC}_{---}(x)\\ + \theta^-_A\theta^-_B\theta^-_C\theta^-_D\epsilon^{ABCD}T_{----}(x) + \ \theta^+_I \text{ terms and cross terms}\end{aligned} \tag{4.73}$$

to which we apply the conformal constraints

$$D^A_-\mathcal{R} = 0, \tag{4.74}$$

and

$$D^I_+\mathcal{L} = 0. \tag{4.75}$$

From these constraints, by virtue of the algebra, one then has

$$\partial_{--}\mathcal{R} = 0, \tag{4.76}$$

$$\partial_{++}\mathcal{L} = 0, \tag{4.77}$$

which yields conservation equations for each term in the supercurrent expansions. Further, one has then

$$D^A_-\mathcal{R} = \frac{\partial}{\partial\theta^-_A}\mathcal{R} + \frac{i}{2}\theta^{-A}\partial_{--}\mathcal{R} = \frac{\partial}{\partial\theta^-_A}\mathcal{R} = 0. \tag{4.78}$$

Thus, all terms in $\mathcal{R}$ that are proportional to θ^-_A are constrained to be zero, and similarly, all terms in $\mathcal{L}$ that are proportional to θ^+_I vanish. Thus, thus, $\mathcal{L}$ and $\mathcal{R}$ consist of only purely left/right moving components,

$$\begin{aligned}\mathcal{R}(x,\theta^+_I) = r(x) + \theta^+_I\psi^I_+(x) + \theta^+_I\theta^+_J\epsilon^{IJ}F_{++}(x)\\ + \theta^+_I\theta^+_J\theta^+_K\epsilon^{IJK}S_{+++}(x) + \theta^+_I\theta^+_J\theta^+_k\theta^+_l\epsilon^{IJKL}T_{++++}(x)\end{aligned} \tag{4.79}$$

$$\begin{aligned}\mathcal{L}(x,\theta^+_I) = l(x) + \theta^-_A\psi^A_-(x) + \theta^-_A\theta^-_B\epsilon^{AB}F_{--}(x)\\ + \theta^-_A\theta^-_B\theta^-_C\epsilon^{ABC}S^{ABC}_{---}(x) + \theta^-_A\theta^-_B\theta^-_C\theta^-_D\epsilon^{ABCD}T_{----}(x).\end{aligned} \tag{4.80}$$

Then, the supercurrent squared operator can be constructed by firstly taking

$$\mathcal{O}_{T^2} = \mathcal{L}\mathcal{R}, \tag{4.81}$$

and then computing the projection

$$\mathrm{O}_{T^2} = \int d^8\theta\mathcal{O}_{T^2}\Big|_{\theta=\bar{\theta}=0} \tag{4.82}$$

This yields

$$\mathrm{O}_{T^2} = T_{++++}T_{----}. \tag{4.83}$$

This novel result indicates, that at least in the conformal case, the supercurrent-squared operator coincides on-shell to $\mathrm{T}\bar{\mathrm{T}}$, indicating that the latter preserves $\mathcal{N} = (4,4)$ SUSY.

5

Construction of $\mathrm{T\bar{T}}$ and $\mathrm{T\bar{T}}$-like Terms for Non-conformal Theories

5.1 General Approach and Procedure

Many of the same ideas mentioned in Section 4.1 follow through to the proceeding analysis. However, we will now make note of, and stress, the differences and complications that arise in the general (non-conformal) case.

The supercurrent multiplets for the non-conformal case are far more complex than the highly symmetric conformal case. In general, it is unknown whether the $\mathrm{T\bar{T}}$ operator preserves extended supersymmetry. Indeed, even in $\mathcal{N} = 1$ $4D$, there can be additional terms in the supercurrrent-squared that are not total derivatives and do not disappear after imposing equations of motion, as seen in [16]. Given the relationship between $4D$ and $2D$ theories, it is thus unknown whether $2D$ extended supersymmetry in theories in general are preserved by $\mathrm{T\bar{T}}$, though all $2D$ theories for which the supercurrent has been classified, are. It may be the case that the supercurrent-squared coincides with $\mathrm{T\bar{T}}$ in general $2D$ extended SUSY only after imposing specific restrictions. Additionally, the supercurrent multiplet is not well-understood in higher extended supersymmetric theories. This makes the study of supercurrent-squared deformations for such theories, difficult.

In this chapter, we will first study the $2D$ $\mathcal{N} = (0, 2)$ theory, and reproduce the known result. We will then study the $2D$ $\mathcal{N} = (0, 4)$ theory, and impose sufficient conditions for the supercurrent-squared operator to be equivalent to $\mathrm{T\bar{T}}$.

5.2 2D $\mathcal{N} = (0,2)$ Supersymmetry

Recall the supercurrents

$$\mathcal{S}_{++}(x,\theta^+,\overline{\theta}^+) = j_{++}(x) + \theta^+ S_{+++}(x) - \overline{\theta}^+\overline{S}_{+++}(x) - \theta^+\overline{\theta}^+ T_{++++}(x) \tag{5.1}$$

and

$$\mathcal{T}_{----}(x,\theta^+,\overline{\theta}^+) = T_{----}(x) + \theta^+ B_{+----}(x) - \overline{\theta}^+\overline{B}_{+----}(x) + \theta^+\overline{\theta}^+ C_{++----}(x) \tag{5.2}$$

discussed for the conformal case of 2D $\mathcal{N} = (0,2)$ supersymmetry. In the non-conformal case, there are additional complex supercurrents

$$\mathcal{W}_-(x,\theta^+,\overline{\theta}^+) = w_-(x) + \theta^+ F_{+-}(x) + \overline{\theta}^+ H_{+-}(x) + \theta^+\overline{\theta}^+ G_{++-}(x),$$

and

$$\overline{\mathcal{W}}_-(x,\theta^+,\overline{\theta}^+) = \overline{w}_-(x) + \theta^+ \overline{H}_{+-}(x) + \overline{\theta}^+ \overline{F}_{+-}(x) + \theta^+\overline{\theta}^+ \overline{G}_{++-}(x). \tag{5.3}$$

Together, these are subject to the constraints [4]

$$\partial_{--}\mathcal{S}_{++} = D_+\mathcal{W}_- - \overline{D}_+\overline{\mathcal{W}}_-, \tag{5.4}$$

$$D_+\mathcal{T}_{----} = \frac{1}{2}\partial_{--}\overline{\mathcal{W}}_-, \tag{5.5}$$

$$\overline{D}_+\mathcal{T}_{----} = \frac{1}{2}\partial_{--}\mathcal{W}_-, \tag{5.6}$$

$$\overline{D}_+\mathcal{W}_- = 0, \tag{5.7}$$

and

$$D_+\overline{\mathcal{W}}_- = 0. \tag{5.8}$$

Solving these equations will yield constraints on the components of the superfields. The final two constraints imply

$$H_{+-} = \overline{H}_{+-} = 0, \quad \overline{G}_{++-} = \frac{i}{2}\partial_{++}\overline{w}_-, \quad G_{++-} = -\frac{i}{2}\partial_{++}w_-. \tag{5.9}$$

Then, (5.5) and (5.6) imply that

$$\begin{aligned} &\overline{\theta}^+ C_{++----} + B_{+----} + (-0.5i)\overline{\theta}^+\theta^+\partial_{++}\Big(B_{+----}\Big) + (-0.5i)\overline{\theta}^+\partial_{++}\Big(T_{----}\Big) \\ &= \frac{1}{2}\partial_{--}\Big(\overline{w}_- + \overline{\theta}^+\overline{F}_{+-} + \frac{i}{2}\theta^+\overline{\theta}^+\partial_{++}\overline{w}_-\Big), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} &\theta^+ C_{++----} + \overline{B}_{+----} + (-0.5i)\theta^+\overline{\theta}^+\partial_{++}\Big(\overline{B}_{+----}\Big) + (0.5i)\theta^+\partial_{++}\Big(T_{----}\Big) \\ &= \frac{1}{2}\partial_{--}\Big(w_- + \theta^+ F_{+-} - \frac{i}{2}\theta^+\overline{\theta}^+\partial_{++}w_-\Big) \end{aligned} \tag{5.11}$$

so,

$$C_{++----} = \frac{1}{4}\partial_{--}\Big(F_{+-} + \overline{F}_{+-}\Big), \quad \partial_{++}T_{----} = \frac{i}{2}\partial_{--}\Big(\overline{F}_{+-} - F_{+-}\Big) \tag{5.12}$$

$$B_{+----} = \frac{1}{2}\partial_{--}\overline{w}_{-}, \quad \partial_{++}B_{+----} = \frac{1}{2}\partial_{--}\partial_{++}\overline{w}_{-}, \tag{5.13}$$

Then, (5.4) yields

$$\begin{aligned}
&\partial_{--}\Big(j_{++}(x) + \theta^{+}S_{+++}(x) - \overline{\theta}^{+}\overline{S}_{+++}(x) + \theta^{+}\overline{\theta}^{+}T_{++++}(x)\Big) \\
&= \overline{\theta}^{+}G_{++-} + F_{+-} + (-0.5i)\overline{\theta}^{+}\theta^{+}\partial_{++}F_{+-} + (-0.5i)\overline{\theta}^{+}\partial_{++}\Big(w_{-}\Big) \\
&\quad - \Bigg(\theta^{+}\overline{G}_{++-} - \overline{F}_{+-} + (0.5i)\theta^{+}\partial_{++}\Big(\overline{w}_{-}\Big) + (0.5i)\theta^{+}\overline{\theta}^{+}\partial_{++}\Big(\overline{F}_{+-}\Big)\Bigg).
\end{aligned} \tag{5.14}$$

From this, one has

$$\partial_{--}S_{+++} = -i\partial_{++}\overline{w}_{-}, \quad \partial_{--}\overline{S}_{+++} = i\partial_{++}w_{-}, \tag{5.15}$$

$$\partial_{--}j_{++} = F_{+-} + \bar{F}_{+-}, \quad -\partial_{--}T_{++++} = \partial_{++}F_{+-} + \partial_{++}\overline{F}_{+-}. \tag{5.16}$$

