Authors' Comments

Blue indicates the authors' comments, red shows the modifications made according to the reviewers' suggestions, and black represents the reviewers' comments.

Reviewer 1: The paper presents a relevant and well-structured study on an adaptive polyalgorithm for solving vehicle suspension models, combining Runge-Kutta methods with anticipatory stiffness detection. The methodology is clearly explained, validated with stiff and non-stiff ODEs, and applied to a quarter-car suspension model, showing good accuracy and stability without excessive computational cost. While the work would be strengthened by additional performance metrics, comparison with standard solvers, and discussion of applicability to more complex vehicle models, these are suggestions for future development rather than mandatory revisions.

We thank the reviewer for the positive evaluation and constructive feedback. We are glad that the study on the adaptive polyalgorithm and its application to the quarter-car model were found relevant and clearly presented. We have considered the suggestions regarding additional performance metrics, comparisons with standard solvers, and discussion of applicability to more complex vehicle models, and revised the manuscript to improve clarity and completeness. Without a doubt, these suggestions will be taken into account in future work. We appreciate your valuable comments, which helped enhance the overall quality of the paper.

Reviewer 2: The paper addresses a relevant topic within the area of numerical methods applied to vehicle engineering. The proposal of an adaptive polyalgorithm for stiffness detection is original and well contextualized in the classical literature. The structure of the article is clearly organized and follows the event's template. The numerical results were applied to a vehicle suspension problem involving speed bumps, which gives practical applicability to the proposed study.

We thank the reviewer for highlighting the relevance of the topic and the originality of the proposed adaptive polyalgorithm for stiffness detection. We appreciate your positive comments on the clear organization of the manuscript and the practical applicability of the numerical results to vehicle suspension problems involving speed bumps.

1. The article was written in technical English; however, some sentences are not natural, such as "the change from RK3(4) to RK3(2) occurred due to stiffness detection" on page 5, penultimate paragraph. There are also long and redundant sentences that compromise the text's fluency. A full revision of the English is recommended.

   We appreciate the reviewer's important remark. The manuscript has undergone a thorough English revision to improve sentence structure, eliminate redundancies, and enhance the overall fluency and clarity of the text.

   On page 3, third paragraph was rewritten as follows: Shampine's criterion uses both the acceptance and rejection of step sizes by the RK1(2) and RK4(5) schemes. A nonstiff behavior is indicated when only RK4(5) accepts the step size, while a stiff

behavior is indicated when both schemes accept it. The number of simultaneously accepted step sizes is checked at the end of each observation window. If more than half of the steps are accepted by both methods, the ODE is classified as stiff.

On page 3, penultimate paragraph was rewritten as follows: The same stiff ODE was solved using the adaptive RK4(5) method in the interval $[0, 10]$ with a tolerance of 10^{-5}. Figure 3 shows the evolution of the coefficient CV, whose curve displays the oscillations associated with stiffness and the adaptations of the step size. These adaptations are also shown in Fig. 4.

On page 5, second paragraph was rewritten as follows: Furthermore, the fact that the user must choose the tolerance and the order can result in several attempts to solve the same equation before achieving a satisfactory solution. To address these challenges, a polyalgorithm is proposed. This strategy consists of three adaptive step size Runge–Kutta methods with different orders, and it automatically adjusts the order of the method in use based on the given tolerance and stiffness detection.

On page 5, penultimate paragraph was rewritten as follows: In Table 4, the final method was RK4(5) in all cases; for 0.1 km/h and 0.3 km/h, the transition from RK3(2) to RK3(4) was driven by excessive step rejections, while RK3(4) subsequently progressed to RK4(5) in response to stiffness detection. For speeds of 0.5 km/h, 0.7 km/h, and 0.9 km/h, the transitions between RK3(2), RK3(4), and RK4(5) were caused solely by excessive step rejections. In Table 5, stiffness detection occurred in all cases, ensuring the use of RK4(5) until the end of the numerical simulation. It is important to emphasize that excessive step rejections can indicate either stiffness or the requirement of high accuracy.

2. The introduction justifies well the importance of numerical methods, but it does not clearly highlight which specific gap in the literature the proposed polyalgorithm addresses in comparison with other adaptive methods.

3. There is a lack of bibliographic references and citations of more recent works (preferably within the last five years).

