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Pipeline Heating Analysis Using the Method of Fundamental Solutions: A Computational Approach

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Abstract. This work proposes an alternative approach to the study presented by Vianna et al. on pipeline heating systems in deep waters, utilizing the Method of Fundamental Solutions (MFS) to solve the direct heat conduction problem. The original study, which employed the finite volume method to model temperature distribution in a pipeline cross-section during shutdown conditions, will be revisited using MFS as a more computationally efficient tool. The MFS offers significant advantages for this type of problem as a meshless method, eliminating the need for physical domain discretization and simplifying the representation of the pipeline's circular geometry. Our aim is to solve the same transient heat conduction problem in a homogeneous medium with constant thermophysical properties, considering the indirect heating system with electrical cables on the external surface. Implementation via MFS is expected to provide high numerical precision and reduced computational cost compared to mesh-based methods, characteristics particularly relevant for real-time applications such as control systems. The results will be validated against analytical solutions and compared with the original study's findings, focusing on computational performance evaluation and accuracy of the proposed method. This study maintains the fundamental objective of adequately modeling the temperature field within the pipeline, an essential step for implementing optimal control strategies that prevent solid deposit formation during deepwater operations.

Keywords: Method of Fundamental Solutions (MFS), Transient Heat Conduction, Temperature Monitoring, Meshless Numerical Methods, Heating Control

1 Introduction

Heat conduction in cylindrical domains plays a critical role in many engineering fields, such as fluid transport, reactors, and thermal insulation. Ensuring reliable and efficient thermal management remains a challenging task, often hindered by the high cost and impracticality of extensive physical experiments [1]. Consequently, computational numerical methods have become the primary tool for investigating such problems, despite the limitations imposed by the expense of commercial software.

Traditional mesh-based techniques like the Finite Element Method (FEM), Finite Volume Method (FVM), and Finite Difference Method (FDM) rely on discretizing the domain into elements or volumes. While effective, these approaches often require significant computational resources and time, particularly for transient heat conduction problems [2, 3]. Meshless methods offer a compelling alternative by avoiding full domain discretization and instead using strategically placed points across the domain and boundaries. Among these, the Method of Fundamental Solutions (MFS) stands out for its simplicity and effectiveness, as demonstrated in steady-state and transient heat conduction studies [4–6].

Applications of MFS have extended to complex scenarios, including backward heat conduction problems and identification of transient thermal sources [7, 8], highlighting its capability in recovering unknown boundary data through numerical iteration [9, 10]. Enhancements such as coupling MFS with Domain Decomposition Methods have improved numerical stability and solution quality in degenerate boundary problems [11, 12], while careful positioning of source points outside the physical domain enhances precision and robustness [13].

Building on these foundations, this work focuses on the transient temperature monitoring within a single cylindrical layer using the MFS. Differing from prior multilayer analyses [14], the present study emphasizes the applicability of MFS for real-time thermal monitoring and control. The study evaluates stability, computational efficiency, and accuracy, comparing the MFS results with established numerical schemes and commercial software benchmarks.

2 Materials and Methods

2.1 Application to Temperature Monitoring

This study addresses temperature monitoring in a single cylindrical layer using the Method of Fundamental Solutions (MFS). The approach adapts the pipeline heating and monitoring framework of Vianna et al. [15], which integrates optimal control and state estimation for transient heat conduction. We employ the MFS to solve the direct transient problem by imposing boundary and initial conditions at collocation points. The resulting unknown coefficients correspond to the system state, estimable from boundary or internal measurements.

This meshless method allows efficient, near-real-time temperature monitoring while inherently satisfying the governing equations. Numerical stability is ensured via Tikhonov regularization applied to the ill-conditioned collocation system. By simplifying the original multilayer model to a single cylindrical layer, our implementation emphasizes accurate temperature estimation for thermal management and control applications.