Thus, the final expressions for the supercurrents are

$$\mathcal{S}_{++}(x, \theta^{+}, \overline{\theta}^{+}) = j_{++}(x) + \theta^{+}S_{+++}(x) - \overline{\theta}^{+}\overline{S}_{+++}(x) - \theta^{+}\overline{\theta}^{+}T_{++++}(x), \tag{5.17}$$

$$\mathcal{T}_{----}(x, \theta^{+}, \overline{\theta}^{+}) = T_{----}(x) + \frac{1}{2}\theta^{+}\partial_{--}\overline{w}_{-} - \frac{1}{2}\overline{\theta}^{+}\partial_{--}w_{-} + \frac{1}{4}\theta^{+}\overline{\theta}^{+}\partial_{--}^{2}j_{++}, \tag{5.18}$$

and

$$\mathcal{W}_{-}(x, \theta^{+}, \overline{\theta}^{+}) = w_{-}(x) + \theta^{+}(\frac{1}{2}\partial_{--}j_{++} + i\mathrm{Im}[F_{+-}(x)]) - \frac{i}{2}\theta^{+}\overline{\theta}^{+}\partial_{++}w_{-}(x). \tag{5.19}$$

It is known that the supercurrent-squared term is given by [4]

$$\mathcal{O}_{T^2} = \mathcal{T}_{----}\mathcal{S}_{++} - \overline{\mathcal{W}}_{-}\mathcal{W}_{-}, \tag{5.20}$$

and then computing the projection

$$\mathrm{O}_{T^2} = \int d\theta d\bar{\theta}\mathcal{O}_{T^2}\Big|_{\theta=\bar{\theta}=0} \tag{5.21}$$

This yields

$$\begin{aligned}
\mathrm{O}_{T^2} = -\Big(&- T_{++++}T_{----} + S_{+++}\overline{B}_{+-----} + C_{++----}j_{++} \\
&+ B_{+----}\overline{S}_{+++} + w_{-}\overline{G}_{++-} + G_{++-}\overline{w}_{--} + F_{+-}\overline{F}_{+-}\Big)
\end{aligned} \tag{5.22}$$

$$
\begin{aligned}
&= -\Big(- T_{++++}T_{----} + S_{+++}\frac{1}{2}\partial_{--}w_{-} + \frac{1}{4}(\partial_{--}^{2}j_{++})j_{++} + \frac{1}{2}(\partial_{--}\overline{w}_{-})\overline{S}_{+++} \\
&\quad + w_{-}\frac{i}{2}\partial_{++}\overline{w}_{-} - \frac{i}{2}(\partial_{++}w_{-})\overline{w}_{--} + \frac{1}{4}(\partial_{--}j_{++})^{2} + \mathrm{Im}(F_{+-})^{2}\Big)
\end{aligned}
\tag{5.23}
$$

$$
\begin{aligned}
&= -\Big(- T_{++++}T_{----} + S_{+++}\frac{1}{2}\partial_{--}w_{-} + \frac{1}{4}(\partial_{--}^{2}j_{++})j_{++} + \frac{1}{2}(\partial_{--}\overline{w}_{-})\overline{S}_{+++} \\
&\quad - w_{-}\frac{1}{2}\partial_{--}S_{+++} - \frac{1}{2}(\partial_{--}\overline{S}_{+++})\overline{w}_{--} + \frac{1}{4}(\partial_{--}j_{++})^{2} + \mathrm{Im}(F_{+-})^{2}\Big)
\end{aligned}
\tag{5.24}
$$

$$
= -\Big(-T_{++++}T_{----}+\frac{1}{2}\partial_{--}\Big(S_{+++}w_{-}\Big)+\frac{1}{4}\partial_{--}(j_{++}\partial_{--}j_{++})+\frac{1}{2}\partial_{--}\Big(\overline{w}_{-}\overline{S}_{+++}\Big)+\mathrm{Im}(F_{+-})^{2}\Big)
\tag{5.25}
$$

Identifying $\mathrm{Im}(F_{+-}) = T_{++--} = \Theta$,

$$
\Longrightarrow \; \mathrm{O}_{T^2} = T_{++++}T_{----} - \Theta^{2} + \text{total derivatives.}
\tag{5.26}
$$

Thus, this supercurrent-squared operator is equivalent to $T\bar{T}$ up to total derivatives generated by supersymmetry.

This result matches that obtained in [4], and thus suggests that our software is able to successfully compute the structure of the supercurrent multiplets, and supercurrent-squared operator.

5.3 2D $\mathcal{N} = (0,4)$ Supersymmetry

For 2D $\mathcal{N} = (0,4)$ supersymmetry, superspace can be parameterised by four real Grassmann coordinates, which we will denote θ^+_{ia}, $i = \{1,2\}$, $a = \{1,2\}$.

These have associated covariant derivatives

$$D^{ia}_+ = \frac{\partial}{\partial \theta^+_{ia}} + \frac{i}{2}\theta^{+ia}\partial_{++}, \tag{5.27}$$

satisfying

$$\{D^{ia}_+, D^{jb}_+\} = i\epsilon^{ij}\epsilon^{ab}\partial_{++}. \tag{5.28}$$

Recall the general form of the supercurrents for 2D $\mathcal{N} = (0,4)$ theories,

$$\begin{aligned}\mathcal{S}(x,\theta^+_{ia}) = j(x) + \theta^+_{ia}\psi^{ia}_+(x) + \theta^+_{ia}\theta^{+i}_b F^{ab}_{++}(x) + \theta^+_{ia}\theta^{+a}_j K^{ij}_{++}(x)\\ + \theta^{+a}_j\theta^+_{ia}\theta^{+i}_b S^{jb}_{+++}(x) + \theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}T_{++++}(x),\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}\mathcal{T}_{----}(x,\theta^+_{ia}) = T_{----}(x) + \theta^+_{ia}A^{ia}_{+----}(x) + \theta^+_{ia}\theta^{+i}_b B^{ab}_{++----}(x) + \theta^+_{ia}\theta^{+a}_j C^{ij}_{++----}(x)\\ + \theta^{+a}_j\theta^+_{ia}\theta^{+i}_b G^{jb}_{+++----}(x) + \theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}H_{++++----}(x).\end{aligned} \tag{5.30}$$

In the most general case, these currents are subject to

$$D^{ia}_+\mathcal{T}_{----} = \mathcal{K}^{ia}_{+----}, \tag{5.31}$$

and

$$\partial_{--}\mathcal{S} = \mathcal{L}_{--}. \tag{5.32}$$

For some arbitary superfields $\mathcal{L}_{--}$ and $\mathcal{K}^{ia}_{+----}$.

In our study of this theory, we make the ansatz that

$$D^{ia}_+\mathcal{T}_{----} = \mathcal{K}^{ia}_{+----} = \partial_{--}\mathcal{W}^{ia}_-, \tag{5.33}$$

$$\partial_{--}\mathcal{S} = \mathcal{L}_{--} = \epsilon_{ij}\epsilon_{ab}D^{ia}_+\mathcal{W}^{jb}_{---}, \tag{5.34}$$

with

$$\mathcal{W}^{ia}_- = D^{ia}_+D^{jb}_+\mathcal{W}_{jb---} - \alpha D^{jb}_+D_{jb+}\mathcal{W}^{ia}_{---} = D^{ia}_+D^{jb}_+\mathcal{W}_{jb---}, \tag{5.35}$$

where $\alpha = 0$ must be taken in order to be consistent with the algebra of covariant derivatives. Note that this may not be the most general relationship between $\mathcal{W}^{ia}_{---}$ and $\mathcal{W}^{ia}_-$, nor the most general ansatz.

These two supercurrents have the generic superfield expansions

$$\begin{aligned}\mathcal{W}^{ia}_-(x,\theta^+_{ia}) = J^{ia}_-(x) + \theta^+_{jb}M^{iajb}_{+-}(x) + \theta^+_{jb}\theta^{+j}_c N^{iabc}_{++-}(x) + \theta^+_{jb}\theta^{+b}_k P^{iajk}_{++-}(x)\\ + \theta^{+c}_j\theta^+_{kc}\theta^{+k}_b Q^{iajb}_{+++-}(x) + \theta^+_{kc}\theta^{+k}_b\theta^{+c}_j\theta^{+jb}R^{ia}_{++++-}(x),\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}\mathcal{W}^{ia}_{---}(x,\theta^+_{ia}) = w^{ia}_{---}(x) + \theta^+_{jb}X^{iajb}_{+---}(x) + \theta^+_{jb}\theta^{+j}_{c}Y^{iabc}_{++---}(x) + \theta^+_{jb}\theta^{+b}_{k}Z^{iajk}_{++---}(x)\\ + \theta^{+c}_{j}\theta^+_{kc}\theta^{+k}_{b}U^{iajb}_{+++---}(x) + \theta^+_{kc}\theta^{+k}_{b}\theta^{+c}_{j}\theta^{+jb}V^{ia}_{++++---}(x).\end{aligned} \tag{5.37}$$

Note that from (5.33), the components of the supercurrents must obey

$$A^i_{+----}{}^a(x) = \partial_{--}J^{ia}_{-}(x), \tag{5.38}$$

$$C^{ij}_{++----}(x) = B^{ab}_{++----}(x) = 0, \tag{5.39}$$

$$\partial_{++}T_{----}(x) = -\frac{i}{2}\partial_{--}M_{+-ia}{}^{ia}(x), \tag{5.40}$$

$$G^{ai}{}_{+++----}(x) = \frac{2}{9}\partial_{--}\Big(P_{++-j}{}^{aij}(x)\Big) - \frac{2}{9}\partial_{--}\Big(N_{++-}{}^{i}{}_{b}{}^{ba}(x)\Big), \tag{5.41}$$

$$H_{++++----}(x) = \frac{1}{16}\partial_{--}Q_{ia}{}^{ia}_{+++-}(x). \tag{5.42}$$

Additionally, from the algebra (5.28), one must have

$$D^{(i}_{+(a}\mathcal{W}^{j)}_{-\,b)} = 0, \tag{5.43}$$

which yields further constraints

$$M^{iabj}_{+-}(x) = \frac{1}{4}\epsilon^{ij}\epsilon^{ab}M_{kc}{}^{kc}_{+-}(x), \tag{5.44}$$

$$Q^{iabj}_{+++-}(x) = \frac{1}{4}\epsilon^{ij}\epsilon^{ab}Q_{kc}{}^{kc}_{+++-}(x), \tag{5.45}$$

$$R^{ia}{}_{++++-}(x) = \frac{i}{36}\partial_{++}\Big(P_{++-j}{}^{aij}(x)\Big) - \frac{i}{36}\partial_{++}\Big(N_{++-}{}^{i}{}_{b}{}^{ba}(x)\Big), \tag{5.46}$$

and

$$\partial_{++}\Big(J_{-}{}^{ia}(x)\Big) = \frac{4i}{3}P_{++-j}{}^{aij}(x) + \frac{4i}{3}N_{++-}{}^{i}{}_{b}{}^{ab}(x). \tag{5.47}$$

Note that these constraints are independent of our choice in how to relate $\mathcal{W}^{ia}_{---}$ and $\mathcal{W}^{ia}_{-}$, and are thus in a sense more fundamental.

