On The Differential Geometry of Torsion-Free Complex Spaces

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1 Notation

A quick note on notational conventions used in this paper. First, I will use pipes | and || for derivatives and covariant derivatives respectively. That is how I first learned differential geometry and it has stuck. Plus it allows me to write faster. Einstein summation notation is implied, and complex conjugate indices are displayed with a bar over them.

2 Introduction

As a physics and mathematics undergraduate, I started studying Quantum Fields early on. Through this, I encountered the Dirac equation and the first gauge theory: Quantum Electrodynamics. At the same time, I was teaching myself General Relativity, learning about Differential Geometry and Tensors. The ideas I read seemed absurd initially. The proposal was straightforward: if we require Dirac's equation to remain invariant under a phase shift of the electron wave function, we must adjust the derivative operator to compensate. This adjustment behaves like Electromagnetic forces, as the Electromagnetic field shifts to cancel the phase shift in the electron.

While the math worked, the idea didn't make sense to me. In General Relativity, we create a new derivative (covariant derivative) that offsets changes brought by coordinate changes. This derivative contains an affine connection term that shifts to cancel the coordinate change effects in tensors. The difference is that all elements in these equations exist in the same spacetime, so any coordinate change affects everything.

However, in Electrodynamics, there is no obvious connection between the electron and the electromagnetic field. Changing one shouldn't affect the other. The electromagnetic field could come from somewhere far from the electron and interacts with different particles at different points. Is it adjusting for each electron differently? A phase change is local for each particle, not global like a coordinate change.

It soon became clear to me that this issue could be resolved by creating a new background on which particle spinor fields sit as simple vectors. We could then give this new background a curvature, like in General Relativity, making the phase change another form of coordinate change.

To start, we needed to extend Riemannian geometry to complex-dimensional spaces. "Spinors as vectors" meant creating tensors in this complex space. However, in 1994, at 17 years old, I had no internet access. The papers I found over the years were written by people uninterested in physics applications, often overly abstract and reliant on narrow constructions. General Relativity provided tensor equations that could be written as differential equations, the tools of a physicist.

I spent many years developing my own mathematics and notations to explore these spaces as a physicist. The mathematical constructions form the core of this paper. The first part covers the fully extensible construction, allowing for conformal invariance (Weyl) and non-analytic spaces. Then, we narrow our focus to analytic cases and create a map between complex dimensional space and Riemannian spaces using "projectors," directly related to Dirac matrices.

Finally, we propose a Lagrangian and explore some possible geometries.

3 Tensors and Conjugation

The biggest challenge in constructing the mathematics is handling the complex conjugate. Each dimension has a conjugate pair, which behaves as its own dimension. This means a vector is not just a series of elements with one per dimension, but rather two sets: one for the regular dimensions and one for the conjugate dimensions. These elements do not need to be directly related as conjugates of each other.

To see this in action, let's write a vector as a sum of its elements multiplied by unit vectors for each dimension. In this case, we would have:

$$\mathbf{V} = \hat{\mathbf{e}}_j \psi^j + \hat{\mathbf{e}}_{\bar{k}} \rho^{\bar{k}} \tag{1}$$

In our notation, a bar over the index indicates a conjugate, and Einstein summation is assumed. The elements ψ and ρ are distinct but together form the complete vector. Each part alone cannot fully describe the vector, and, as we will see, the different parts transform into one another.

To understand how distances and metric tensors are calculated, we need to square the above vector.

$$\mathbf{V}^2 = \frac{1}{2}(\hat{\mathbf{e}}_j \hat{\mathbf{e}}_k + \hat{\mathbf{e}}_k \hat{\mathbf{e}}_j)\psi^j \psi^k + \frac{1}{2}(\hat{\mathbf{e}}_{\bar{m}} \hat{\mathbf{e}}_{\bar{n}} + \hat{\mathbf{e}}_{\bar{n}} \hat{\mathbf{e}}_{\bar{m}})\rho^{\bar{m}} \rho^{\bar{n}} + (\hat{\mathbf{e}}_j \hat{\mathbf{e}}_{\bar{n}} + \hat{\mathbf{e}}_{\bar{n}} \hat{\mathbf{e}}_j)\psi^j \rho^{\bar{n}} \tag{2}$$

Using this we can construct our tensor equation and metric tensors.

$$l^2 = s_{jk}\psi^j \psi^k + s_{\bar{m}\bar{n}}\rho^{\bar{m}}\rho^{\bar{n}} + 2g^{j\bar{n}}\psi_j \rho_{\bar{n}} \tag{3}$$