2.2 Problem Formulation and Governing Equation

Consider a single cylindrical layer defined in the radial interval $r \in (r_0, r_1)$. The transient heat conduction within this layer, without internal heat generation, is governed by the axisymmetric heat equation in cylindrical coordinates:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad (r,t) \in (r_0, r_1) \times [0, T], \tag{1}$$

where $u(r,t)$ denotes the temperature distribution, and α is the thermal diffusivity of the material.

The boundary conditions applied at the inner ($r = r_0$) and outer ($r = r_1$) surfaces of the cylindrical layer may be of Dirichlet, Neumann, or Robin type:

- Dirichlet (prescribed temperature):

$$\underbrace{u(r_b,t) = g(t)}_{\text{Dirichlet}} \qquad \underbrace{-k\frac{\partial u}{\partial r}(r_b,t) = q(t)}_{\text{Neumann}} \qquad \underbrace{-k\frac{\partial u}{\partial r}(r_b,t) = h\big(u(r_b,t) - u_\infty(t)\big)}_{\text{Robin}} \tag{2}$$

where $r_b \in \{r_0, r_1\}$, k is the thermal conductivity, h is the convective heat transfer coefficient, and $g(t)$, $q(t)$, and $u_\infty(t)$ are given time-dependent boundary functions.

This formulation reduces the general multilayer model to a single homogeneous layer scenario, appropriate for the present study focusing on temperature monitoring within a single cylindrical domain.

2.3 Method of Fundamental Solutions

The Method of Fundamental Solutions (MFS) is a meshless numerical technique for solving boundary value problems governed by partial differential equations. This approach approximates the solution by a linear combination of fundamental solutions that inherently satisfy the governing differential equations.

For the transient heat conduction problem in cylindrical coordinates in a single homogeneous layer, the fundamental solution is given by:

$$F(r,t;\rho,\tau) = \frac{H(t-\tau)}{4\pi\alpha(t-\tau)} \exp\left(-\frac{r^2+\rho^2}{4\alpha(t-\tau)} \right) I_0\left(\frac{r\rho}{2\alpha(t-\tau)} \right), \tag{3}$$

where $H(t-\tau)$ is the Heaviside function, ensuring $F = 0$ for $t \leq \tau$. The thermal diffusivity is α, r and ρ denote the spatial coordinates of the collocation and source points, respectively, and t and τ are temporal variables. The function I_0 is the modified Bessel function of the first kind of order zero.

2.4 Numerical Implementation

The temperature distribution in the layer is approximated by a linear combination of these fundamental solutions:

$$u(r,t) \approx \sum_{j=1}^{N} c_j \, F(r,t;\rho_j,\tau_j), \quad (r,t) \in (r_0, r_1) \times [0, T], \tag{4}$$

where c_j are unknown coefficients to be determined.

The source points $\{(\rho_j, \tau_j)\}$ are placed outside the physical spatial domain at a small distance δ from the boundaries:

$$\rho_j = \begin{cases} r_0 - \delta, j = 1, \ldots, M \\ r_1 + \delta, j = M+1, \ldots, 2M \end{cases} \tag{5}$$

where $N = 2M$ and $\delta > 0$ is chosen to ensure the sources lie outside the physical domain.

The temporal coordinates τ_j of the source points are distributed symmetrically about zero to ensure causality and proper time offset:

$$\tau_j = \frac{T(2j - 1 - 2M)}{2M}, \quad j = 1, \ldots, 2M. \tag{6}$$

To enhance numerical stability, the source points are arranged symmetrically with respect to the time interval $t \in [0, T]$, which helps to mitigate the effects of potential singularities. The collocation points, where the boundary conditions are enforced, are distributed in time and space according to:

$$t_i = \frac{iT}{P_t}, \quad i = 0, \ldots, P_t, \qquad s_i = r_0 + \frac{(r_1 - r_0)i}{1 + P_x}, \quad i = 1, \ldots, P_x, \tag{7}$$

where P_t and P_x denote the number of temporal and spatial collocation points, respectively. Here, s_i are the radial positions of the collocation points within the single cylindrical layer defined by (r_0, r_1).