Then from (5.35), the components of $\mathcal{W}^{ia}_{-}$ and $\mathcal{W}^{ia}_{---}$ are related by the equations

$$J_{-}{}^{ia}(x) = -2(Z_{++---j}{}^{aji}(x) + Y_{++---}{}^{i}{}_{b}{}^{ba}(x)) + \frac{i}{2}\partial_{++}w_{---}{}^{ia}(x), \tag{5.48}$$

$$M_{+-ia}{}^{ia}(x) = 2i\partial_{++}X_{+---ia}{}^{ia}(x), \tag{5.49}$$

$$N_{++-}{}^{i}{}_{b}{}^{ba}(x) = -9V_{++++---}{}^{ia}(x) + \frac{3i}{2}\partial_{++}Y_{++---}{}^{i}{}_{b}{}^{ba}(x) + \frac{3}{16}\partial^2_{++}w_{---}{}^{ia}(x), \tag{5.50}$$

$$P_{++-j}{}^{aij}(x) = 9V_{++++---}{}^{ia}(x) + \frac{3i}{2}\partial_{++}\Big(Z_{++---j}{}^{aij}(x)\Big) + \frac{3}{16}\partial_{++}^2 w_{---}{}^{ia}(x), \tag{5.51}$$

$$Q_{+++-ia}{}^{ia}(x) = 2i\partial_{++}U_{+++---ia}{}^{ia}(x), \tag{5.52}$$

$$R^{ia}_{++++-}(x) = \frac{i}{2}\partial_{++}V^{ia}_{++++---}(x) - \frac{1}{48}\partial_{++}^2(Z_{++---j}{}^{aij}(x) - Y_{++---b}{}^{iab}(x)). \tag{5.53}$$

Finally, from the defining relation (5.34), there are the constraints

$$\partial_{--}j(x) = X_{+---ia}{}^{ia}(x), \tag{5.54}$$

$$\partial_{--}\psi_+{}^{ia}(x) = -2(Z_{++---j}{}^{aji}(x) + Y_{++---}{}^{i}{}_{b}{}^{ab}(x)) + \frac{i}{2}\partial_{++}w_{---}{}^{ia}(x), \tag{5.55}$$

$$\partial_{--}F_{++}{}^{ab}(x) = \partial_{--}K_{++}{}^{ij}(x) = 0, \tag{5.56}$$

$$\partial_{--}S_{+++}{}^{ai}(x) = 4V_{++++---}{}^{ia}(x) + \frac{i}{3}\partial_{++}(Z_{++---j}{}^{aji}(x) - Y_{++---}{}^{i}{}_{b}{}^{ab}(x)), \tag{5.57}$$

$$\partial_{--}T_{++++}(x) = \frac{i}{8}\partial_{++}U_{+++---ia}{}^{ai}(x). \tag{5.58}$$

The relations

$$Q_{+++-iabj}(x)X_{+---}{}^{iajb}(x) = \frac{1}{4}Q_{+++-ia}{}^{ia}(x)X_{+---ia}{}^{ia}(x) \tag{5.59}$$

and

$$M_{+-iabj}(x)U_{+++---}{}^{iajb}(x) = \frac{1}{4}M_{+-ia}{}^{ia}(x)U_{+++---ia}{}^{ia}(x) \tag{5.60}$$

are useful also.

We can then construct a supercurrent-squared operator with the combination

$$\mathcal{O}_{T^2} = \mathcal{T}_{----}\mathcal{S} - \epsilon_{ab}\epsilon_{ij}\mathcal{W}^{ia}_{-}\mathcal{W}^{jb}_{---}, \tag{5.61}$$

and then compute the projection

$$\mathrm{O}_{T^2} = \int d\theta^4 \mathcal{O}_{T^2}\Big|_{\theta=0}. \tag{5.62}$$

This yields

$$\begin{aligned}
\mathrm{O}_{T^2} &= H_{++++----}(x)j(x) + 0.25G_{+++----ai}(x)\psi_+{}^{ia}(x) - 0.25A_{+----ia}(x)S_{+++}{}^{ai}(x) \\
&+ T_{----}(x)T_{++++}(x) - R_{++++-ia}(x)w_{---}{}^{ia}(x) + 0.25Q_{+++-iabj}(x)X_{+---}{}^{iajb}(x) \\
&- \frac{1}{3}P_{++-iajk}(x)Z_{++---}{}^{iajk}(x) + \frac{1}{3}N_{++-iabc}(x)Y_{++---}{}^{iabc}(x) \\
&- 0.25M_{+-iajb}(x)U_{+++---}{}^{iabj}(x) - J_{-ia}(x)V_{++++---}{}^{ia}(x).
\end{aligned} \tag{5.63}$$

The supercurrents are then constrained to be

$$\begin{aligned}
\mathcal{S}(x,\theta^+_{ia}) = j(x) &+ \theta^+_{ia}\psi^{ia}_+(x) + \theta^+_{ia}\theta^{+i}_b F^{ab}_{++}(x) + \theta^+_{ia}\theta^{+a}_j K^{ij}_{++}(x) \\
&+ \theta^{+a}_j\theta^+_{ia}\theta^{+i}_b S^{jb}_{+++}(x) + \theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}T_{++++}(x),
\end{aligned} \tag{5.64}$$

$$\begin{aligned}
\mathcal{T}_{----}(x,\theta^+_{ia}) = T_{----}(x) &+ \theta^+_{ia}\Bigg(\partial_{--}\Big(-2(Z_{++---j}{}^{aji}(x) + Y_{++---}{}^{i}{}_{b}{}^{ba}(x)) \\
&+ \frac{i}{2}\partial_{++}w_{---}{}^{ia}(x)\Big)\Bigg) + \theta^{+a}_j\theta^+_{ia}\theta^{+i}_b\partial_{--}\Big(4V_{++++---}{}^{ia}(x) \\
&+ \frac{i}{3}\partial_{++}(Z_{++---j}{}^{aij}(x) - Y_{++---}{}^{i}{}_{b}{}^{ba}(x))\Big) \\
&+ \frac{1}{16}\theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}\partial_{--}Q_{ia}{}^{ia}_{+++-}(x),
\end{aligned} \tag{5.65}$$

$$\begin{aligned}
\mathcal{W}^{ia}_-(x,\theta^+_{ia}) = &\Big(-2(Z_{++---j}{}^{aji}(x) + Y_{++---}{}^{i}{}_{b}{}^{ba}(x)) + \frac{i}{2}\partial_{++}w_{---}{}^{ia}(x)\Big) \\
&+ \frac{i}{2}\theta^{ia}\partial_{++}\partial_{--}j(x) + \theta^+_{jb}\theta^{+j}_c\Bigg(\frac{3}{2}\epsilon^{ba}\partial_{--}S_{+++}{}^{ai}(x) + i\partial_{++}Y_{++---}{}^{iabc}(x) \\
&+ \frac{i}{2}\epsilon^{ba}\partial_{++}Z_{++---j}{}^{cji}(x) - \frac{i}{2}\partial_{++}\epsilon^{ba}\partial_{++}Y_{++---}{}^{i}{}_{d}{}^{dc}(x) - \frac{1}{4}\epsilon^{ba}\partial^2_{++}w_{---}{}^{ic}(x)\Bigg) \\
&+ \theta^+_{jb}\theta^{+b}_k\Bigg(-\frac{3}{2}\epsilon^{ji}\partial_{--}S_{+++}{}^{ka}(x) + \frac{i}{2}\epsilon^{ji}\partial_{++}Y_{++---}{}^{k}{}_{d}{}^{da}(x) \\
&+ i\partial_{++}Z_{++---}{}^{iajk}(x) - \frac{i}{2}\partial_{++}\epsilon^{ji}\partial_{++}Z_{++---}{}^{k}{}_{l}{}^{la}(x) - \frac{1}{4}\epsilon^{ji}\partial^2_{++}w_{---}{}^{ka}(x)\Bigg) \\
&+ \frac{i}{2}\theta^{+ci}\theta^+_{kc}\theta^{+ka}\partial_{++}U_{+++---jb}{}^{jb}(x) + \theta^+_{kc}\theta^{+k}_b\theta^{+c}_j\theta^{+jb}\Big(\frac{i}{8}\partial_{++}\partial_{--}S_{+++}{}^{ai}(x) \\
&- \frac{1}{24}\partial^2_{++}(Z_{++---j}{}^{aij}(x) - Y_{++---}{}^{i}{}_{b}{}^{ab}(x))\Big),
\end{aligned} \tag{5.66}$$

and

$$\begin{aligned}\mathcal{W}^{ia}_{---}(x, \theta^+_{ia}) = w^{ia}_{---}(x) + \theta^{+ia}\partial_{--}j(x) + \theta^+_{jb}\theta^{+j}_c Y^{iabc}_{++---}(x) + \theta^+_{jb}\theta^{+b}_k Z^{iajk}_{++---}(x) \\ + \theta^{+c}_j\theta^+_{kc}\theta^{+k}_b U^{iajb}_{+++---}(x) + \frac{1}{4}\theta^+_{kc}\theta^{+k}_b\theta^{+c}_j\theta^{+jb}\Big(\partial_{--}S_{+++}{}^{ai}(x) \\ - \frac{i}{3}\partial_{++}(Z_{++---j}{}^{aji}(x) - Y_{++---}{}^{i}{}_{b}{}^{ab}(x))\Big).\end{aligned} \tag{5.67}$$

Amongst these terms, we expect $U^{iajb}_{+++---}(x)$ and $M_{+-ia}{}^{ia}(x)$ to be related to Θ, and the supersymmetry current to be related to the difference between $Z^{iajk}_{++---}(x)$ and $Y^{iabc}_{++---}(x)$, according to our construction.