The letter s was chosen to mean "symmetric" while g was chosen as it will play a role very similar to the metric in General Relativity later. From (2) you can easily see the symmetry properties of the metric tensors.

$$\begin{aligned} s_{jk} &= s_{kj} \\ g_{j\bar{k}} &= g_{\bar{k}j} \end{aligned} \tag{4}$$

A simple analysis also shows that taking the complex conjugate of (3) has the effect of swapping ψ and ρ. If ψ is equal to ρ then the length squared is a real value. Vectors of zero length we will call "neutrinos" for reasons that will become obvious later.

Next let's construct the covariant version of our vector, and the relationship between covariant and contra-variant versions of the metrics

$$
\begin{aligned}
\psi_j &\equiv s_{jk}\psi^k + g_{j\bar{k}}\rho^{\bar{k}} \\
\rho_{\bar{k}} &\equiv s_{\bar{j}\bar{k}}\rho^{\bar{j}} + g_{j\bar{k}}\psi^{\bar{j}}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
s_{jk}s^{mk} + g_{j\bar{k}}g^{m\bar{k}} &= \delta^m_j \\
g_{k\bar{j}}s^{mk} + s_{\bar{j}\bar{k}}g^{m\bar{k}} &= 0
\end{aligned}
\tag{6}
$$

The metric element s raises and lowers indices while maintaining the vector portion, while g swaps between them.

4 Covariant Derivatives

First let's consider coordinate transformations and their effect on the ordinary derivatives of a vector.

$$
\begin{aligned}
\psi_{m'} &= \frac{\partial z^a}{\partial z^{m'}}\psi_a + \frac{\partial z^{\bar{b}}}{\partial z^{m'}}\rho_{\bar{b}} \\
\rho_{\bar{n}'} &= \frac{\partial z^{\bar{b}}}{\partial z^{\bar{n}'}}\rho_{\bar{b}} + \frac{\partial z^a}{\partial z^{\bar{n}'}}\psi_a \\
\frac{\partial}{\partial z^{m'}} &= \frac{\partial z^a}{\partial z^{m'}}\frac{\partial}{\partial z^a} + \frac{\partial z^{\bar{b}}}{\partial z^{m'}}\frac{\partial}{\partial z^{\bar{b}}} \\
\frac{\partial}{\partial z^{\bar{n}'}} &= \frac{\partial z^{\bar{b}}}{\partial z^{\bar{n}'}}\frac{\partial}{\partial z^{\bar{b}}} + \frac{\partial z^a}{\partial z^{\bar{n}'}}\frac{\partial}{\partial z^a}
\end{aligned}
\tag{7}
$$

Now we can take the derivative of a vector and see where affine connections are needed. This mimics what we've seen in standard Riemannian geometry but now contains the complexity of conjugate elements. Here we will examine only one scenario.

$$
\begin{aligned}
\psi_{m'|j'} &= \frac{\partial z^s}{\partial z^{j'}}\frac{\partial}{\partial z^s}(\frac{\partial z^a}{\partial z^{m'}}\psi_a + \frac{\partial z^{\bar{b}}}{\partial z^{m'}}\rho_{\bar{b}}) + \frac{\partial z^{\bar{t}}}{\partial z^{j'}}\frac{\partial}{\partial z^{\bar{t}}}(\frac{\partial z^a}{\partial z^{m'}}\psi_a + \frac{\partial z^{\bar{b}}}{\partial z^{m'}}\rho_{\bar{b}}) \\
&= \frac{\partial z^a}{\partial z^{m'}}\frac{\partial z^s}{\partial z^{j'}}\psi_{a|s} + \frac{\partial z^a}{\partial z^{m'}}\frac{\partial z^{\bar{t}}}{\partial z^{j'}}\psi_{a|\bar{t}} + \frac{\partial z^{\bar{b}}}{\partial z^{m'}}\frac{\partial z^s}{\partial z^{j'}}\rho_{\bar{b}|s} + \frac{\partial z^{\bar{b}}}{\partial z^{m'}}\frac{\partial z^{\bar{t}}}{\partial z^{j'}}\rho_{\bar{b}|\bar{t}} \\
&+ \frac{\partial^2 z^a}{\partial z^{j'}\partial z^{m'}}\psi_a + \frac{\partial^2 z^{\bar{b}}}{\partial z^{j'}\partial z^{m'}}\rho_{\bar{b}}
\end{aligned}
\tag{8}
$$

