On the LP Breakeven:
Risk Analysis of Uniswap and Leveraged Uniswap

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1 Abstract

In this paper, we analyze liquidity provider (LP) return-risk profiles in Uniswap's automated market maker (AMM) [1], and compare it to that of a modification we define called *leveraged Uniswap AMM* - a much more capital efficient AMM that introduces a *leverage factor* (λ) for the liquidity pool. We found that, in the limit, the leveraged LP return-risk is similar to underwriting an at-the-money (ATM) put option with zero premium. In practice, this means that any asset holder should consider the action of participating as an LP in a given time duration, as a choice to collect one of the following return streams:

1. upfront premium from underwriting an ATM put, with expiry at end of duration.
2. LP trading fees (earnings share) from the liquidity pool.

In other words, LPs need to earn return from fees commensurate to the upfront premium an asset holder receives from underwriting of ATM puts just to break even. A better alternative is needed.

2 Setup

Following the setup in previous work [3], we will assume a two token economy with $X as the base token and $Y as the quote token, and without further specification, *price* would mean the price of

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\$X in \$Y. We also assume we have the *trading function* $f(.)$ for the AMM. For $f(.)$ to be a valid trading function, it needs to satisfy $f'(.) < 0$ and that $f'(.)$ is increasing, i.e. f is monotonically decreasing and convex. Suppose the initial state is (x_0, y_0) with initial price $p_0 = -f'(x_0)$. Suppose later the price changes to p_1.

3 Analysis

3.1 Uniswap

The trading function of a vanilla Uniswap AMM [1] is of the form $f(x) = k/x$ for some constant k. Plugging in $y_0 = f(x_0)$, we can solve for $k = x_0 y_0$. Also, given $p_0 = -f'(x_0) = k/x_0^2 = y_0/x_0$, we can solve for $y_0 = p_0 x_0$, so $f(x) = p_0 x_0^2/x$. We define the following value quantities:

1. $V_{00} = p_0 x_0 + y_0$, which is the value of the initial (x_0, y_0) marked to the price p_0 at t_0
2. $V_{01} = p_1 x_0 + y_0$, which is the value of the initial (x_0, y_0) marked to the price p_1 at t_1. This is also known as the "HODLER's value".
3. $V_{11} = p_1 x_1 + y_1$, which is the value of the current (x_1, y_1) marked to the price p_1 at t_1. This is known as the LP's value at t_1. Assuming an equilibrium where the pool price aligns with external price p_1 at t_1, we can solve for x_1 and y_1: $x_1 = x_0\sqrt{p_0/p_1}$, $y_1 = p_0 x_0 \sqrt{p_1/p_0}$.

We can solve for V_{00}, V_{01}, V_{11} in terms of x_0, p_0, and p_1:

$$
\begin{align}
V_{00} &= 2p_0 x_0 \tag{1}\\
V_{01} &= p_0 x_0 (p_1/p_0 + 1) = p_0 x_0 (p + 1) \tag{2}\\
V_{11} &= 2p_0 x_0 \sqrt{p_1/p_0} = 2p_0 x_0 \sqrt{p} \tag{3}
\end{align}
$$

where $p := p_1/p_0$ is defined as the price ratio. With this, we can calculate:

$$
\begin{align}
V_{01} - V_{00} &= p_0 x_0 (p - 1) \tag{4}\\
V_{11} - V_{01} &= p_0 x_0 (2\sqrt{p} - (p + 1)) = p_0 x_0 (-(\sqrt{p} - 1)^2) \tag{5}\\
V_{11} - V_{00} &= 2p_0 x_0 (\sqrt{p} - 1) \tag{6}
\end{align}
$$

Note that $V_{11} - V_{01}$ from equation (5) is properly defined as the *impermanent loss* of the Uniswap AMM (the difference between LP's value and HODLER's value). It is clear from the expression that the quantity is always non-positive, in fact always negative except when $p_1/p_0 = p = 1$ (i.e. no price change) in which case it is 0. This means that impermanent loss is always a loss and never a gain, whether price moves up or down.