It should once again be noted that this may not be the most general construction, both in terms of the initial ansatz, nor in terms of the relation (5.35). Further work must be done to determine the full set of possible forms that the multiplet can take, and which form is the most general.

5.3.1 Sufficient Conditions

Note that from the algebra, we must have (5.28). A sufficient condition for this to hold is when

$$D^{(i}_{+a}\mathcal{W}^{j)}_{-\,b} = D^{i}_{+(a}\mathcal{W}^{j}_{-b)} = 0. \tag{5.68}$$

When this condition is imposed on $\mathcal{W}^{ia}_{-}$, there are the additional constraints on the supercurrent components

$$P_{++-j}{}^{aij}(x) = N_{++-}{}^{i}{}_{b}{}^{ba}(x) = \partial_{++}J_{-}{}^{ia}(x) = R_{++++-}{}^{ia}(x) = G_{+++----}{}^{ia}(x) = 0. \tag{5.69}$$

This then reduces the expression for the supercurrent-squared to

$$\begin{aligned}\mathrm{O}_{T^2} = H_{++++----}(x)j(x) + 0.25G_{+++----ai}(x)\psi_+{}^{ia}(x) - 0.25A_{+----ia}(x)S_{+++}{}^{ai}(x) \\ + T_{----}(x)T_{++++}(x) - R_{++++-ia}(x)w_{---}{}^{ia}(x) + 0.25Q_{+++-iabj}(x)X_{+---}{}^{iajb}(x) \\ - \frac{1}{3}P_{++-iajk}(x)Z_{++---}{}^{iajk}(x) + \frac{1}{3}N_{++-iabc}(x)Y_{++---}{}^{iabc}(x) \\ - 0.25M_{+-iajb}(x)U_{+++---}{}^{iabj}(x) - J_{-ia}(x)V_{++++---}{}^{ia}(x)\end{aligned} \tag{5.70}$$

$$\begin{aligned}= T_{----}(x)T_{++++}(x) - \frac{1}{16}M_{+-ia}{}^{ia}(x)U_{+++---ia}{}^{ia}(x) \quad + \frac{1}{4}\partial_{--}(J_{-ia}(x)S_{+++}{}^{ai}(x)) \\ + \frac{1}{16}\partial_{--}(Q_{ia}{}^{ia}_{+++-}(x)j(x)) + \frac{i}{12}\partial_{++}\Big(J_{-ia}(x)(Z_{++---j}{}^{aji}(x) - Y_{++---}{}^{i}{}_{b}{}^{ab}(x))\Big)\end{aligned} \tag{5.71}$$

Recalling the conservation equations

$$\partial_{--}T_{++++}(x) = \frac{i}{8}\partial_{++}U_{+++---ia}{}^{ai}(x) \tag{5.72}$$

and

$$\partial_{++}T_{----}(x) = -\frac{i}{2}\partial_{--}M_{+-ia}{}^{ia}(x), \tag{5.73}$$

suggests the identifications

$$T_{++--}(x) \equiv \frac{i}{2}M_{+-ia}{}^{ia}(x) \tag{5.74}$$

and

$$T_{--++}(x) \equiv -\frac{i}{8}U_{+++---ia}{}^{ia}(x). \tag{5.75}$$

Then, the supercurrent-squared term becomes

$$\begin{aligned}\mathrm{O}_{T^2} = T_{----}(x)T_{++++}(x) - T_{++--}(x)T_{--++}(x) \quad &+ \frac{1}{4}\partial_{--}(J_{-ia}(x)S_{+++}{}^{ai}(x)) \\ +\frac{1}{16}\partial_{--}(Q_{ia}{}^{ia}_{+++-}(x)j(x)) + \frac{i}{12}\partial_{++}\Big(J_{ia}(x)(Z_{++---j}{}^{aji}(x) &- Y_{++---}{}^{i}{}_{b}{}^{ab}(x))\Big)\end{aligned} \tag{5.76}$$

$$= T_{----}(x)T_{++++}(x) - \Theta(x)^2 \quad + \text{total derivs.} \tag{5.77}$$

Thus, under these conditions, the supercurrent-squared term for $\mathcal{N} = (0,4)$ supersymmetry reduces to the $\mathrm{T\bar{T}}$ term up to a total derivative.

The supercurrents also reduce to

$$\begin{aligned}\mathcal{S}(x,\theta^+_{ia}) = j(x) + \theta^+_{ia}\psi^{ia}_+(x) + \theta^+_{ia}\theta^{+i}_bF^{ab}_{++}(x) + \theta^+_{ia}\theta^{+a}_jK^{ij}_{++}(x) \\ + \theta^{+a}_j\theta^+_{ia}\theta^{+i}_bS^{jb}_{+++}(x) + \theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}T_{++++}(x),\end{aligned} \tag{5.78}$$

$$\begin{aligned}\mathcal{T}_{----}(x,\theta^+_{ia}) = T_{----}(x) + \theta^+_{ia}\Bigg(\partial_{--}\Big(-2(Z_{++---j}{}^{aji}(x) + Y_{++---}{}^{i}{}_{b}{}^{ba}(x)) \\ + \frac{i}{2}\partial_{++}w_{---}{}^{ia}(x)\Big)\Bigg) + \frac{1}{16}\theta^+_{ia}\theta^{+i}_b\theta^{+a}_j\theta^{+jb}\partial_{--}Q_{ia}{}^{ia}_{+++-}(x),\end{aligned} \tag{5.79}$$

$$\begin{aligned}\mathcal{W}^{ia}_-(x,\theta^+_{ia}) = \Big(-2(Z_{++---j}{}^{aji}(x) + Y_{++---}{}^{i}{}_{b}{}^{ba}(x)) + \frac{i}{2}\partial_{++}w_{---}{}^{ia}(x)\Big) \\ + \frac{i}{2}\theta^{ia}\partial_{++}\partial_{--}j(x) + \frac{i}{2}\theta^{+ci}\theta^+_{kc}\theta^{+ka}\partial_{++}U_{+++---jb}{}^{jb}(x),\end{aligned} \tag{5.80}$$

and

$$\begin{aligned}\mathcal{W}^{ia}_{---}(x,\theta^+_{ia}) = w^{ia}_{---}(x) + \theta^{+ia}\partial_{--}j(x) + \theta^+_{jb}\theta^{+j}_cY^{iabc}_{++---}(x) \\ + \theta^+_{jb}\theta^{+b}_kZ^{iajk}_{++---}(x) + \theta^{+c}_j\theta^+_{kc}\theta^{+k}_bU^{iajb}_{+++---}(x) \\ - \frac{i}{12}\theta^+_{kc}\theta^{+k}_b\theta^{+c}_j\theta^{+jb}\partial_{++}(Z_{++---j}{}^{aji}(x) - Y_{++---}{}^{i}{}_{b}{}^{ab}(x)).\end{aligned} \tag{5.81}$$

To summarise, under these very strict conditions, the supercurrent-squared term reduces to the $\mathrm{T\bar{T}}$ operator. However, it may be that these conditions are too strong, and thus unphysical. Further analysis should be performed to see if this formulation is physically meaningful, or whether a more general result holds.

6 Conclusion

The $\mathrm{T\bar{T}}$ deformation presents a powerful tool to study the UV behaviour of $2D$ QFTs. Given that QFTs are notoriously difficult to solve, and that most of our knowledge is confined to low energy approximations, the remarkability of the $\mathrm{T\bar{T}}$ operator to make high-energy predictions for our $2D$ theories cannot be overstated. A full classification of the supersymmetric analogue to $\mathrm{T\bar{T}}$, the supercurrent-squared operator, will allow for a better understanding of the high-energy behaviour of $2D$ supersymmetric QFTs possessing extended supersymmetry, and whether or not the flow generated by $\mathrm{T\bar{T}}$ preserves supersymmetry.

In this thesis, we focused on extending the classification of supercurrent-squared deformations to $2D$ $\mathcal{N} = (0, 4)$ and $\mathcal{N} = (4, 4)$ supersymmetry. These theories have important roles in the understanding of $\mathcal{N} = 4$ strings and little string theories [26], and $\mathcal{N} = (0, 4)$ theories are relevant to heterotic theories on Calabi-Yau manifolds. It would be interesting to investigate whether supersymmetric $\mathrm{T\bar{T}}$ plays a role in improving the understanding of these models with extended supersymmetry, which still remains mysterious. It is well known that little string theory and linear dilaton backgrounds have shown to arise in the analysis of single trace TTbar deformations [27], though many technicalities are still unknown. Further work extending this thesis would help push the boundary of our understanding of these string models and models of quantum gravity.

Furthermore, a better understanding of extended SUSY $\mathrm{T\bar{T}}$ in general could give an answer on whether in higher dimensions these deformations only apply to effective field theories, or to all theories. With extended supersymmetry, TTbar-like operators should lie within short multiplets. This would allow them to be free of short-distance divergencies and be quantum mechanically well-defined. Understanding this property is key to studying deformations in $d > 2$. It would be very interesting to understand whether maximally supersymmetric theories (e.g., $4D$ $\mathcal{N} = 4$, and $3D$ $\mathcal{N} = 6, 8$ gauge theories) possess quantum mechanically well-defined, though irrelevant, deformations. By dimensional grounds, we know that such deformations would be part of short multiplets and then enjoy enhanced quantum behaviours.