The last two terms prevent the derivative from transforming like a tensor. In the usual way, we can add an affine connection term that transforms in just the right way to offset this and give us back a tensor form of the derivative.

$$
\begin{aligned}
\psi_{m||j} &\equiv \psi_{m|j} - \Gamma^l{}_{mj}\psi_l - \Gamma^{\bar{k}}{}_{mj}\rho_{\bar{k}} \\
\psi_{m||\bar{k}} &\equiv \psi_{m|\bar{k}} - \Gamma^l{}_{m\bar{k}}\psi_l - \Gamma^{\bar{l}}{}_{m\bar{k}}\rho_{\bar{l}} \\
\rho_{\bar{m}||j} &\equiv \rho_{\bar{m}|j} - \Gamma^{\bar{l}}{}_{\bar{m}j}\rho_{\bar{l}} - \Gamma^l{}_{\bar{m}j}\psi_l \\
\rho_{\bar{m}||\bar{k}} &\equiv \rho_{\bar{m}|\bar{k}} - \Gamma^{\bar{l}}{}_{\bar{m}\bar{k}}\rho_{\bar{l}} - \Gamma^l{}_{\bar{m}\bar{k}}\psi_l
\end{aligned}
\tag{9}
$$

Transforming under a general coordinate change as follows

$$
\Gamma^{l'}{}_{m'j'} = \frac{\partial z^{l'}}{\partial z^p}\frac{\partial z^n}{\partial z^{m'}}\frac{\partial z^k}{\partial z^{j'}}\Gamma^p{}_{nk} + \frac{\partial z^{l'}}{\partial z^p}\frac{\partial^2 z^p}{\partial z^{j'}\partial z^{m'}} + \frac{\partial z^{l'}}{\partial z^{\bar{q}}}\frac{\partial z^n}{\partial z^{m'}}\frac{\partial z^k}{\partial z^{j'}}\Gamma^{\bar{q}}{}_{nk} + \frac{\partial z^{l'}}{\partial z^{\bar{q}}}\frac{\partial^2 z^{\bar{q}}}{\partial z^{j'}\partial z^{m'}}
\tag{10}
$$

The complete list of transformations is long and complex, but mimics the format of (10). Before we can construct the covariant derivatives of our metric tensors, we first need to consider the conformal extension of (3) that allows the length to vary over the space (Weyl). This is accomplished by looking at the conformal transformation of the metric tensors.

$$\begin{aligned} s'_{mn} &= \epsilon s_{mn} \\ g'_{m\bar{n}} &= \epsilon g_{m\bar{n}} \end{aligned} \tag{11}$$

$$\begin{aligned} g_{m\bar{n}||j} &= g_{m\bar{n}|j} - \Gamma^{l}{}_{mj} g_{l\bar{n}} - \Gamma^{\bar{k}}{}_{\bar{n}j} g_{m\bar{k}} - \Gamma^{\bar{k}}{}_{mj} s_{\bar{k}\bar{n}} - \Gamma^{l}{}_{\bar{n}j} s_{ml} = 2\phi_j g_{m\bar{n}} \\ g_{m\bar{n}||\bar{k}} &= g_{m\bar{n}|\bar{k}} - \Gamma^{l}{}_{m\bar{k}} g_{l\bar{n}} - \Gamma^{\bar{p}}{}_{\bar{n}\bar{k}} g_{m\bar{p}} - \Gamma^{\bar{p}}{}_{m\bar{k}} s_{\bar{p}\bar{n}} - \Gamma^{l}{}_{\bar{n}\bar{k}} s_{ml} = 2\phi_{\bar{k}} g_{m\bar{n}} \\ s_{mn||j} &= s_{mn|j} - \Gamma^{l}{}_{mj} s_{ln} - \Gamma^{\bar{k}}{}_{nj} g_{m\bar{k}} - \Gamma^{\bar{k}}{}_{mj} g_{n\bar{k}} - \Gamma^{l}{}_{nj} s_{ml} = 2\phi_j s_{mn} \\ s_{mn||\bar{k}} &= s_{mn|\bar{k}} - \Gamma^{l}{}_{m\bar{k}} s_{ln} - \Gamma^{\bar{p}}{}_{n\bar{k}} g_{m\bar{p}} - \Gamma^{\bar{p}}{}_{m\bar{k}} g_{n\bar{p}} - \Gamma^{l}{}_{n\bar{k}} s_{ml} = 2\phi_{\bar{k}} s_{mn} \end{aligned} \tag{12}$$