We thank the reviewer for suggestions and comments 2 and 3. As pointed out, the introduction properly justifies the relevance of numerical methods, and the proposed approach is well contextualized within the classical literature. Indeed, the methods employed are well established in the field, which explains the publication dates of the references used. This applies to both the polyalgorithm and the stiffness detection. The polyalgorithm was specifically designed and implemented as a practical framework to validate and demonstrate the effectiveness of the proposed stiffness detection criterion, which constitutes the main contribution and original idea of this work. In this sense, the gap addressed is not the development of a new numerical method per se, but rather the proposal of an embedded stiffness detection strategy that can be integrated into existing adaptive schemes, providing a systematic way to identify stiffness and guide the automatic selection of suitable solvers. Although the references cited are not from the last five years, they remain fundamental and widely

used in the field, which is why we chose to retain them to ensure that the discussion and methodological background are built upon well-established and authoritative sources. Finally, this concept is part of a broader research project currently under development, aiming to extend and refine the proposed methodology in more complex and realistic scenarios.

4. The model adopted is the linearized quarter-car. This simplification is valid but should be better justified in terms of the model's limitations.

   We thank the reviewer for this valuable observation. The manuscript has been revised to provide a clearer discussion of the assumptions and limitations associated with the adopted model.

   In Section 2, the text was added as follows: The model is a single degree-of-freedom (1-DOF) system, where m denotes the lumped mass corresponding to one quarter of the vehicle mass, representing the wheel together with that portion of the car body. The parameter k_e is the equivalent stiffness, combining the tire and suspension spring stiffnesses into a single elastic element, while c_e is the equivalent damping, accounting for the dissipative effect of the damper. With these definitions, the wheel and the associated body mass behave as a single mass connected to one spring k_e and one damper c_e, a simplification that also makes the model's limitations evident [6].

5. The comfort analysis is based only on maximum vertical acceleration. It would be interesting to cite standards (e.g., ISO 2631) or metrics derived from this parameter.

   We thank the reviewer for the suggestion. Relevant standards (e.g., ISO 2631) and metrics derived from maximum vertical acceleration have been cited and discussed in the revised manuscript.

   In Section 2, the sentence has been revised as follows: Although vibration exposure can affect health, its most common effect is discomfort. According to ISO 2631-1 (1997), the vertical acceleration of the system classifies the comfort level as follows: below $0.8\,\mathrm{m/s^2}$ is considered comfortable, between $0.8\,\mathrm{m/s^2}$ and $1.6\,\mathrm{m/s^2}$ uncomfortable, between $1.6\,\mathrm{m/s^2}$ and $2.5\,\mathrm{m/s^2}$ very uncomfortable, and above $2.5\,\mathrm{m/s^2}$ extremely uncomfortable [3].

6. For the explanation regarding stiffness detection via the coefficient of variation (CV), stronger mathematical grounding is required, as well as clarification on: why was the threshold of 3.2 chosen? Is there a sensitivity analysis?

   We thank the reviewer for this valuable comment. The manuscript has been revised to strengthen the mathematical formalism underlying the coefficient of variation (CV).

   In Section 3, the explanations has been restructured as follows: Let SD denote the standard deviation and M_A the arithmetic mean of the N stages, and define the coefficient of variation CV as in eq.(3):

$$CV = \frac{SD}{M_A}. \tag{1}$$

In particular, if $SD > M_A$, then $CV > 1$, which indicates a high variability among the N stages.

In addition, we restructured paragraph 5, page 3, as follows: As an alternative, an approach based on monitoring stages dispersion is proposed, which allows early detection without additional evaluations of the function $f(t, y)$. Geometrically, stiffness is manifested in the solution plot as an abrupt change in the slope of the tangent vector, where the slope varies suddenly in magnitude at a given instant, resulting in high dispersion within the sliding window. This approach is compatible with adaptive step-size Runge-Kutta methods and relies solely on well-established explicit methods from the literature, preserving the main advantages of the method proposed by Shampine.

We thank the reviewer for the comment. Regarding the threshold of 3.2 and the sensitivity analysis, these points are already addressed in Section 3 (page 4, paragraphs 4 and 5, and page 3, antepenultimate paragraph), where we clarify that several values of the coefficient CV were tested to identify those that differentiate stiff from nonstiff systems. In particular, we note that $CV = 3.2$ is the largest value that distinguishes a stiff ODE system from a nonstiff one, and we provide the corresponding sensitivity analysis of CV with respect to stiffness.

7. The description of the switching between RK3(2), RK3(4), and RK4(5) is insufficiently detailed. More elaboration is needed.