In this thesis, we have demonstrated sufficient conditions that under which the supercurrent-squared operator in the $2D$ $\mathcal{N} = (0, 4)$ takes the same form as the $\mathrm{T\bar{T}}$ operator. Immediate further works should firstly explore whether these sufficient conditions are necessary conditions for the supercurrent to obey, and whether or not the resulting supercurrent multiplet is physical. If they are not, the reduction from the supercurrent-squared operator to the $\mathrm{T\bar{T}}$ operator should be studied under more general conditions.

Additionally, we have examined $4D$ $\mathcal{N} = 2$ supersymmetry in the conformal setting. This is an important step towards understanding the $2D$ $\mathcal{N} = (4, 4)$ theory. By extending the analysis of $4D$ $\mathcal{N} = 2$ supersymmetry to the non-conformal case, in a method analogous to the $2D$ $\mathcal{N} = (0, 4)$ case presented in this thesis, the result can be dimensionally reduced to study the $2D$ $\mathcal{N} = (4, 4)$ theory. This is perhaps the easiest way to study the $2D$ $\mathcal{N} = (4, 4)$ theory, as the supercurrent multiplets of this theory are poorly understood in the general case.

Additionally, as part of this thesis, we have created a new piece of software that is able to perform algebraic manipulations of tensors. Particularly, it is able to handle graded objects in a consistent manner, while being fast, and easy to use for those without a programming background. Such a piece of software is useful for a far broader set of applications than solely the study of supersymmetry. Future works extending this thesis could aim to incorporate additional useful functions into the software, making it even more powerful as a tool to tackle difficult algebra.

A

Notation and Convention

A.1 2D Notation and Conventions

A.1.1 Lightcone Conventions

We define the lightcone coordinates

$$x^{\pm\pm} = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right), \tag{A.1}$$

with associated derivatives

$$\partial_{\pm\pm} = \frac{1}{\sqrt{2}}\left(\partial_0 \pm \partial_1\right) \tag{A.2}$$

obeying

$$\partial_{\pm\pm} x^{\pm\pm} = 1 \text{ and } \partial_{\pm\pm} x^{\mp\mp} = 0. \tag{A.3}$$

Spinors carry a single lightcone index, which is raised and lowered by

$$\psi^+ = -\psi_-, \quad \psi^- = \psi_+. \tag{A.4}$$

The two-dimensional metric is $\eta_{ab} = \mathrm{diag}(-1, +1)$. The Levi-Civita tensor satisfies $\epsilon^{01} = 1$.

A.1.2 $\mathcal{N} = (0, 2)$ Conventions

In flat $2D$ $\mathcal{N} = (0, 2)$, we parametrise superspace by

$$x^M = (x^{\pm\pm}, \theta^+, \overline{\theta}^+), \tag{A.5}$$

where θ^+ is a complex Grassmann coordinate and $\bar{\theta}^+$ is its complex conjugate. The spinor covariant derivatives and supercharges are given by

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - \frac{\mathrm{i}}{2}\bar{\theta}^+\partial_{++}, \quad \overline{\mathcal{D}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + \frac{\mathrm{i}}{2}\theta^+\partial_{++}, \tag{A.6}$$

$$\mathcal{Q}_+ = \frac{\partial}{\partial\theta^+} + \frac{\mathrm{i}}{2}\bar{\theta}^+\partial_{++}, \quad \overline{\mathcal{Q}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} - \frac{\mathrm{i}}{2}\theta^+\partial_{++}, \tag{A.7}$$

and obey the anti-commutation relations

$$\left\{\mathcal{D}_+, \overline{\mathcal{D}}_+\right\} = \mathrm{i}\partial_{++}, \quad \left\{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\right\} = -\mathrm{i}\partial_{++}. \tag{A.8}$$

with all the other (anti-)commutators between the $\mathcal{D}$s, $\mathcal{Q}$s, and $\partial_{\pm\pm}$ being identically zero. Given an $\mathcal{N} = (0,2)$ superfield [3] $\mathcal{F}(\zeta) = \mathcal{F}(\sigma,\theta)$ its supersymmetry transformations are given by

$$\delta_Q\mathcal{F} := \mathrm{i}\epsilon^+\mathcal{Q}_+\mathcal{F}(\sigma,\theta) - \mathrm{i}\epsilon^+\overline{\mathcal{Q}}_+\mathcal{F}(\sigma,\theta). \tag{A.9}$$

A.1.3 $\mathcal{N} = (2,2)$ Conventions

In flat $2D$ $\mathcal{N} = (0,2)$, we parametrise superspace by

$$x^M = (x^{\pm\pm}, \theta^\pm, \overline{\theta}^\pm), \tag{A.10}$$

where $\theta^\pm$ is a complex Grassmann coordinate and $\bar{\theta}^\pm$ is its complex conjugate. The spinor covariant derivatives are given by $D_A = \left(\partial_a, D_\pm, \bar{D}_\pm\right)$, are defined by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - \frac{i}{2}\bar{\theta}^\pm\partial_{\pm\pm}, \quad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + \frac{i}{2}\theta^\pm\partial_{\pm\pm}, \tag{A.11}$$

and satisfy

$$\left\{D_\pm, \bar{D}_\pm\right\} = i\partial_{\pm\pm}, \tag{A.12}$$

with all other (anti-)commutators vanishing. We define for convenience

$$\partial_\pm \equiv \frac{\partial}{\partial\theta^\pm}, \quad \overline{\partial}_\pm \equiv \frac{\partial}{\partial\bar{\theta}^\pm}. \tag{A.13}$$

Additionally, the supercharges are given by

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + \frac{i}{2}\bar{\theta}^\pm\partial_{\pm\pm}, \quad \overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - \frac{i}{2}\theta^\pm\partial_{\pm\pm}, \tag{A.14}$$

satisfying

$$\left\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\right\} = -i\partial_{\pm\pm}, \tag{A.15}$$

and commuting with the covariant derivatives D_A. These generate supersymmetry transformations for an $\mathcal{N} = (2,2)$ superfield $\mathcal{F}(\zeta) = \mathcal{F}\left(x^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm\right)$ given by

$$\delta_Q\mathcal{F} := i\epsilon^+\mathcal{Q}_+\mathcal{F} + i\epsilon^-\mathcal{Q}_-\mathcal{F} - i\bar{\epsilon}^+\overline{\mathcal{Q}}_+\mathcal{F} - i\bar{\epsilon}^-\overline{\mathcal{Q}}_-\mathcal{F}. \tag{A.16}$$

A.1.4 $\mathcal{N} = (0,4)$ Conventions

In flat 2D $\mathcal{N} = (0,4)$ supersymmetry, superspace can be parametrised by 4 real Grassmann coordinates,

$$x^M = (x^{\pm\pm}, \theta^+_{ia}), \tag{A.17}$$

where $i = \{1,2\}$, $a = \{1,2\}$ are $SU(2)$ indices.

These have associated covariant derivatives

$$D^{ia}_+ = \frac{\partial}{\partial\theta^+_{ia}} + \frac{i}{2}\theta^{+ia}\partial_{++}, \tag{A.18}$$

satisfying

$$\{D^{ia}_+, D^{jb}_+\} = i\epsilon^{ij}\epsilon^{ab}\partial_{++}\ , \quad [\partial_{\pm\pm}, D^{ia}_+] = 0. \tag{A.19}$$

A.1.5 $\mathcal{N} = (4,4)$ Conventions

In flat $2D$ $\mathcal{N} = (4,4)$, we parameterise superspace by 8 Grassmann coordinates

$$x^M = (x^{\pm\pm}, \theta^+_I, \theta^-_A), \tag{A.20}$$

where θ^+_I and θ^-_A are real Grassmann coordinates and $I = \{1,2,3,4\}$, $A = \{1,2,3,4\}$ are $SO(4)$ indices.

These have associated covariant derivatives

$$D^I_+ = \frac{\partial}{\partial\theta^+_I} + \frac{i}{2}\theta^{+I}\partial_{++}, \tag{A.21}$$

and

$$D^A_- = \frac{\partial}{\partial\theta^-_A} + \frac{i}{2}\theta^{-A}\partial_{--}. \tag{A.22}$$

One can see that these obey

$$\{D^I_+, D^J_+\} = i\delta^{IJ}\partial_{++}, \tag{A.23}$$

and

$$\{D^A_-, D^B_-\} = i\delta^{AB}\partial_{--}, \tag{A.24}$$

while they commute with $\partial_{\pm\pm}$.

A.2 4D Notation and Conventions

4D $\mathcal{N} = 2$ Minkowski superspace is defined in terms of the superspace coordinates

$$z^M = \left(x^a, \theta^\alpha_i, \bar\theta^i_{\dot\alpha}\right), \quad a = 0,1,2,3, \quad \alpha = +,-, \quad \dot\alpha = \dot+,-, \quad i = 1,2, \tag{A.25}$$

with Minkowski metric

$$\eta_{ab} = \mathrm{diag}[-1,1,1,1]. \tag{A.26}$$

We denote spacetime indices with Latin letters $a, ..., h$, and $SU(2)$ indices with Latin letters $i, ...$ onwards, including uppercase. We denote spinor indices with Greek letters, with dotted indices for the anti-fundamental representation, and undotted for the fundamental representation.