$$\begin{aligned} \Gamma^{l}{}_{mj} &= \frac{1}{2} s^{lp}(s_{mp|j} + s_{pj|m} - s_{mj|p}) + \frac{1}{2} g^{l\bar{k}}(g_{m\bar{k}|j} + g_{j\bar{k}|m} - s_{mj|\bar{k}}) - \delta^{l}_{m}\phi_j - \delta^{l}_{j}\phi_m + s_{mj}\phi^{l} \\ \Gamma^{\bar{k}}{}_{mj} &= \frac{1}{2} g^{p\bar{k}}(s_{mp|j} + s_{pj|m} - s_{mj|p}) + \frac{1}{2} s^{\bar{k}\bar{p}}(g_{m\bar{p}|j} + g_{j\bar{p}|m} - s_{mj|\bar{p}}) + s_{mj}\phi^{\bar{k}} \\ \Gamma^{l}{}_{\bar{n}j} &= \frac{1}{2} s^{lp}(s_{pj|\bar{n}} + g_{p\bar{n}|j} - g_{j\bar{n}|p}) + \frac{1}{2} g^{l\bar{k}}(s_{\bar{n}\bar{k}|j} + g_{j\bar{k}|\bar{n}} - g_{j\bar{n}|\bar{k}}) - \delta^{l}_{j}\phi_{\bar{n}} + g_{j\bar{n}}\phi^{l} \\ \Gamma^{\bar{k}}{}_{\bar{n}j} &= \frac{1}{2} g^{p\bar{k}}(s_{pj|\bar{n}} + g_{p\bar{n}|j} - g_{j\bar{n}|p}) + \frac{1}{2} s^{\bar{k}\bar{p}}(s_{\bar{n}\bar{p}|j} + g_{j\bar{p}|\bar{n}} - g_{j\bar{n}|\bar{p}}) - \delta^{\bar{k}}_{\bar{n}}\phi_j + g_{j\bar{n}}\phi^{\bar{k}} \end{aligned} \tag{13}$$

$$\begin{gathered} d\tau^2 = s_{jk} dz^j dz^k + s_{\bar{j}\bar{k}} dz^{\bar{j}} dz^{\bar{k}} + 2 g_{j\bar{k}} dz^j dz^{\bar{k}} \\ \delta \int d\tau = 0 \\ \ddot{z}^m + \tilde{\Gamma}^{m}{}_{jk} \dot{z}^j \dot{z}^k + \tilde{\Gamma}^{m}{}_{\bar{j}\bar{k}} \dot{z}^{\bar{j}} \dot{z}^{\bar{k}} + 2\tilde{\Gamma}^{m}{}_{j\bar{k}} \dot{z}^j \dot{z}^{\bar{k}} = 0 \\ \ddot{z}^{\bar{n}} + \tilde{\Gamma}^{\bar{n}}{}_{jk} \dot{z}^j \dot{z}^k + \tilde{\Gamma}^{\bar{n}}{}_{\bar{j}\bar{k}} \dot{z}^{\bar{j}} \dot{z}^{\bar{k}} + 2\tilde{\Gamma}^{\bar{n}}{}_{j\bar{k}} \dot{z}^j \dot{z}^{\bar{k}} = 0 \end{gathered} \tag{14}$$

$$g_{\mu\nu} = N \sigma_{\mu}{}^{j\bar{k}} \sigma_{\nu}{}^{m\bar{n}} (s_{jm} s_{\bar{k}\bar{n}} + g_{j\bar{n}} g_{m\bar{k}}) \tag{15}$$

$$j_{\mu} = \sigma_{\mu}{}^{j\bar{k}} (\psi_j \psi_{\bar{k}} + \rho_j \rho_{\bar{k}}) \tag{16}$$

$$\begin{aligned} g_{m\bar{n}} &= (\epsilon^{A}_{m} \epsilon^{\bar{B}}_{\bar{n}} + \tau^{\bar{B}}_{m} \tau^{A}_{\bar{n}}) g_{A\bar{B}} + \tau^{\bar{A}}_{m} \epsilon^{\bar{B}}_{\bar{n}} s_{\bar{A}\bar{B}} + \epsilon^{A}_{m} \tau^{B}_{\bar{n}} s_{AB} \\ s_{mn} &= (\epsilon^{A}_{m} \tau^{\bar{B}}_{n} + \tau^{\bar{B}}_{m} \epsilon^{A}_{n}) g_{A\bar{B}} + \tau^{\bar{A}}_{m} \tau^{\bar{B}}_{n} s_{\bar{A}\bar{B}} + \epsilon^{A}_{m} \epsilon^{B}_{n} s_{AB} \end{aligned} \tag{17}$$