2.5 Solution Procedure and Regularization

The imposition of boundary, initial, and interface conditions leads to a linear system of equations represented by:

$$A\mathbf{c} = b, \tag{8}$$

where the matrix A contains the fundamental solutions evaluated at the collocation points, the vector $\mathbf{c}$ holds the unknown coefficients to be determined, and the vector b contains the values of the imposed boundary and initial conditions.

Due to the potential ill-conditioning of matrix A, the **Tikhonov regularization** method is employed exclusively to stabilize the solution process. This technique consists of solving the minimization problem:

$$\min_{\mathbf{c}} \left(\|A\mathbf{c} - b\|^2 + \lambda \|\mathbf{c}\|^2 \right), \tag{9}$$

where $\lambda > 0$ is the regularization parameter that balances the trade-off between solution accuracy and numerical stability. The corresponding regularized linear system is solved by:

$$\left(A^T A + \lambda^2 I \right) \mathbf{c} = A^T b, \tag{10}$$

Where the algorithm used to calculate the MFS is found in Algorithm 1.

Algorithm 1 Transient MFS for Heat Conduction in a Single Cylindrical Layer with Heating Control

procedure MFS($\mathcal{X}_c, \mathcal{X}_s, t_0, t_f, \Delta t, \alpha, k, \lambda, T_{crit}, T_{target}$)
  Initialize $t \leftarrow t_0$, $\mathbf{T} \leftarrow \mathbf{T}_0$, heating_on $\leftarrow$ **false**
  while $t \leq t_f$ **do**
    $T_{surf} \leftarrow T(r_{max}, t)$
    if $T_{surf} < T_{crit}$ **then** heating_on $\leftarrow$ **true**
    else if $T_{surf} \geq T_{target}$ **then** heating_on $\leftarrow$ **false**
    Build fundamental solution matrix $\mathbf{A}$ (Eq. (3))
    Form linear system $\mathbf{Ac} = \mathbf{b}$ with boundary conditions including heating_on
    Regularize and solve: $\mathbf{A}_{reg} \leftarrow \mathbf{A}^T\mathbf{A} + \lambda^2\mathbf{I}$, $\mathbf{b}_{reg} \leftarrow \mathbf{A}^T\mathbf{b}$; solve $\mathbf{c}$ (Eq. (10))
    Compute $T(r, t + \Delta t) \leftarrow \sum_j c_j F(r, t + \Delta t; \rho_j, \tau_j, \alpha)$ (Eq. (4))
    $t \leftarrow t + \Delta t$
    Update boundary conditions
  end while
  return temperature field $T(r, t)$ for $r \in (r_0, r_1)$, $t \in [t_0, t_f]$
end procedure

2.6 Finite Difference Scheme for Comparison

To validate and benchmark the results obtained via the Method of Fundamental Solutions, an explicit finite difference scheme was implemented to solve the transient heat conduction equation in cylindrical coordinates for the single-layer domain.

The governing equation is discretized in space and time using a forward difference in time and central differences in space. For a uniform radial grid with spacing Δr and time step Δt, the temperature at the i-th radial node and n-th time step, u_i^n, is updated as:

$$u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{(\Delta r)^2} \left[\left(1 - \frac{\Delta r}{2r_i}\right) u_{i-1}^n - 2u_i^n + \left(1 + \frac{\Delta r}{2r_i}\right) u_{i+1}^n \right], \tag{11}$$

where $r_i = r_0 + i\Delta r$ is the radial coordinate of node i. Boundary and initial conditions are applied at each time step accordingly. This explicit scheme provides a simple and direct numerical solution, serving as a reference to assess the accuracy and stability of the MFS-based solution under comparable discretization and time stepping parameters.

3 Results and Discussion

3.1 Numerical Experiments and Accuracy Verification

The numerical experiments were conducted on a system with an AMD Ryzen 7 5700X CPU, 16 GB DDR4 RAM, and a GALAX GeForce RTX 3060Ti GPU. To verify the accuracy of the Method of Fundamental Solutions (MFS), the Root Mean Square Error (RMSE) and the relative error percentage between the MFS results and reference solutions were computed as follows:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(u_i - \hat{u}_i)^2} \qquad E_{rel} = \frac{|T_{MFS} - T_{FDM}|}{|T_{FDM}|} \times 100\% \tag{12}$$

where u_i denotes analytical solution values, $\hat{u}_i$ the MFS approximations, and T_{MFS} and T_{FDM} are temperatures obtained by MFS and Finite Difference Method, respectively. This comparison evaluates the robustness and applicability of the MFS using the established FDM as benchmark.