These satisfy the following reality conditions

$$(x^m)^* = x^m, \quad (\theta_i^\alpha)^* = \bar{\theta}^{\dot{\alpha}i}, \quad \left(\bar{\theta}^i_{\dot{\alpha}}\right)^* = \theta_{\alpha i}, \quad \left(\theta^{\alpha i}\right)^* = -\bar{\theta}^{\dot{\alpha}}_i, \quad \left(\bar{\theta}_{\dot{\alpha}i}\right)^* = -\theta^i_\alpha \tag{A.27}$$

and

$$\begin{gathered}(\varepsilon_{\alpha\beta})^* = \varepsilon_{\dot{\alpha}\dot{\beta}}, \quad \left(\varepsilon^{\alpha\beta}\right)^* = \varepsilon^{\dot{\alpha}\dot{\beta}}, \quad \left(\varepsilon_{\dot{\alpha}\dot{\beta}}\right)^* = \varepsilon_{\alpha\beta}, \quad \left(\varepsilon^{\dot{\alpha}\dot{\beta}}\right)^* = \varepsilon^{\alpha\beta}, \quad \left(\delta^\beta_\alpha\right)^* = \delta^{\dot{\beta}}_{\dot{\alpha}}, \quad \left(\delta^{\dot{\beta}}_{\dot{\alpha}}\right)^* = \delta^\beta_\alpha \\ \left(\varepsilon^{ij}\right)^* = -\varepsilon_{ij}, \quad (\varepsilon_{ij})^* = -\varepsilon^{ij}, \quad \left(\delta^j_i\right)^* = \delta^i_j,\end{gathered} \tag{A.28}$$

where

$$\varepsilon^{12} = \varepsilon_{21} = 1. \tag{A.29}$$

Spinor and SU(2) indices are raised and lowered with

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \psi^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}, \quad \psi_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}}, \quad \psi^i = \varepsilon^{ij}\psi_j, \quad \psi_i = \varepsilon_{ij}\psi^j \tag{A.30}$$

where it holds that

$$\begin{aligned}\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha_\beta, \quad \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon_{\dot{\gamma}\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}}, \quad \varepsilon^{ik}\varepsilon_{kj} = \delta^i_j \\ \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta^\beta_\alpha, \quad \varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}, \quad \varepsilon_{ik}\varepsilon^{kj} = \delta^j_i.\end{aligned} \tag{A.31}$$

The superspace covariant derivatives are given by

$$D_M = \left(\partial_m, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i\right) \tag{A.32}$$

where

$$\partial_m = \frac{\partial}{\partial x^m}, \quad D^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i} - \mathrm{i}\left(\sigma^b\right)_\alpha^{\ \dot{\beta}} \bar{\theta}^i_{\dot{\beta}}\partial_b, \quad \bar{D}^{\dot{\alpha}}_i = \frac{\partial}{\partial \bar{\theta}^i_{\dot{\alpha}}} - \mathrm{i}\left(\sigma^b\right)_\beta^{\ \dot{\alpha}}\theta^\beta_i\partial_b \tag{A.33}$$

that satisfy the algebra

$$\begin{gathered}\left[\partial_a, D^i_\alpha\right] = \left[\partial_a, \bar{D}^{\dot{\alpha}}_i\right] = 0, \quad \left\{D^i_\alpha, D^j_\beta\right\} = \left\{\bar{D}^{\dot{\alpha}}_i, \bar{D}^{\dot{\beta}}_j\right\} = 0 \\ \left\{D^i_\alpha, \bar{D}^{\dot{\beta}}_j\right\} = -2\mathrm{i}\delta^i_j\left(\sigma^c\right)^{\ \dot{\beta}}_\alpha\partial_c = -2\mathrm{i}\delta^i_j\partial^{\dot{\beta}}_\alpha, \\ \left\{D^i_\alpha, \bar{D}^j_{\dot{\beta}}\right\} = 2\mathrm{i}\varepsilon^{ij}\left(\sigma^c\right)_{\alpha\dot{\beta}}\partial_c = 2\mathrm{i}\varepsilon^{ij}\partial_{\alpha\dot{\beta}}, \quad \left\{D_{\alpha i}, \bar{D}_{\dot{\beta}j}\right\} = -2\mathrm{i}\varepsilon_{ij}\left(\sigma^c\right)_{\alpha\dot{\beta}}\partial_c = -2\mathrm{i}\varepsilon_{ij}\partial_{\alpha\dot{\beta}}.\end{gathered} \tag{A.34}$$

These also commute with the supercharges that, on a superfields $U(z) = U(x, \theta, \bar{\theta})$, act as a differential operator

$$\begin{gathered}Q^i_\alpha = \mathrm{i}\frac{\partial}{\partial \theta^\alpha_i} - \left(\sigma^b\right)_\alpha^{\ \dot{\beta}}\bar{\theta}^i_{\dot{\beta}}\partial_b, \quad \bar{Q}^{\dot{\alpha}}_i = \mathrm{i}\frac{\partial}{\partial \bar{\theta}^i_{\dot{\alpha}}} - \left(\sigma^b\right)_\beta^{\ \dot{\alpha}}\theta^\beta_i\partial_b \\ \left[D_A, Q_B\right] = 0, \quad \delta U := U'(z) - U(z) = \mathrm{i}\left(\epsilon^\alpha_i Q^i_\alpha + \bar{\epsilon}^i_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}_i\right)U.\end{gathered} \tag{A.35}$$

Note that

$$\begin{aligned}\bar{D}_i^{\dot\alpha}\bar{D}_j^{\dot\beta} &= \frac{1}{2}\left[\bar{D}_i^{\dot\alpha},\bar{D}_j^{\dot\beta}\right] + \frac{1}{2}\left\{\bar{D}_i^{\dot\alpha},\bar{D}_j^{\dot\beta}\right\} \\ \implies \bar{D}_i^{\dot\alpha}\bar{D}_j^{\dot\beta} &= \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar{D}_{ij} - \frac{1}{2}\varepsilon_{ij}\bar{D}^{\dot\alpha\dot\beta}\end{aligned} \tag{A.36}$$

where we have made the definitions

$$\bar{D}^{\dot\alpha\dot\beta} := \bar{D}_k^{(\dot\alpha}\bar{D}^{\dot\beta)k}, \quad \bar{D}_{ij} := \bar{D}_{\dot\gamma(i}\bar{D}_{j)}^{\dot\gamma}. \tag{A.37}$$

Analogously,

$$\bar\theta_i^{\dot\alpha}\bar\theta_j^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta_{ij} - \frac{1}{2}\varepsilon_{ij}\bar\theta^{\dot\alpha\dot\beta} \tag{A.38}$$

where we have defined

$$\bar\theta^{\dot\alpha\dot\beta} := \bar\theta_k^{(\dot\alpha}\bar\theta^{\dot\beta)k}, \quad \bar\theta_{ij} := \bar\theta_{\dot\gamma(i}\bar\theta_{j)}^{\dot\gamma}. \tag{A.39}$$

Thus there are two independent θ^2 structures. Importantly,

$$\bar{D}_i^{\dot\alpha}\bar{D}_{jk} = \frac{2}{3}\varepsilon_{i(j}\bar{D}^{\dot\alpha q}\bar{D}_{k)q}, \quad \bar{D}_i^{\dot\alpha}\bar{D}^{\dot\beta\dot\gamma} = \frac{2}{3}\varepsilon^{\dot\alpha(\dot\beta}\bar{D}^{\dot\gamma)k}\bar{D}_{ik}, \tag{A.40}$$

and analogously,

$$\bar\theta_i^{\dot\alpha}\bar\theta_{jk} = \frac{2}{3}\varepsilon_{i(j}\bar\theta^{\dot\alpha q}\bar\theta_{k)q} \quad \bar\theta_i^{\dot\alpha}\bar\theta^{\dot\beta\dot\gamma} = \frac{2}{3}\varepsilon^{\dot\alpha(\dot\beta}\bar\theta^{\dot\gamma)k}\bar\theta_{ik}. \tag{A.41}$$

Note however, that due to the anticommuting nature of the Grassmann variables, these products both reduce to the single θ^3 independent structure

$$\bar\theta^{i\dot\alpha}\bar\theta_{ij} = \bar\theta^{\dot\alpha} = \bar\theta_{j\dot\beta}\bar\theta^{\dot\alpha\dot\beta}. \tag{A.42}$$

Also,

$$\bar{D}^4 := \bar{D}^{ij}\bar{D}_{ij}, \quad \bar{D}^{\dot\alpha\dot\beta}\bar{D}_{\dot\alpha\dot\beta} = -\bar{D}^4, \tag{A.43}$$

are the 4-derivative structures and analogously,

$$\begin{aligned}\bar\theta^4 &:= \bar\theta^{ij}\bar\theta_{ij} \\ \bar\theta^{\dot\alpha\dot\beta}\bar\theta_{\dot\alpha\dot\beta} &= -\bar\theta^4.\end{aligned} \tag{A.44}$$

Analogous equations hold for the complex conjugates, with the definitions

$$\theta^{\alpha\beta} := \theta^{(\alpha k}\theta_k^{\beta)}, \quad \theta_{ij} := \theta_{(i}^{\gamma}\theta_{\gamma j)}. \tag{A.45}$$

Given V_a and $F_{ab} = -F_{ba}$ we use the bispinor convention, given by

$$\begin{aligned} V_{\alpha\dot\beta} = (\sigma^a)_{\alpha\dot\beta}V_a, &\quad V_a = -\frac{1}{2}(\tilde\sigma_a)^{\dot\beta\alpha}V_{\alpha\dot\beta} \\ F_{\alpha\beta} = \frac{1}{2}\left(\sigma^{ab}\right)_{\alpha\beta}F_{ab}, &\quad F_{\dot\alpha\dot\beta} = -\frac{1}{2}\left(\tilde\sigma^{ab}\right)_{\dot\alpha\dot\beta}F_{ab} \\ F_{ab} = (\sigma_{ab})^{\alpha\beta}F_{\alpha\beta} &- (\tilde\sigma_{ab})^{\dot\beta\dot\beta}F_{\dot\alpha\dot\beta}, \end{aligned} \tag{A.46}$$

where

$$\begin{aligned}(\sigma_{ab})_\alpha^{\ \beta} &= -\frac{1}{4}\left((\sigma_a)_{\alpha\dot\beta}(\tilde\sigma_b)^{\dot\beta\beta} - (\sigma_b)_{\alpha\dot\beta}(\tilde\sigma_a)^{\dot\beta\beta}\right) \\ (\tilde\sigma_{ab})^{\dot\alpha}_{\ \dot\beta} &= -\frac{1}{4}\left((\tilde\sigma_a)^{\dot\alpha\beta}(\sigma_b)_{\beta\dot\beta} - (\tilde\sigma_b)^{\dot\alpha\beta}(\sigma_a)_{\beta\dot\beta}\right).\end{aligned} \tag{A.47}$$