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We can also calculate the *impermanent loss rate*, defined as a function of p:

$$L(p) = \frac{V_{11} - V_{01}}{V_{01}} = \frac{-(\sqrt{p}-1)^2}{p+1} = \frac{2\sqrt{p}}{p+1} - 1 \tag{7}$$

This is the underlying function for the graph that is often seen on online posts regarding impermanent loss discussions. It has certain properties:

1. $-1 \leq L(p) \leq 0$
2. $L(1) = 0$
3. $\lim_{p \to 0} L(p) = \lim_{p \to \infty} L(p) = -1$
4. $L(p)$ is monotonically increasing for $p \in [0, 1]$ and monotonically decreasing for $p \in [1, \infty)$
5. $L(p) = L(1/p)\ \forall p$

The last property implies that, for a Uniswap AMM, a 100% increase in price would yield the same impermanent loss rate as a 50% decrease in price.

We can also decompose the LP's return from t_0 to t_1 by:

$$\begin{aligned} R(p) &= \frac{V_{11} - V_{00}}{V_{00}} = \sqrt{p} - 1 \\ &= \frac{V_{01} - V_{00}}{V_{00}} + \frac{V_{11} - V_{01}}{V_{00}} \\ &= \frac{p-1}{2} + \frac{-(\sqrt{p}-1)^2}{2} \end{aligned} \tag{8}$$

where the first part of equation (8) is the HODLER's return purely due to external price movement (in fact it is easy to see that it is linear in price movement, i.e. an r% change in price would yield a 0.5r% HODLER's return, where the 0.5 factor comes from the portfolio weight of the "risky" asset \$X), and the second part of equation (8) comes from impermanent loss (the return difference between HODLER and LP) which is always non-positive.

For instance, a 50% increase in price, would yield $R = 25\% + -2.53\% = 22.47\%$, where 25% comes from the gain from price movement, and the -2.53% comes from impermanent loss. A 50% decrease in price would yield $R = -25\% + -4.29\% = -29.29\%$, where -25% comes from the loss from price movement, and the -4.29% comes from impermanent loss. Note that impermanent loss is not symmetric with respect to price movement but rather symmetric with respect to log price movement, therefore the 2 cases yield different impermanent losses.

We can also calculate the percentage of return that is contributed by impermanent loss:

$$\alpha(p) = \left| \frac{V_{11} - V_{01}}{V_{11} - V_{00}} \right| = \left| \frac{\sqrt{p} - 1}{2} \right| \tag{9}$$

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We can see that, as $p \to 0$ starting from 1, the percentage goes from 0 to 50%, i.e. impermanent loss constitutes 0% of total return in beginning and approaches 50% of the total -100% loss when $p \to 0$. On the other hand, as $p \to \infty$ starting from 1, the percentage goes from 0 to ∞. This seems intuitively strange, but it is a reflection of the fact that as $p \to \infty$, LP return only increases at a square root rate, while HODLER's return grows at a linear rate (with a portfolio weight factor of 0.5), and all the "missing" returns come from impermanent loss.

The decomposition of LP return in equation (8) will be useful for analyzing the risk of a leveraged Uniswap AMM.

3.2 Leveraged Uniswap

In a *leveraged Uniswap* AMM, the pool may only have balances $\tilde{x}_0$ of \$X and $\tilde{y}_0$ of \$Y, but yet is providing the same liquidity as a regular Uniswap with balances x_0 of \$X and y_0 of \$Y. We will assume the leveraged Uniswap pool balance to follow the same ratio as the regular Uniswap, i.e. $\frac{\tilde{x}_0}{\tilde{y}_0} = \frac{x_0}{y_0}$, therefore $\tilde{y}_0 = p_0\tilde{x}_0$, and we define $\lambda = \frac{x_0}{\tilde{x}_0} = \frac{y_0}{\tilde{y}_0}$ as the *leverage factor*. For the rest of the analysis, we will assume $\lambda \geq 1$. We can define similar value quantities as the vanilla Uniswap:

1. $\tilde{V}_{00} = p_0\tilde{x}_0 + \tilde{y}_0$, which is the value of the initial $(\tilde{x}_0, \tilde{y}_0)$ marked to the price p_0 at t_0
2. $\tilde{V}_{01} = p_1\tilde{x}_0 + \tilde{y}_0$, which is the value of the initial $(\tilde{x}_0, \tilde{y}_0)$ marked to the price p_1 at t_1. This is also known as the "HODLER's value".
3. $\tilde{V}_{11} = p_1\tilde{x}_1 + \tilde{y}_1$, which is the value of the current $(\tilde{x}_1, \tilde{y}_1)$ marked to the price p_1 at t_1. This is known as the LP's value at t_1. Assuming an equilibrium where the pool price aligns with external price p_1 at t_1, we can solve for $\tilde{x}_1$ and $\tilde{y}_1$: $\tilde{x}_1 = \tilde{x}_0 + \Delta x = \tilde{x}_0 + (x_1 - x_0) = \tilde{x}_0 + x_0(\sqrt{p_0/p_1} - 1) = \tilde{x}_0 + \lambda\tilde{x}_0(\sqrt{p_0/p_1} - 1)$, $\tilde{y}_1 = \tilde{y}_0 + \Delta y = \tilde{y}_0 + (y_1 - y_0) = p_0\tilde{x}_0 + p_0x_0(\sqrt{p_1/p_0} - 1) = p_0\tilde{x}_0 + \lambda p_0\tilde{x}_0(\sqrt{p_1/p_0} - 1)$.

We can solve for $\tilde{V}_{00}$, $\tilde{V}_{01}$, $\tilde{V}_{11}$ in terms of $\tilde{x}_0$, p_0, p_1, and λ:

$$\tilde{V}_{00} = 2p_0\tilde{x}_0 = p_0\tilde{x}_0 \cdot 2 \tag{10}$$
$$\tilde{V}_{01} = p_0\tilde{x}_0(p_1/p_0 + 1) = p_0\tilde{x}_0 \cdot (p + 1) \tag{11}$$
$$\tilde{V}_{11} = p_0\tilde{x}_0 \cdot [(p + 1) - \lambda(\sqrt{p} - 1)^2] \tag{12}$$

where $p := p_1/p_0$ is defined as the price ratio. With this, we can calculate:

$$\tilde{V}_{01} - \tilde{V}_{00} = p_0\tilde{x}_0 \cdot (p - 1) \tag{13}$$
$$\tilde{V}_{11} - \tilde{V}_{01} = p_0\tilde{x}_0 \cdot (-\lambda(\sqrt{p} - 1)^2) \tag{14}$$
$$\tilde{V}_{11} - \tilde{V}_{00} = p_0\tilde{x}_0 \cdot ((p - 1) - \lambda(\sqrt{p} - 1)^2) \tag{15}$$

Note that the impermanent loss amount of the leveraged Uniswap AMM is the same as the vanilla Uniswap AMM (by the relationship $x_0 = \lambda\tilde{x_0}$ and comparing with equation (5)), but the impermanent loss rate is magnified by the leverage factor λ:

$$\tilde{L}(p) = \frac{\tilde{V}_{11} - \tilde{V}_{01}}{\tilde{V}_{01}} = \frac{-\lambda(\sqrt{p}-1)^2}{p+1} = \lambda(\frac{2\sqrt{p}}{p+1} - 1) \tag{16}$$

Similar properties hold for the $\tilde{L}(.)$ function as the $L(.)$ function in the previous section for vanilla Uniswap.

The LP's return from t_0 to t_1 can be similarly decomposed as:

$$\begin{aligned}\tilde{R}(p) &= \frac{\tilde{V}_{11} - \tilde{V}_{00}}{\tilde{V}_{00}} \\ &= \frac{\tilde{V}_{01} - \tilde{V}_{00}}{\tilde{V}_{00}} + \frac{\tilde{V}_{11} - \tilde{V}_{01}}{\tilde{V}_{00}} \\ &= \frac{p-1}{2} + \frac{-\lambda(\sqrt{p}-1)^2}{2}\end{aligned} \tag{17}$$

where the first part of equation (17) is the HODLER's return purely due to external price movement and the second part comes from impermanent loss. Comparing against equation (8), we can see that the first part is identical while the second part has an extra leverage factor λ. Therefore we can say that in the normal case (when the reserves are not yet run out) the leveraged Uniswap AMM has the same "delta 1" risk (the first part that grows linearly with p) but a leveraged second order risk (the second part that grows square root with p).