3.2 Input Variables and Nondimensionalization

The model evaluation focused on the temperature distribution at the duct surface and the heating power profile over time. Heating is activated when the surface temperature drops below a critical threshold T_{crit} and deactivated upon reaching a target temperature T_{target}. Simulations used physical parameters: initial temperature $T_0 = 80°C$, ambient temperature $T_e = 4°C$, duct radius $r_{ext} = 0.0841\,m$, thermal conductivity $k = 12.54\,W/m°C$, density $\rho = 933.59\,kg/m^3$, specific heat $c_p = 1826.80\,J/kg°C$, and convective coefficient defined by $Bi = 1.0$. Both T_{crit} and T_{target} were nondimensionalized in the simulations. To maintain consistency between the Method of Fundamental Solutions (MFS) and Finite Difference Method (FDM), variables were nondimensionalized as:

$$\theta = \frac{T - T_e}{T_0 - T_e}, \quad Fo = \frac{\alpha t}{r_{ext}^2}, \quad Q^* = \frac{q}{k(T_0 - T_e)/r_{ext}},$$

with $\alpha = k/(\rho c_p)$ representing thermal diffusivity.

3.3 Discussion

Table 1 summarize the error metrics and computational times obtained for the temperature distribution on the duct surface and the heating power profile, respectively. The Method of Fundamental Solutions (MFS) was evaluated for increasing numbers of source points M and collocation points N, with a fixed number of iterations dedicated to the heating control. The Root Mean Square Error (RMSE) values for the temperature distribution remain below 0.032 across all tested discretizations, demonstrating strong agreement between the MFS solution and the Finite Difference Method (FDM) benchmark.

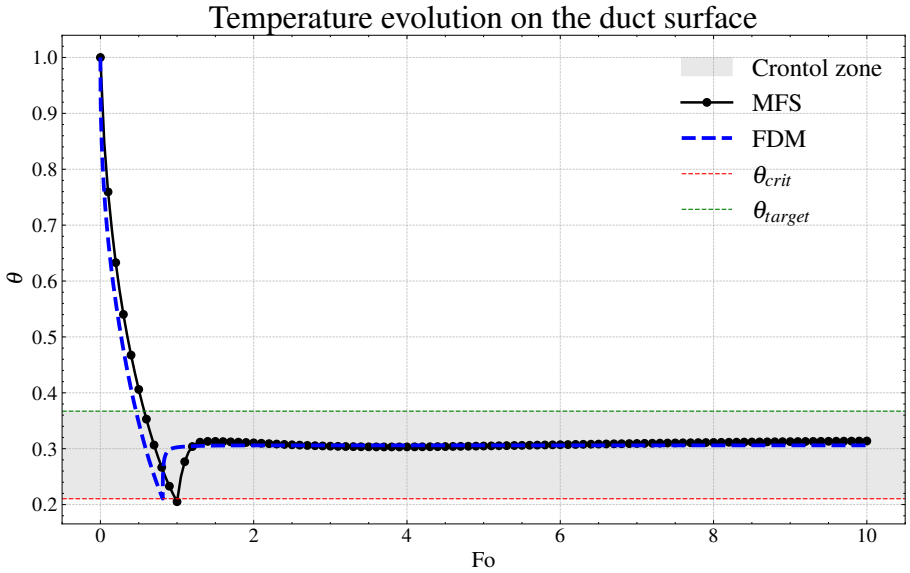


Figure 1. Comparison of the transient temperature evolution at the duct surface obtained by the Method of Fundamental Solutions (MFS) and the Finite Difference Method (FDM)

Additionally, the Mean Squared Error (MSE) decreases as the discretization is refined, indicating convergence and improved accuracy. Although computational time increases with problem size, the MFS remains significantly more efficient compared to the total simulation time required by the FDM approach. Regarding the heating power control, RMSE values are slightly higher, ranging approximately from 0.07 to 0.08. This reflects the increased complexity and sensitivity inherent to the power profile estimation. Nonetheless, these errors remain within acceptable limits for effective practical monitoring.