$$\begin{gathered} \epsilon^{A}_{m} \epsilon^{m}_{B} + \tau^{A}_{\bar{n}} \tau^{\bar{n}}_{B} = \delta^{A}_{B} \\ \epsilon^{A}_{m} \tau^{m}_{\bar{B}} + \tau^{A}_{\bar{n}} \epsilon^{\bar{n}}_{\bar{B}} = 0 \\ \epsilon^{m}_{A} \epsilon^{A}_{n} + \tau^{m}_{\bar{A}} \tau^{\bar{A}}_{n} = \delta^{m}_{n} \\ \epsilon^{m}_{A} \tau^{A}_{\bar{n}} + \tau^{m}_{\bar{A}} \epsilon^{\bar{A}}_{\bar{n}} = 0 \end{gathered} \tag{18}$$

$$\begin{aligned}
\epsilon^A_{m||j} &= \epsilon^A_{m|j} - \Gamma^l{}_{mj}\epsilon^A_l - \Gamma^{\bar{k}}{}_{mj}\tau^A_{\bar{k}} + \omega_j{}^A{}_B\epsilon^B_m + \omega_j{}^A{}_{\bar{B}}\tau^{\bar{B}}_m = \phi_j\epsilon^A_m \\
\tau^A_{\bar{n}||j} &= \tau^A_{\bar{n}|j} - \Gamma^{\bar{k}}{}_{\bar{n}j}\tau^A_{\bar{k}} - \Gamma^l{}_{\bar{n}j}\epsilon^A_l + \omega_j{}^A{}_B\tau^B_{\bar{n}} + \omega_j{}^A{}_{\bar{B}}\epsilon^{\bar{B}}_{\bar{n}} = \phi_j\tau^A_{\bar{n}} \\
\epsilon^{\bar{B}}_{\bar{n}||j} &= \epsilon^{\bar{B}}_{\bar{n}|j} - \Gamma^{\bar{k}}{}_{\bar{n}j}\epsilon^{\bar{B}}_{\bar{k}} - \Gamma^l{}_{\bar{n}j}\tau^{\bar{B}}_l + \omega_j{}^{\bar{B}}{}_{\bar{C}}\epsilon^{\bar{C}}_{\bar{n}} + \omega_j{}^{\bar{B}}{}_C\tau^C_{\bar{n}} = \phi_j\epsilon^{\bar{B}}_{\bar{n}} \\
\tau^{\bar{B}}_{m||j} &= \tau^{\bar{B}}_{m|j} - \Gamma^l{}_{mj}\tau^{\bar{B}}_l - \Gamma^{\bar{k}}{}_{mj}\epsilon^{\bar{B}}_{\bar{k}} + \omega_j{}^{\bar{B}}{}_{\bar{C}}\tau^{\bar{C}}_m + \omega_j{}^{\bar{B}}{}_C\epsilon^C_m = \phi_j\tau^{\bar{B}}_m
\end{aligned} \tag{19}$$