We can also calculate the price range within which the leveraged Uniswap is properly functioning, i.e. does not run out of reserves. We need the following inequalities to hold:

$$\begin{aligned}\tilde{x_1} &= \tilde{x_0} + \lambda\tilde{x_0}(\sqrt{p_0/p_1} - 1) = \tilde{x_0}(1 + \lambda(\sqrt{1/p} - 1)) \geq 0 \\ \tilde{y_1} &= p_0\tilde{x_0} + \lambda p_0\tilde{x_0}(\sqrt{p_1/p_0} - 1) = p_0\tilde{x_0}(1 + \lambda(\sqrt{p} - 1)) \geq 0\end{aligned}$$

Solving for p, we get:

$$\left(1 - \frac{1}{\lambda}\right)^2 \leq p \leq \left(1 - \frac{1}{\lambda}\right)^{-2} \tag{18}$$

For large λ, this can be approximated by $1 - 2/\lambda \leq p \leq 1 + 2/\lambda$. For instance, for $\lambda = 100$, roughly a price movement of 2% would cause the leveraged Uniswap AMM to run out of reserves.

On the other hand, given a pre-specified range of p, we can calculate the maximum leverage that a leveraged Uniswap can have without running out of reserves, by solving the above inequalities for λ:

$$\lambda \leq \min\left(\frac{1}{1-\sqrt{1/p}}, \frac{1}{1-\sqrt{p}}\right) \tag{19}$$

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For instance, for a $\pm 30\%$ price movement, we need $\lambda \leq \min\left(\frac{1}{1-\sqrt{1/1.3}}, \frac{1}{1-\sqrt{0.7}}\right) \approx 6.12$.

Next, we will consider the situation where the leveraged Uniswap AMM is at depletion of one reserve. Define $\underline{p} = (1-\frac{1}{\lambda})^2 \leq 1$ and $\overline{p} = (1-\frac{1}{\lambda})^{-2} \geq 1$. Note that for $\lambda = 1$, $\underline{p} = 0$ and $\overline{p} = \infty$. Then from inequality (18), we know that for $p \leq \underline{p}$, the pool will run out of \$Y and for $p \geq \overline{p}$, the pool will run out of \$X.

Consider the first case $p = \overline{p}$: in this case $\lambda = \frac{1}{1-\sqrt{1/\overline{p}}}$, plugging into equation (17), we get

$$\begin{aligned}\tilde{R}(\overline{p}) &= \frac{\overline{p}-1}{2} + \frac{\sqrt{\overline{p}}-\overline{p}}{2} = \frac{2\lambda-1}{2(\lambda-1)^2} + \frac{-\lambda}{2(\lambda-1)^2} \\ &= \frac{\sqrt{\overline{p}}-1}{2} = \frac{1}{2(\lambda-1)}\end{aligned} \tag{20}$$

When $p > \overline{p}$, the leveraged Uniswap AMM will have run out of \$X, and therefore the return stay constant at $\tilde{R}(\overline{p})$ and will not change any more as price keeps further increasing.

Consider the second case $p = \underline{p}$: in this case $\lambda = \frac{1}{1-\sqrt{\underline{p}}}$, plugging into equation (17), we get

$$\begin{aligned}\tilde{R}(\underline{p}) &= \frac{\underline{p}-1}{2} + \frac{\sqrt{\underline{p}}-1}{2} = \frac{1-2\lambda}{2\lambda^2} + \frac{-1}{2\lambda} \\ &= \frac{\underline{p}+\sqrt{\underline{p}}-2}{2} = \frac{1-3\lambda}{2\lambda^2}\end{aligned} \tag{21}$$

When $p < \underline{p}$, the leveraged Uniswap AMM will have run out of \$Y, and its \$X balance will be equal to the balance of \$X when $p = \underline{p}$, i.e., $\tilde{x}_1 = \tilde{x}_0(1+\lambda(\sqrt{1/\underline{p}}-1))$ and $\tilde{y}_1 = 0$. Then following the definition of $\tilde{R}(p)$, we have

$$\begin{aligned}\tilde{R}(p) &= \frac{\tilde{V}_{11}-\tilde{V}_{00}}{\tilde{V}_{00}} \\ &= \frac{p_0\tilde{x}_0 \cdot p(1+\lambda(\sqrt{1/\underline{p}}-1)) + 0 - p_0\tilde{x}_0 \cdot 2}{p_0\tilde{x}_0 \cdot 2} \\ &= \frac{p(1+\lambda(\sqrt{1/\underline{p}}-1))-2}{2} \\ &= \frac{p(1+1/\sqrt{\overline{p}})-2}{2} = p \cdot \left(\frac{2\lambda-1}{2(\lambda-1)}\right) - 1\end{aligned} \tag{22}$$

Plugging in $p = \underline{p} = (1-\frac{1}{\lambda})^2$, we can verify that equation (22) agrees with equation (21).