We appreciate the reviewer's valuable comment. The manuscript has been revised to provide a more detailed description of the switching process between RK3(2), RK3(4), and RK4(5), clarifying the conditions and criteria involved in each transition.

In Section 3, the explanation has been restructured as follows: The user specifies the tolerance that controls the local error. The polyalgorithm begins with the least expensive scheme, RK3(2). If excessive step-size rejections occur or stiffness is detected, the algorithm switches to RK3(4). If RK3(4) remains insufficient to satisfy the tolerance due to continued rejections or persistent stiffness, the polyalgorithm then switches to RK4(5). At each switch, the step-size adjustment recommended by the outgoing method is adopted as the initial step size for the incoming method.

In addition, we added the following text at the end of Section 3, on page 5: In this way, the developed polyalgorithm performs a dynamic selection among RK3(2), RK3(4), and RK4(5) according to the local complexity of the problem. This allows it to maintain a low computational cost where the solution is smooth, without compromising accuracy or stability.

In the adaptive Runge–Kutta method RK$p(q)$, the step size h is updated adaptively based on the estimated local error, using the classical step-size control formula given by eq. 4 [10].

$$h_{new} = \gamma \left(\frac{tol}{|y^{(p)} - y^{(q)}|} \right)^{\frac{1}{p+1}} h_{old}, \tag{2}$$

where $\gamma = 0.9$ is a safety factor, tol is the tolerance, $y^{(p)}$ is the solution estimated with order p, and $y^{(q)}$ is the solution estimated with order q. If the estimated error, $|y^{(p)} - y^{(q)}|$, exceeds the tolerance, the current step is rejected and recomputed with the new step size. This adaptive mechanism, guided by the actual behavior of the solution, represents a significant advantage compared to using a fixed method, which applies a uniform computational effort throughout the entire integration process [10].

8. The tested speeds (0.1 to 0.9 km/h) are extremely low and unrealistic for vehicles. This must be discussed: would the model or the algorithm fail at typical vehicle speeds (e.g., 20–50 km/h)?

We appreciate the reviewer's important comment. A discussion has been added to the manuscript addressing the use of low testing speeds and clarifying the applicability of both the model and the algorithm at typical vehicle speeds, such as 30 km/h and 50 km/h.

In Section 4, paragraph 6, the explanation was added as follows: These tested speeds are sufficient to demonstrate the decrease in comfort as the vehicle speed increases. Furthermore, this range of speeds is proportional to those tested for speed bumps [3]. However, when considering typical speeds of 30.0 km/h and 50.0 km/h, we obtain, respectively, 98.16 m/s^2 and 166.43 m/s^2 for the harmonic speed bump, and 102.38 m/s^2 and 162.10 m/s^2 for the trapezoidal speed bump. These results can be attributed to the limitations of the suspension model used, such as the number of degrees of freedom, the parameter choices, and the assumption that the wheel remains in contact with the ground at all times, as well as to the fact that a speed bump, referred to as a destructive miniature hump, produces a more severe impact than a speed hump [3].

9. The comparison between the polyalgorithm and reference methods (RK4(5) or RK3(2)) is not clear in terms of computational cost (e.g., CPU time, number of function calls).

We are grateful for the reviewer's valuable observation. The manuscript now presents a clearer comparison between the polyalgorithm and the reference methods regarding computational cost, with particular emphasis on the number of function evaluations per step.

In Section 3, the explanation has been restructured as follows: The proposed polyalgorithm combines the Runge-Kutta methods RK3(2), RK3(4), and RK4(5), which compute respectively four, five, and six stages per step. Since the computational cost is directly related to the number of stages, consequently, the methods are employed in order of increasing computational cost.

10. Figures (5 to 8) show only position and velocity. Would it not be more relevant to present the vertical acceleration curves directly or comfort metrics?

We thank the reviewer for this valuable suggestion. The manuscript has been revised to include the vertical acceleration curves in addition to position and velocity.

In Section 4, the graphs os speeds were modified to include acceleration curves as follows: Figures 1, 2, 3, and 4 show the position, velocity and acceleration of the mass m as functions of time for a vehicle speed of $v_{car} = 0.5$ km/h.

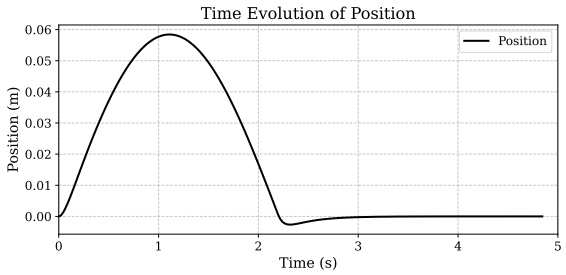


Figure 1. Position - Harmonic.