$$\begin{aligned}
\omega_j{}^A{}_B &= -\epsilon^A_{m|j}\epsilon^m_B - \tau^A_{\bar{n}|j}\tau^{\bar{n}}_B + \Gamma^l{}_{mj}\epsilon^A_l\epsilon^m_B + \Gamma^{\bar{k}}{}_{mj}\tau^A_{\bar{k}}\epsilon^m_B + \Gamma^l{}_{\bar{n}j}\epsilon^A_l\tau^{\bar{n}}_B + \Gamma^{\bar{k}}{}_{\bar{n}j}\tau^A_{\bar{k}}\tau^{\bar{n}}_B + \phi_j\delta^A_B \\
\omega_j{}^A{}_{\bar{B}} &= -\epsilon^A_{m|j}\tau^m_{\bar{B}} - \tau^A_{\bar{n}|j}\epsilon^{\bar{n}}_{\bar{B}} + \Gamma^l{}_{mj}\epsilon^A_l\tau^m_{\bar{B}} + \Gamma^{\bar{k}}{}_{mj}\tau^A_{\bar{k}}\tau^m_{\bar{B}} + \Gamma^l{}_{\bar{n}j}\epsilon^A_l\epsilon^{\bar{n}}_{\bar{B}} + \Gamma^{\bar{k}}{}_{\bar{n}j}\tau^A_{\bar{k}}\epsilon^{\bar{n}}_{\bar{B}} \\
\omega_j{}^{\bar{A}}{}_B &= -\tau^{\bar{A}}_{m|j}\epsilon^m_B - \epsilon^{\bar{A}}_{\bar{n}|j}\tau^{\bar{n}}_B + \Gamma^l{}_{mj}\tau^{\bar{A}}_l\epsilon^m_B + \Gamma^{\bar{k}}{}_{mj}\epsilon^{\bar{A}}_{\bar{k}}\epsilon^m_B + \Gamma^l{}_{\bar{n}j}\tau^{\bar{A}}_l\tau^{\bar{n}}_B + \Gamma^{\bar{k}}{}_{\bar{n}j}\epsilon^{\bar{A}}_{\bar{k}}\tau^{\bar{n}}_B \\
\omega_j{}^{\bar{A}}{}_{\bar{B}} &= -\tau^{\bar{A}}_{m|j}\tau^m_{\bar{B}} - \epsilon^{\bar{A}}_{\bar{n}|j}\epsilon^{\bar{n}}_{\bar{B}} + \Gamma^l{}_{mj}\tau^{\bar{A}}_l\tau^m_{\bar{B}} + \Gamma^{\bar{k}}{}_{mj}\epsilon^{\bar{A}}_{\bar{k}}\tau^m_{\bar{B}} + \Gamma^l{}_{\bar{n}j}\tau^{\bar{A}}_l\epsilon^{\bar{n}}_{\bar{B}} + \Gamma^{\bar{k}}{}_{\bar{n}j}\epsilon^{\bar{A}}_{\bar{k}}\epsilon^{\bar{n}}_{\bar{B}} + \phi_j\delta^{\bar{A}}_{\bar{B}}
\end{aligned} \tag{20}$$

$$\begin{aligned}
\frac{\partial}{\partial x^\alpha} &= \frac{\partial z^j}{\partial x^\alpha}\frac{\partial}{\partial z^j} + \frac{\partial z^{\bar{k}}}{\partial x^\alpha}\frac{\partial}{\partial z^{\bar{k}}} \\
\psi_{j||\alpha} &= \frac{\partial z^m}{\partial x^\alpha}\psi_{j||m} + \frac{\partial z^{\bar{n}}}{\partial x^\alpha}\psi_{j||\bar{n}} = \psi_{j|\alpha} - \Gamma^l{}_{j\alpha}\psi_l - \Gamma^{\bar{l}}{}_{j\alpha}\rho_{\bar{l}}
\end{aligned} \tag{21}$$

$$\begin{aligned}
\psi_{j||\alpha||\beta} - \psi_{j||\beta||\alpha} &= R^l{}_{j\alpha\beta}\psi_l + R^{\bar{l}}{}_{j\alpha\beta}\rho_{\bar{l}} \\
R^l{}_{j\alpha\beta} &\equiv \Gamma^l{}_{j\beta|\alpha} - \Gamma^l{}_{j\alpha|\beta} + \Gamma^m{}_{j\beta}\Gamma^l{}_{m\alpha} + \Gamma^{\bar{n}}{}_{j\beta}\Gamma^l{}_{\bar{n}\alpha} - \Gamma^m{}_{j\alpha}\Gamma^l{}_{m\beta} - \Gamma^{\bar{n}}{}_{j\alpha}\Gamma^l{}_{\bar{n}\beta} \\
R^{\bar{l}}{}_{j\alpha\beta} &\equiv \Gamma^{\bar{l}}{}_{j\beta|\alpha} - \Gamma^{\bar{l}}{}_{j\alpha|\beta} + \Gamma^m{}_{j\beta}\Gamma^{\bar{l}}{}_{m\alpha} + \Gamma^{\bar{n}}{}_{j\beta}\Gamma^{\bar{l}}{}_{\bar{n}\alpha} - \Gamma^m{}_{j\alpha}\Gamma^{\bar{l}}{}_{m\beta} - \Gamma^{\bar{n}}{}_{j\alpha}\Gamma^{\bar{l}}{}_{\bar{n}\beta}
\end{aligned} \tag{22}$$