Finally, by combining equation (17), (20), and (22), we have the comprehensive expression for the LP return of a leveraged Uniswap AMM:

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$$\tilde{R}(p) = \begin{cases} p \cdot \left(\frac{2\lambda-1}{2(\lambda-1)}\right) - 1 & \text{if } p < \underline{p} \\ \frac{p-1}{2} + \frac{-\lambda(\sqrt{p}-1)^2}{2} & \text{if } \underline{p} \le p \le \overline{p} \\ \frac{1}{2(\lambda-1)} & \text{if } p > \overline{p} \end{cases}$$

where $\underline{p} = (1 - \frac{1}{\lambda})^2$ and $\overline{p} = (1 - \frac{1}{\lambda})^{-2}$ are defined as above.

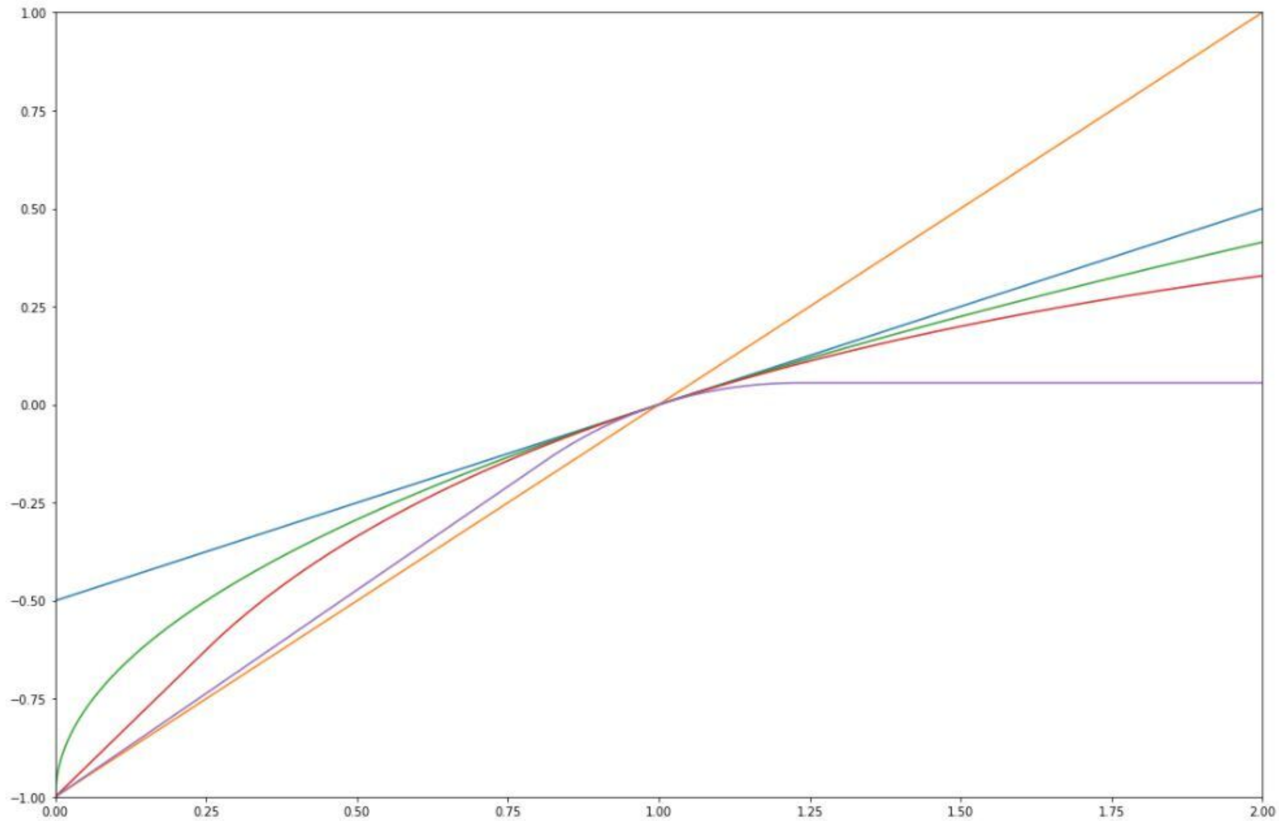
This seemingly complicated expression has a few noteworthy properties:

1. $\tilde{R}(p)$ is continuous at $\underline{p}$ and $\overline{p}$ (and clearly everywhere else too)
2. $\tilde{R}(p)$ is differentiable at $\underline{p}$ and $\overline{p}$ (and clearly everywhere else too)
3. $\tilde{R}(p)$ is non decreasing, which is saying the more price increases (drops), the better (worse) LP's return is.
4. $\tilde{R}(p)$ is capped at $\frac{1}{2(\lambda-1)}$, this is because beyond $\overline{p}$ the pool has sold out all its \$X and therefore cannot earn any further return if price continues to go up. Note that when $\lambda = 1$ it is not capped; it is only capped when $\lambda > 1$.
5. HODLER's return $H(p) = \frac{p-1}{2}$ is always greater than $\tilde{R}(p)$, where the difference is impermanent loss which is always negative, except at $p = 1$ where the two are equal and impermanent loss is 0.
6. $\tilde{R}'(1) = \frac{1}{2}\ \forall\lambda$, which means that for tiny price movement, LP's return is locally approximately HODLER's return, i.e. impermanent loss is a second order effect.
7. For $p \le 1$, $\tilde{R}(p) \ge p - 1$, which means that when price drops, LP's return is no worse than the percentage of price drop.
8. For $p \le \underline{p}$, $\tilde{R}'(p) = \frac{2\lambda-1}{2(\lambda-1)} > 1$ which is a constant. For large λ, this constant is close to 1 while $\underline{p}$ is also close to 1, and LP's return is approximately the price drop amount when price drops.
9. For all λ_1, λ_2 such that $1 \le \lambda_1 < \lambda_2$, $\tilde{R}(p; \lambda_1) \le \tilde{R}(p; \lambda_2)$ for all p, with equality holds only when $p = 1$ and $p = 0$. This is because the leverage factor λ is a magnifier of impermanent loss, which is always a loss and never a gain.

The following is the graph of various returns: x-axis is p, y-axis is return.

1. Blue line is HODLER's return: $H(p) = \frac{p-1}{2}$
2. Orange line is price return, i.e. return of holding 100% \$X: $P(p) = p - 1$

3. Green line is vanilla Uniswap's LP return (or leveraged Uniswap with $\lambda = 1$): $R(p) = \sqrt{p} - 1$
4. Red line is 2x leveraged Uniswap's LP return: $\tilde{R}(p; \lambda = 2)$
5. Purple line is 10x leveraged Uniswap's LP return: $\tilde{R}(p; \lambda = 10)$



3.3 Discussion

We observe through the shapes of the 2x leveraged Uniswap and 10x leveraged Uniswap return curves, that applying additional leverage, in the limit, makes a leveraged liquidity provider return-risk profile resemble that of underwriting an at-the-money (ATM) put option with zero premium. In practice, this means that any asset holder considering the action of participating as an LP in a given time duration with start time t_0 and end time t_1 has a choice to collect one of the following return streams, the latter of which is the actual LP return profile:

1. upfront premium received at t_0 from underwriting an ATM put, with option expiry at t_1 (end of duration).

2. LP trading fees (earnings share) from the liquidity pool, accrued throughout $[t_0, t_1)$.

The commensurate American put option underwriter (case 1) is betting on net rise of the underlying asset price and/or volatility contraction in the time duration between the position open and close. On the other hand, a participating LP (case 2) must decide when to deposit (open) and withdraw (close) their position inside the AMM liquidity pool, and receives the accrued trading fee earnings during that period net of any impermanent loss incurred at close [2].

In other words, just to break even in this decision framework, LPs need their variable fee stream to exceed the commensurate upfront premium an asset holder receives from underwriting ATM puts for the same period, while facing similar risks. A better alternative is needed.

References

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[3] Jamie E. Young. On equivalence of automated market maker and limit order book systems, 2020.

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