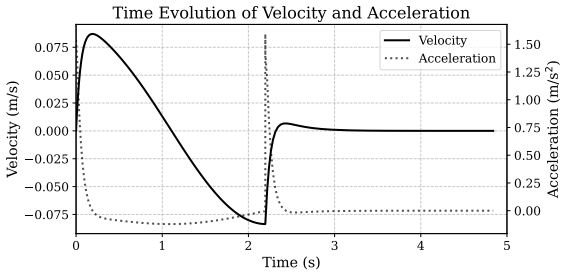


Figure 2. Speed and Accceleration - Harmonic.

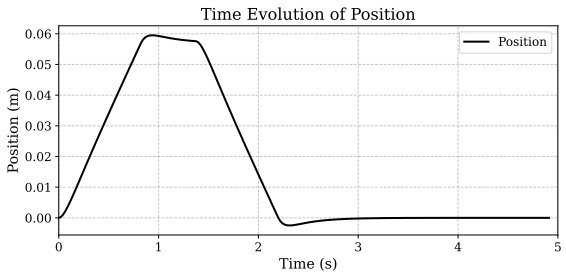


Figure 3. Position - Trapezoidal.

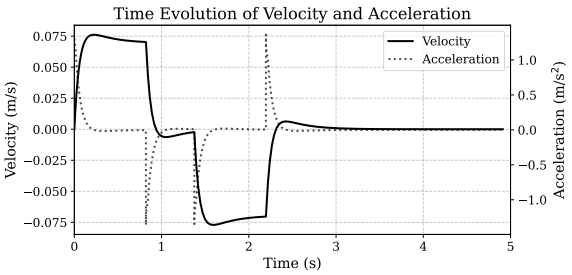


Figure 4. Speed and Accceleration - Trapezoidal.

11. Review and standardize all references according to the CILAMCE template.

We thank the reviewer for this observation. All references have been carefully reviewed and standardized to fully comply with the CILAMCE template.

To standardize the work, whole introduction was restructured to cite the references only in the and of each sentence as follows:

The mathematical modeling of vehicle suspension systems leads to ordinary differential equations (ODEs) that describe the vertical dynamics of the vehicle. These ODEs are generally solved using explicit methods, such as Runge-Kutta formulas and their variants, which are widely employed due to their accuracy and simplicity [2].

Different speed bump profiles have been analyzed [3-5]. Various combinations of vehicle speed and suspension parameters are investigated to determine configurations

that optimize ride comfort. These studies involve multiple simulations to evaluate the system's dynamic behavior under different scenarios.

The efficiency of the classical fourth-order Runge-Kutta, Dormand-Prince, and Bulirsch-Stoer numerical methods has been discussed [6]. In this work, the classical Runge-Kutta method was applied with a relatively small step size. A mass-spring-damper system exhibiting stiff behavior has also been presented, reinforcing the possibility of stiffness in vibratory system models, such as vehicle suspension systems [7] These studies highlight the need for numerical methods capable of detecting stiffness and adapting the integration strategy, ensuring computational efficiency and stability when small time steps are required.

In this context, the present work proposes an adaptive polyalgorithm that performs automatic and early stiffness detection based on the analysis of stage dispersion in adaptive step-size Runge-Kutta methods. The algorithm switches between different numerical methods according to the local characteristics of the solution. This approach aims to optimize computational performance and ensure robustness, particularly when the stiffness of the problem is not known *a priori* or varies during the simulation [8].

This article is organized as follows: Section 2 describes the vehicle suspension system; Section 3 presents the development of the adaptive polyalgorithm; Section 4 discusses the numerical results; and Section 5 provides the conclusions.

In Section 2, the penultimate paragraph was rewritten as follows: These elevations are commonly modeled as speed humps, whose main function is to act as a speed reducer. In this context, the speed humps (longer and smoother structures) and the speed bumps (shorter and more abrupt structures) are distinguished. The speed bumps act like miniature speed hump and impose abrupt excitations to the vehicle system [3-6].

In Section 3, the second paragraph was rewritten as follows: The method of stiffness detection uses a Runge-Kutta adaptive step size scheme, such as RK4(5)[2],[11].

The paragraph following Table 3 in Section 3 was rewritten as follows: The strategy developed by Shampine can be used with any adaptive method, as previously mentioned [11].