B
Example Code Usage

B.1 4D $\mathcal{N} = 2$ Supercurrent Code

```
include("alakazam.jl")
include("superspace.jl")

######################################
######################################
## setup and definitions
######################################

#define index sets
flavour_indices = IndexSet("flavour", 2,lower)
spinor_indices = IndexSet("spinor", 2)
spacetime_indices = IndexSet("spacetime", 4,lower)
spinor_conj_indices = IndexSet("spinor_conj", 2,lower)

#define indices
i = Index("i", flavour_indices, 0)
j = Index("j", flavour_indices)
k = Index("k", flavour_indices, 0)
l = Index("l", flavour_indices)
m = Index("m", flavour_indices)
n = Index("n", flavour_indices)
p = Index("p", flavour_indices)
q = Index("q", flavour_indices)
```

```
r = Index("r", flavour_indices)
s = Index("s", flavour_indices)
t = Index("t", flavour_indices)
u = Index("u", flavour_indices)
v = Index("v", flavour_indices)
y = Index("y", flavour_indices)
z = Index("z", flavour_indices)
A = Index("A", flavour_indices)
B = Index("B", flavour_indices)
C = Index("C", flavour_indices)
D = Index("D", flavour_indices)
E = Index("E", flavour_indices)
F = Index("F", flavour_indices)

α = Index("α", spinor_indices, 1)
β = Index("β", spinor_indices, 1)
γ = Index("γ", spinor_indices, 1)
μ = Index("μ", spinor_indices, 1)
ν = Index("ν", spinor_indices, 1)
ρ = Index("ρ", spinor_indices, 1)
τ = Index("τ", spinor_indices, 1)
σ = Index("σ", spinor_indices, 1)
ω = Index("ω", spinor_indices, 1)
χ = Index("χ", spinor_indices, 1)
π = Index("π", spinor_indices, 1)
η = Index("η", spinor_indices, 1)

αdot = Index("α̇", spinor_conj_indices, 1)
βdot = Index("β̇", spinor_conj_indices, 1)
γdot = Index("γ̇", spinor_conj_indices, 1)
ρdot = Index("ρ̇", spinor_conj_indices, 1)
μdot = Index("μ̇", spinor_conj_indices, 1)
νdot = Index("ν̇", spinor_conj_indices, 1)
τdot = Index("τ̇", spinor_conj_indices, 1)
σdot = Index("σ̇", spinor_conj_indices, 1)
ωdot = Index("ω̇", spinor_conj_indices, 1)
πdot = Index("π̇", spinor_conj_indices, 1)
ηdot = Index("η̇", spinor_conj_indices, 1)

a = Index("a", spacetime_indices, 0)
b = Index("b", spacetime_indices)
c = Index("c", spacetime_indices)
d = Index("d", spacetime_indices)
e = Index("e", spacetime_indices)
```

```
f = Index("f", spacetime_indices)
g = Index("g", spacetime_indices)

#define coordinates, tensors will be functions of these
#coordinates (for use in derivatives)
x = Coordinate("x", spacetime_indices)
θ = Coordinate("θ", [flavour_indices, spinor_indices])
θbar = Coordinate("Θ", [flavour_indices, spinor_conj_indices])

# define our grassman basis ¯θ
θiα=Tensor("θ",[i=>lower, α=>upper],θ)
barθαi=Tensor("Θ",[i=>upper,αdot=>lower],θbar)
barθij=Tensor("Θ",[i=>lower, νdot=>upper],θbar)*Tensor("Θ",[j=>lower,
νdot=>lower],θbar)
barθupij=Tensor("Θ",[i=>upper, μdot=>upper],θbar)*Tensor("Θ",
[j=>upper, μdot=>lower],θbar)
θij=Tensor("θ",[i=>lower, ν=>upper],θ)*Tensor("θ",[j=>lower,
ν=>lower],θ)
θupij=Tensor("θ",[i=>upper, μ=>upper],θ)*Tensor("θ",[j=>upper,
μ=>lower],θ)
θjα=Tensor("θ",[j=>lower, α=>upper],θ)
θ4=Tensor("θ",[m=>lower, ν=>upper],θ)*Tensor("θ",[n=>lower,
ν=>lower],θ)*Tensor("θ",[m=>upper, μ=>upper],θ)*Tensor("θ",[n=>upper,
μ=>lower],θ)
θbar4=Tensor("Θ",[p=>lower, νdot=>upper],θbar)*Tensor("Θ",[q=>lower,
νdot=>lower],θbar)*Tensor("Θ",[p=>upper, μdot=>upper],θbar)*Tensor("Θ",
[q=>upper, μdot=>lower],θbar)
θᵅᵝ=Tensor("θ",[y=>upper, α=>upper],θ)*Tensor("θ",[y=>lower,
β=>upper],θ)
θᵦᵧ =Tensor("θ",[y=>upper, β=>lower],θ)*Tensor("θ",[y=>lower,
γ=>lower],θ)
barθᵅᵝ=Tensor("Θ",[z=>upper, αdot=>upper],θbar)*Tensor("Θ",
[z=>lower, βdot=>upper],θbar)
barθᵦᵧ =Tensor("Θ",[z=>upper, βdot=>lower],θbar)*Tensor("Θ",
[z=>lower, γdot=>lower],θbar)
θiβ=Tensor("θ",[i=>lower, β=>upper],θ)
barθjα=Tensor("Θ",[j=>upper,αdot=>lower],θbar)
barθjk=Tensor("Θ",[j=>lower, νdot=>upper],θbar)*Tensor("Θ",[k=>lower,
νdot=>lower],θbar)
θjk=Tensor("θ",[j=>lower, ν=>upper],θ)*Tensor("θ",[k=>lower, ν=>lower],θ)
θʲᵏ=Tensor("θ",[j=>upper, ν=>upper],θ)*Tensor("θ",[k=>upper, ν=>lower],θ)
```

```
# defining action of the derivatives on grassmann variables
grassman_deriv_normalrep=Derivative("∂",zero, [θ],
[j => upper, β => lower])
grassman_deriv_conjrep=Derivative("��",zero, [θbar],
[j => lower, βdot => upper])
spatial_deriv=Derivative("∇",zero, [x], [b => lower])
#note that the derivative isn't defined on literally "θiα", but on tensors
#with the same index patterns
add_derivative_action(grassman_deriv_normalrep,θiα,KroneckerDelta([i=>lower,
j=>upper])*KroneckerDelta([β=>lower,α=>upper]))
add_derivative_action(grassman_deriv_normalrep,Tensor("θ",[i=>lower, α=>lower],θ),
KroneckerDelta([i=>lower,j=>upper])*EpsilonTensor([α=>lower,β=>lower]))
add_derivative_action(grassman_deriv_normalrep,Tensor("θ",[i=>upper, α=>upper],θ),
EpsilonTensor([i=>upper,j=>upper])*KroneckerDelta([β=>lower,α=>upper]))
add_derivative_action(grassman_deriv_normalrep,Tensor("θ",[i=>upper, α=>lower],θ),
EpsilonTensor([i=>upper,j=>upper])*EpsilonTensor([α=>lower,β=>lower]))

add_derivative_action(grassman_deriv_conjrep,barθαi,KroneckerDelta([i=>upper,
j=>lower])*KroneckerDelta([βdot=>upper,αdot=>lower]))
add_derivative_action(grassman_deriv_conjrep,Tensor("Θ",[i=>lower,
αdot=>lower],θbar),EpsilonTensor([i=>lower,j=>lower])*KroneckerDelta([βdot=>upper,
αdot=>lower]))
add_derivative_action(grassman_deriv_conjrep,Tensor("Θ",[i=>upper,
αdot=>upper],θbar),KroneckerDelta([i=>upper,j=>lower])*EpsilonTensor([αdot=>upper,
βdot=>upper]))
add_derivative_action(grassman_deriv_conjrep,Tensor("Θ",[i=>lower,
αdot=>upper],θbar),EpsilonTensor([i=>lower,j=>lower])*EpsilonTensor([αdot=>upper,
βdot=>upper]))


#create our superfield expansion

SUPERFIELD=(Tensor("j", x)
+θiα*Tensor("ψ",[i=>upper, α=>lower],x)
+θij*SymmetricTensor("F",[i=>upper, j=>upper],x)
+(θjα*θupij)*Tensor("G",[i=>lower, α=>lower],x)
+θij*θupij*Tensor("H",x)
+θjα*barθαi*Tensor("K",[j=>upper, α=>lower,i => lower, αdot =>upper],x)
+Tensor("θ",[k=>lower, α=>upper],θ)*barθupij*Tensor("L",
[k=>upper, α =>lower, i=>lower,j => lower],x)
+Tensor("θ",[k=>lower, α=>upper],θ)*barθαi*barθij*Tensor("M",
[k=>upper, j=>upper, α =>lower, αdot=> upper],x)
+θiα*θbar4*Tensor("N",[i=>upper, α=>lower],x)
+θij*Tensor("Θ",[m=>upper, νdot=>upper],θbar)*Tensor("Θ",
```

```
[n=>upper, νdot=>lower],θbar)*Tensor("P",[i=>upper,j=>upper,m=>lower,
n=>lower],x)
+θij*Tensor("Θ",[k=>upper, αdot=>lower],θbar)*Tensor("Θ",
[k=>lower, νdot=>upper],θbar)*Tensor("Θ",[l=>lower, νdot=>lower],
θbar)*Tensor("Q",[i=>upper,j=>upper,l=>upper,αdot=>upper],x)
+θij*θbar4*SymmetricTensor("R",[i=>upper, j=>upper],x)
+θupij*Tensor("θ",[j=>lower, α=>upper],θ)*Tensor("Θ",[k=>upper,
αdot=>lower],θbar)*Tensor("Θ",[k=>lower, νdot=>upper],θbar)*Tensor("Θ",
[l=>lower, νdot=>lower],θbar)*Tensor("S",[i=>lower,α=>lower,l=>upper,
αdot=>upper],x)
+Tensor("θ",[j=>lower, α=>upper],θ)*θupij*θbar4*Tensor("E",[i=>lower,
α=>lower],x)
+θ4*θbar4*Tensor("V",x)
+barθαi*Tensor("¯ψ",[i=>lower, αdot=>upper],x)
+barθupij*Tensor("¯F",[i=>lower, j=>lower],x)
+barθαi*barθij*Tensor("¯G",[j=>upper, αdot=>upper],x)
+θbar4*Tensor("¯H",x)
+barθαi*Tensor("θ",[j=>lower, ν=>upper],θ)*Tensor("θ",[k=>lower,
ν=>lower],θ)*Tensor("¯L",[j=>upper,k=>upper, i=>lower, αdot=>upper],x)
+barθαi*Tensor("θ",[k=>lower, β=>upper],θ)*Tensor("θ",[j=>upper,
ν=>upper],θ)*Tensor("θ",[k=>upper, ν=>lower],θ)*Tensor("¯M",[j=>lower,
i=>lower,β=>lower, αdot=>upper],x)
+barθαi*θ4*Tensor("¯N",[i=>lower, αdot=>upper],x)
+barθupij*Tensor("θ",[k=>upper, μ=>upper],θ)*Tensor("θ",[l=>upper,
μ=>lower],θ)*Tensor("θ",[l=>lower, α=>upper],θ)*Tensor("¯Q",[i=>lower,
j=>lower,α =>lower, k=>lower ],x)
+barθupij*θ4*SymmetricTensor("¯R",[i=>lower, j=>lower],x)
+Tensor("Θ",[j=>upper, αdot=>lower],θbar)*barθij*θ4*Tensor("¯E",[i=>upper,
αdot=>upper],x)
+θᵅᵝ*SymmetricTensor("Ω",[α=>lower,β=>lower],x)
+barθᵦᵧ*SymmetricTensor("¯Ω",[βdot=>upper,γdot=>upper],x)
+θᵅᵝ*barθαi*Tensor("A",[α=>lower, β=>lower,i=>lower,αdot=>upper],x)
+θᵅᵝ*barθupij*Tensor("B",[α=>lower, β=>lower,i=>lower,j=>lower],x)
+θᵅᵝ*barθαi*barθij*Tensor("U",[α=>lower, β=>lower,j=>upper,αdot=>upper],x)
+θᵅᵝ*θbar4*SymmetricTensor("C",[α=>lower, β=>lower],x)
+θᵅᵝ*barθᵦᵧ*SymmetricTensor("T",[α=>lower, β=>lower,βdot=>upper, γdot=>upper],x)
+barθᵦᵧ*θiα*Tensor("¯A",[i=>upper, α =>lower,βdot=>upper,γdot=>upper],x)
+barθᵦᵧ*θij*Tensor("¯B",[i=>upper, j =>upper,βdot=>upper,γdot=>upper],x)
+barθᵦᵧ*θjα*θupij*Tensor("¯U",[i=>lower, α =>lower,βdot=>upper,γdot=>upper],x)
+barθᵦᵧ*θ4*SymmetricTensor("¯C",[βdot=>upper,γdot=>upper],x)
)

custom_sort_order(["θ","Θ","ϵ"])
```

```



#######################################
#######################################
## computing the constraint equations
#######################################

covar_expansion=grassman_deriv_normalrep-im*Tensor("σ",
[b =>upper, γdot=>upper, β =>lower])*Tensor("Θ",[j=>upper,
γdot=>lower],θbar)*spatial_deriv
covar=Derivative("D",zero, [x,θ], [j => upper, β => lower])
covar_expansion_bar=grassman_deriv_conjrep-im*Tensor("σ",[b =>upper,
βdot=>upper, β =>lower])*Tensor("θ",[j=>lower,β=>upper],θ)*spatial_deriv
covar_bar=Derivative("¯D",zero, [x,θbar], [j => lower, βdot => upper])
println("our superfield expansions is: ", SUPERFIELD)

#apply our derivative
 println("Computing Symmetrised deriv: ")

D_SUPERFIELD=Derivative("D",SUPERFIELD, [x,θbar], [s => upper, τ => lower])
out=expand_derivative(D_SUPERFIELD,covar,covar_expansion)
out=apply_derivative(out)
out=simplify(out)
DD_SUPERFIELD=Derivative("D",out,[x,θbar], [r => upper, ρ => lower])
out=expand_derivative(DD_SUPERFIELD,covar,covar_expansion)
out=apply_derivative(out)
out=EpsilonTensor([τ=>upper,ρ=>upper])*out
out=eliminate_epsilon(out)


D_SUPERFIELD2=Derivative("D",SUPERFIELD, [x,θbar], [r => upper, τ => lower])
out2=expand_derivative(D_SUPERFIELD2,covar,covar_expansion)
out2=apply_derivative(out2)
out2=simplify(out2)
DD_SUPERFIELD2=Derivative("D",out2,[x,θbar], [s => upper, ρ => lower])
out2=expand_derivative(DD_SUPERFIELD2,covar,covar_expansion)
out2=apply_derivative(out2)
out2=EpsilonTensor([τ=>upper,ρ=>upper])*out2
out2=eliminate_epsilon(out2)


println("Result:")
res2=simplify(out+out2)/2
```

```
println(res2)


#example extraction of a constraint

#theta=2, thetabar=0 term in expansion:
tt=2
tb=0
constraint=rename_dummies(get_weighted_terms(get_weighted_terms(res2,
"θ",tt),"Θ",tb))
#get the constraint on the 2nd independant structure associated with θ^2
constraint=project_component(constraint,θ,[i=>upper, α=>lower],tt,tb,θbar,
[i=>lower, αdot=>upper],2,1)
constraint=rename_dummies(contract_metrics(simplify(
eliminate_epsilon(eliminate_kronecker(constraint)))))


######################################
######################################
## compute the supercurrent squared
######################################


SC²=SUPERFIELD × SUPERFIELD
out=get_weighted_terms(get_weighted_terms(SC²,"θ",4),"Θ",4)
#integrate over superspace to obtain the supercurrent-squared term
ttb=project_component(out,θ,[i=>upper, α=>lower],4,4,θbar,[i=>lower,
αdot=>upper],1,1)
println(ttb)
```