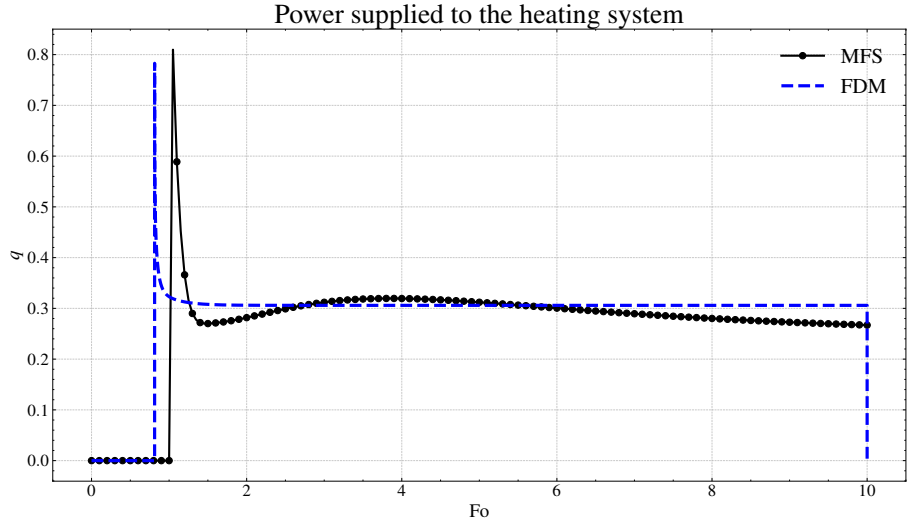


Figure 2. Heating power profile over time as predicted by the MFS and FDM.

The temporal domain was discretized with an adimensional time step $\Delta t = 5 \times 10^{-5}$, resulting in 200,000 iterations to reach a final dimensionless time $t_f = 10$. Each simulation required on average 9 seconds of CPU time. This fine temporal discretization ensured stability and convergence of the explicit finite difference method, as confirmed through sensitivity analyses.

Table 1. Temperature Distribution and Heating Power Results obtained by MFS

Temperature Distribution						**Heating Power**					
M	N	Iterations	RMSE	MSE (%)	CPU (s)	M	N	Iterations	RMSE	MSE(%)	CPU (s)
20	40	250	0.03034	1.980	1.22790	40	120	200	0.06949	3.511	5.09094
24	48	250	0.02778	1.788	1.85960	45	90	200	0.07169	4.056	4.25296
28	56	250	0.02675	1.626	2.33600	45	135	200	0.08461	3.340	7.15868
40	80	250	0.02244	1.378	4.95350	50	100	200	0.07985	4.332	5.19382
40	120	200	0.02196	1.015	5.09094	55	100	200	0.08359	4.929	6.47312

The heating activation, triggered when the surface temperature falls below the critical threshold, and deactivation near the target temperature, were consistently captured by both methods. This control behavior is illustrated in Figures 1 and 2, where the evolution of the heating power supplied and the temperature on the duct surface are plotted, respectively. The close match between MFS and FDM results in both figures validates the reliability of the MFS for transient thermal monitoring and active control applications.

4 Conclusion

This study demonstrated the effective application of the Method of Fundamental Solutions (MFS) for transient temperature monitoring and active heating control in a single cylindrical layer. The MFS provided accurate temperature predictions and heating power profiles in close agreement with the Finite Difference Method benchmark, while requiring significantly less computational effort. The use of Tikhonov regularization ensured numerical stability during the solution process. Overall, the results confirm that MFS is a reliable and efficient alternative for real-time thermal monitoring and control in cylindrical systems.

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