References

[1] F. A. Smirnov and A. B. Zamolodchikov. *On the space of integrable quantum field theories*. arXiv preprint arXiv:1608.05499v1 (2016).

[2] T. T. Dumitrescu and N. Seiberg. *Supercurrents and brane currents in diverse dimensions*. Journal of High Energy Physics **2011**(7) (2011). URL

[3] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli, and H. Walsh. *On ttbar deformations and supersymmetry*. Journal of High Energy Physics **2019**(6) (2019). URL

[4] H. Jiang, A. Sfondrini, and G. Tartaglino-Mazzucchelli. *Ttbar deformations with n=(0,2) supersymmetry*. Physical Review D **100**(4) (2019). URL

[5] C.-K. Chang, C. Ferko, S. Sethi, A. Sfondrini, and G. Tartaglino-Mazzucchelli. *Ttbar flows and (2, 2) supersymmetry*. Physical Review D **101**(2) (2020). URL

[6] C. Ferko. *Supersymmetry and irrelevant deformations*. arXiv preprint arXiv:2112.14647v2 (2021).

[7] C. Ferko, L. Smith, and G. Tartaglino-Mazzucchelli. *Stress Tensor Flows, Birefringence in Non-Linear Electrodynamics, and Supersymmetry* (2023). `2301.10411`.

[8] *Lightcone in 2d minkowski spacetime*. `https://i.pinimg.com/originals/03/3c/19/033c19f3c7e7d95430b54b8b86e3d139.png` (2023). Accessed: February 17, 2023.

[9] *Einstein*. `https://filebox.ece.vt.edu/ jbhuang/teaching/ ece5554-4554/fa16/images/hybrid_image.jpg` (2016). Accessed: February 17, 2023.

[10] T. Lancaster and S. Blundell. *Quantum Field Theory for the Gifted Amateur* (Oxford University Press, 2014).

[11] A. Shomer. *A pedagogical explanation for the non-renormalizability of gravity*. arXiv preprint arXiv:0709.3555 (2008).

[12] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory* (CRC Press, 2019).

[13] A. Schellekens. *Introduction to Conformal Field Theory* (1995).

[14] A. B. Zamolodchikov. *Expectation value of composite field ttbar in two-dimensional quantum field theory*. arXiv preprint hep-th/0401146 (2004).

[15] Y. Jiang. *A pedagogical review on solvable irrelevant deformations of 2d quantum field theory.* arXiv preprint arXiv:1904.13376 (2021).

[16] C. Ferko, H. Jiang, S. Sethi, and G. Tartaglino-Mazzucchelli. *Non-linear supersymmetry and ttbar-like flows* (2019). 1910.01599.

[17] G. Bonelli, N. Doroud, and M. Zhu. *TTbar-deformations in closed form* (2018). 1804.10967.

[18] S. Coleman and J. Mandula. *All possible symmetries of the s matrix.* Phys. Rev. **159**, 1251 (1967). https://doi.org/10.1103/PhysRev.159.1251.

[19] R. Haag, J. T. Łopuszański, and M. Sohnius. *All possible generators of supersymmetries of the s-matrix.* Nuclear Physics B **88**(2), 257 (1975).

[20] S. J. J. Gates. *Superspace, or One thousand and one lessons in supersymmetry* (Westview Press, 1999).

[21] K. Peeters. *Cadabra2: computer algebra for field theory revisited.* Journal of Open Source Software **3**(32), 1118 (2018). URL https://doi.org/10.21105/joss.01118.

[22] P. Kuusela. *"gammamap" - a mathematica package for clifford algebras, gamma matrices and spinors* (2019). 1905.00429.

[23] J. Bezanson, S. Karpinski, V. B. Shah, and A. Edelman. *Julia: A fast dynamic language for technical computing* (2012). 1209.5145.

[24] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah. *Julia: A fresh approach to numerical computing* (2015). 1411.1607.

[25] Y. Korovin, S. M. Kuzenko, and S. Theisen. *The conformal supercurrents in diverse dimensions and conserved superconformal currents* (2016). 1604.00488.

[26] O. Aharony. *A brief review of 'little string theories'.* Classical and Quantum Gravity **17**(5), 929 (2000). URL https://dx.doi.org/10.1088/0264-9381/17/5/302.

[27] S. Chakraborty, A. Giveon, and D. Kutasov. *Comments on single-trace $t\bar{t}$ holography* (2023). 2303.12422.